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MDCCCLXXXVII.

Erratum in Professor Ewing's Paper on Experimental Researches in Magnetism.

'Phil. Trans.,' vol. 176 (1885, Part II.).

In formula at foot of p. 538, for $4lm$ read $4m$, and for $4lH_7^2$ read $4H_7^2$.

Adjudication of the Medals of the Royal Society for the year 1886, by the President and Council.

The Copley Medal to Professor Franz Ernst Neumann, For. Mem. R.S., for his researches in Theoretical Optics and Electro-dynamics.

A Royal Medal to Francis Galton, F.R.S., for his Statistical Inquiries into Biological Phenomena.

A Royal Medal to Professor Peter Guthrie Tait for his various Mathematical and Physical researches.

The Rumford Medal to Professor Samuel P. Langley for his researches on the Spectrum by means of the Bolometer.

The Davy Medal to Professor Jean Charles Galissard de Marignac, For. Mem. R.S., for his researches on Atomic Weights.

The Bakerian Lecture, "Colour Photometry," was delivered by Captain Abney, R.E., F.R.S., and Major-General Festing, R.E.

The Croonian Lecture, "On the Coagulation of the Blood," was delivered by L. C. Wooldridge, M.B., D.Sc.

X. *Effects of Stress and Magnetisation on the Thermoelectric Quality of Iron.*

By Professor J. A. Ewing, *B.Sc., F.R.S.E., University College, Dundee.*

Communicated by Sir William Thomson, *F.R.S.*

Received February 24,—Read March 11, 1886.

[Plates 21–23.]

§ 1. In May, 1881, the writer submitted to the Royal Society a paper entitled "Effects of Stress on the Thermoelectric Quality of Metals, Part I.," an abstract of which was published in the 'Proceedings,' vol. 32 (1881), p. 399. The paper was described as incomplete, and its further publication was postponed until additional results should be submitted. The experiments were continued in 1882–3, in the physical laboratory of the University of Tokiô, in conjunction with others on the magnetisation of iron. These latter, which were communicated to the Royal Society in January, 1885, and are now being published in the 'Philosophical Transactions,'* have a very intimate relation to the subject of this paper, and the writer has for this reason deferred the publication of the thermoelectric experiments until the appearance of the experiments in magnetism. The present paper embodies the results of the paper referred to above as Part I. (sufficiently fully to make separate publication of that part unnecessary), along with those of subsequent work, in which the writer has to acknowledge the very valuable help of one of his Japanese students, Mr. S. Sakai.

§ 2. In the earlier experiments the thermoelectric effects of stress on an iron wire were studied by exposing a piece of the wire to longitudinal pull, while one of the junctions between the stressed and the unstressed part was kept at a temperature of about 100° C., the other junction being at the atmospheric temperature, or about 15° C. to 20° C. The wire was held at one end of the stressed portion by being twisted round a fixed hook. Various plans of heating this junction were tried; the plan finally selected, and followed throughout all the experiments, was to immerse the hook in a bath of hot oil, whose temperature was kept very nearly uniform by a regulated flame under it. The other end of the stressed portion of the wire was connected to a cord which passed over a fixed pulley, and from which a light water-tank hung. Pull was applied to the wire by running water into the tank, and could be relaxed by allowing water to escape through a stopcock at the foot of the tank. Water was run in at a sensibly uniform rate, and the pull on the wire was given by

* "Experimental Researches in Magnetism," Phil. Trans., vol. 176 (1885), p. 523.

the reading of a glass gauge-tube at the side of the tank, which was graduated to show each kilogramme of pull on the wire. In all the experiments described in Part I. the water-tank was used as the means of applying load; but in later work it was found more convenient, and not less satisfactory, to load the wire by stringing discs of lead on a vertical rod which took the place of the tank.

§ 3. The wire, which was a piece of moderately soft iron, was well annealed, to begin with, throughout its whole length by heating it to bright redness and allowing it to cool in the air. Its slack ends were soldered to copper leading wires, and these were connected to a mirror galvanometer whose resistance was 0·25 ohm. The effects of stress in altering the thermoelectric quality of the iron were measured by the deflection of the galvanometer, that being proportional to the electromotive force due to the difference of temperature of the two ends of the stressed part. This electromotive force was so small that the galvanometer required to be adjusted to a high degree of sensibility by using a controlling magnet to weaken very much the earth's directing field on the needle. In consequence of this the zero position of the needle became somewhat variable, and to eliminate error from this cause, double readings of the deflection were always taken by manipulating a reversing-key in the circuit. At the two copper-iron junctions a very slight difference of temperature would have given rise to an electromotive force greater than that which it was the object of the experiments to measure, and much care had to be taken to keep their temperature the same. They were immersed close together (but, of course, without contact) in a large beaker of paraffin oil, which was stirred from time to time, and was screened from radiation. In later experiments the value of the thermoelectric electromotive force was determined in absolute measure. In those, however, which have first to be described its value is stated only in the arbitrary unit which corresponds to one scale division of the galvanometer.

§ 4. The effect is called positive when the piece of iron affected has its thermoelectric position (with the assigned temperatures of the junctions) shifted towards bismuth in the thermoelectric series. In other words, the effect is called positive when it causes a current to pass from the part affected to the part unaffected through the hot junction.

The following (§§ 5–16) is a summary of the results stated in the paper referred to above as "Part I.," illustrated by figures taken from that paper.

§ 5. The first effects of loading are in many cases somewhat uncertain. This may be ascribed to the fact that continuous portions of any sample of wire differ more or less in thermoelectric quality, and the yielding which occurs during the first loading is apt to make some variation in the precise position along the wire of the hot junction. On the whole, however, the experiments agree in showing that the first effect of loading an annealed iron wire is *negative;* that is to say, if there is no current before the load is applied, the effect of a moderate amount of load is to make a current flow from the unstressed to the stressed portion of the wire through the hot junction.

This is in agreement with observations published as long ago as 1856 by Sir W. THOMSON, who has laid the foundation of the subject in Part III. of his great series of papers on the "Electrodynamic Qualities of Metals." (Phil. Trans., vol. 146 (1856), p. 709, or Reprint of Papers, vol. II.)

§ 6. If, however, the application of load be continued up to and beyond the limit of elasticity of the metal, the negative effect passes a maximum, becomes much reduced, and generally even changes to positive before the wire breaks.*

A characteristic example of this action is given in fig. 1, Plate 21, which shows the thermoelectric effect of loading an annealed iron wire up to the breaking point. The initial electromotive force (which has a small negative value) is due to accidental defect of homogeneity in the wire. The maximum negative E.M.F. is reached at or near the limit of elasticity (12 kilos.), and the subsequent drawing out is associated with a rapid change towards positive. After the wire was broken the load was removed, and the broken ends joined. A strong positive deflection then showed that the part which had suffered longitudinal strain, and was now freed from stress, was thermoelectrically positive to the other portions. This result has also been previously pointed out by Sir W. THOMSON† as well as by MAGNUS.‡

§ 7. Another experiment is given in fig. 2, where $a\ a$ shows the effect of the first loading of another piece of annealed iron wire of larger section than the last. In this case the loading was stopped at 35 kilos., before the wire broke, and the load was gradually removed. The thermoelectric changes which took place during the removal are shown by the curve $b\ b$. During removal of load the E.M.F. again changed to negative, passed through a negative maximum, changed to positive, and finally, when the load was entirely removed, exhibited the positive value which MAGNUS and THOMSON have remarked.

The load was then gradually reapplied, and this resulted in the complex changes of E.M.F. shown by the curve $c\ c$—a curve by no means coincident with that found during the removal of the load. The loading was continued past 35 kilos. At 36 kilos. a sharp change of gradient occurred as the wire (hardened by the previous stretching) began again to draw. At 38 kilos. the wire broke.

§ 8. Fig. 3 shows the results of another experiment of the same kind, in which another piece of the same iron wire was stretched (after annealing), first by the application and removal of 35 kilos., then of 40, and finally broken, in the third loading, by a stress of 43 kilos. The thermoelectric currents were observed for every kilogramme of load, during application and removal. In the first loading ($a\ a$) while the wire was still soft the currents were irregular, but at 30 kilos., when the wire began to

* *Cf.* a paper by Mr. G. W. VON TUNZELMANN on "The Production of Thermoelectric Currents in Wires subjected to Mechanical Strain," Phil. Mag., ser. 5, vol. 5 (1878), p. 339, where this result has been previously stated. In other respects the present writer's observations do not altogether agree with those of Mr. VON TUNZELMANN.

† *Loc. cit.*, § 109.

‡ POGGEND. Annal., vol. 83 (1851), p. 469.

draw, they became steady, and the curves of E.M.F. and load were well defined throughout the remainder of the experiment, and have the same character as those already described. In this instance the whole diagram lies, to an unusual degree, on the positive side, probably because of some local peculiarity in the metal at the hot junction.

Here, again, the effect of each step in the process of drawing out is well marked (first at 30 kilos., then at 37 kilos., and last at 42 kilos.). At each extension the stretched portion becomes more and more positive. The want of coincidence between each "off" curve (or curve of E.M.F. and load during removal of load), and the succeeding "on" curve (or curve of E.M.F. and load during application of load), is as conspicuous as in fig. 2.

§ 9. This last is the most noteworthy of the phenomena now under review. In drawing attention to it in the paper submitted to the Royal Society in 1881, the writer was not aware that it had been previously noticed by any observer. He is now glad to be able to refer to an important paper by Herr EMIL COHN,* who has anticipated him in the discovery of this very interesting characteristic of the curves of stress and thermoelectric quality. In the figure which Herr COHN has given to illustrate his experiments with annealed iron wires, the process of loading has not been carried far enough to pass the negative maximum of E.M.F. shown in figs. 1, 2, and 3, and for this reason the curves have a comparatively simple form. But the "on" and "off" curves differ from one another in just the way which is shown by the independent evidence of the present writer's experiments.

§ 10. For the next group of experiments another piece of the same wire was taken, and, after annealing, was loaded with 21 kilos., which caused some permanent extension. This load was left on for a time, and was then removed. In all the subsequent loadings of this wire the load never exceeded 21 kilos., so that subsequent loadings caused no other than elastic strain.

The load of 21 kilos. was then repeatedly applied and removed. This brought the changes of E.M.F. to a cyclic state. They are shown in fig. 4, Plate 21, for the cycle of loads 0—21—0. Numerous experiments with other specimens have shown that the curves of fig. 4 are thoroughly characteristic of the behaviour of a piece of iron which, after annealing, has been stretched beyond its limit of elasticity, and is then alternately loaded and unloaded.

The great difference between the "on" and "off" curves will be realised when it is noticed that the negative maximum occurs at about 18 kilos. on the "on" curve, but at near 8 kilos. on the "off" curve, and that a load such as 4 kilos., when reached from a lower value, has associated with it a strongly *positive* quality on the part of the stressed piece, while if approached from a considerably higher value it is associated with an even more strongly *negative* quality.

* "Ueber das thermo-electrische Verhalten gedehnter Drähte," WIED. Annal., vol. 6 (1879), p. 385. A few similar observations have more recently been published (three years after the present writer's first paper on the subject was read before the Royal Society) by R. OVERBECK, WIED. Annal., vol. 22 (1884), p. 344.

§ 11. One conspicuous feature in this difference of thermoelectric quality during loading and unloading is an apparent lagging of the thermoelectric change behind the change of stress. This aspect of the action may justify the use, for the sake of brevity in referring to it, of the name *hysteresis*, introduced by the writer in speaking of a similar phenomenon which presents itself whenever changes of magnetisation are caused by changes of magnetising force or by changes of stress.*

This hysteresis is (as Herr COHN has already pointed out) an essentially *static* phenomenon. It is in no way affected by the speed at which load is applied and removed. In some of the writer's experiments the rate of loading and unloading was varied tenfold without causing any perceptible change in the form of the curves or in the area included between them. In other experiments a particular value of the load was kept for a long time constant, but the mere lapse of time appeared to be without effect. The value of the E.M.F. reached by any process of loading or unloading remains constant if the wire be undisturbed. The E.M.F. associated with any particular load is, of course, capable of assuming any value within a wide range, in dependence on the particular mode of loading by which the assigned load is reached. Any point within the area enclosed by the curves expresses a possible relation between E.M.F. and load, and the actual relation depends not only on the actual load, but on all the preceding states of load, especially on those which have immediately preceded the actual state. The effect of adding any load is by no means necessarily the same as the effect of removing that load, unless the operation be repeated, by itself, often enough to reduce the corresponding thermoelectric changes to a cyclic state.

§ 12. Figs. 5, 6, 7, 8, and 9, Plate 21, further illustrate the character of this hysteresis in the relation of thermoelectric quality to stress. They all represent experiments made on the wire of fig. 4. In fig. 5 the cycle 0—14—0 was performed, with the result that although the "on" branch had stopped short of the negative maximum (at 18 kilos. in fig. 4), nevertheless there was a very distinct negative maximum on the "off" branch. Fig. 5 is similar to the figure given by COHN as representative of the behaviour of annealed iron. Other experiments showed the same thing to be true if the process of loading was stopped anywhere on the steeply-descending portion of the "on" curve. Fig. 6 shows the loop formed by superposing on the main cycle, 0—21—0, the small cycle, 11—3—11, starting at the point 11 kilos. on the main "on" curve. The dotted line in fig. 6 gives the result of a separate experiment, showing the effect of stopping unloading at 12 kilos. on the "off" curve and reapplying the full load. It illustrates how the points which define the relation of E.M.F. to load are not even limited to fall within the main "on" and "off" curves. Fig. 7 shows the curious changes of E.M.F. which occurred when at the point 16 kilos. on the "on" curve, the load was reduced to $7\frac{1}{2}$, then reapplied, and the loading continued to 21 kilos. In the same way, fig. 8 shows the effect of loads 0—12—6—15. Fig. 9

* Hysteresis, from ὑστερέω, *vide* Proc. Roy. Soc., vol. 33 (1881), p. 22; vol. 34 (1882), p. 39; vol. 36 (1883), p. 123; and Phil. Trans., vol. 176 (1885), p. 524.

shows the formation of a loop on the "off" curve at the point 4 kilos., by superposing the small cycle 4—10—4.

§ 13. The characteristic difference between the "on" and "off" curves is not, however, a mere lagging of changes of thermoelectric quality behind changes of stress, such as the lagging of magnetic change which is observed when the state of stress of a magnetised piece of iron is varied.* It is that, but it is something more. An inspection of the figures will make it evident that, in general, a reversal from loading to unloading, or from unloading to loading, causes at first a continuation of the same kind of thermoelectric change as has been going on before the reversal takes place. Thus on the "off" curve, towards the close of the removal of load, a rapid change of E.M.F. towards positive is going on, and when loading begins the change towards positive continues in the earliest part of the "on" curve. And it must be borne in mind that this happens irrespective of the speed of unloading and loading, and even if a long interval of time be allowed to elapse between the unloading and the next loading. Again (figs. 5 to 9), to stop loading at a point on the "on" curve, when a rapid change of E.M.F. towards negative is going on, and to begin unloading, causes the metal to continue at first to change towards negative. The only case in which the figures do not clearly bear out this remark is where, after the application of a load equal or nearly equal to the greatest value used in these experiments, the process of unloading begins. But careful experiments made with the express object of elucidating this point have given grounds for believing that, there as elsewhere, the very first effects of reversal of the process from loading to unloading were of the same sign as the effects of the process which was reversed. Certainly the phenomenon in question is much less well marked with high than with low values of the load, and is not (with high values) easily distinguished from inevitable errors of experiment. But it may be affirmed with confidence that generally, if not universally, the earliest effect of reversal from loading to unloading, or *vice versâ*, is to continue the same kind of thermoelectric change as has been going on just before the reversal takes place.

§ 14. *Effects of Vibration.*—These were studied by briskly tapping the wire during the application and removal of load, at the same time observing the thermoelectric currents. It was found that vibration destroys the effects of hysteresis nearly, though not quite, completely, and causes the "on" and "off" curves to approach towards coincidence. Fig. 10, Plate 21, illustrates this by showing the effect of applying to the same piece of wire as before the cycle of loads 4—19—4 without and with vibration. The wire was kept vibrating by tapping it briskly with a piece of wood, and the load was not reduced below 4 kilos., in order that the wire might remain tense enough for the tapping to take effect. Fig. 10A shows the changes of thermoelectric quality when the cycle 4—19—4 was gone through in the usual way without vibration, after several applications of the same cycle had brought about a cyclic state. Fig. 10B shows the effect of the same cycle, with vibration during the application and removal

* Phil. Trans., vol. 176 (1885), p. 587, § 75. Proc. Roy. Soc., vol. 34 (1882), p. 43.

of the load. The influence of hysteresis is still observable, though to a far less degree than when the wire was undisturbed. The change caused by vibration will be best understood by considering the figs. 10A and 10B in their relation to the curves which have been already given for the complete cycle of loads, 0—21—0.

§ 15. In another experiment the influence of vibration in removing the accumulated effect of hysteresis was examined as follows:—The wire was loaded, without vibration, to 9 kilos. The galvanometer deflection, measuring the thermoelectric current, was then −99. Then, with the load of 9 kilos. kept constant, the wire was vigorously tapped, and the deflection changed to −144. Next the load was increased to 21 kilos. and then reduced to 9 kilos. The deflection was then −175. On tapping the wire, with the load of 9 kilos. kept constant, this changed to −150. Thus the total difference due to hysteresis of 175−99 or 76 scale-divisions was reduced by tapping to 6 divisions.

§ 16. In a supplement to his original communication to the Royal Society in 1881, the writer described a further experiment which showed that changes of magnetisation had an effect like that of vibration in removing the influence of hysteresis in the relation of thermoelectric quality to stress. When a constant load was applied, and the wire was gently stroked along its length by a bar magnet, the thermoelectric current changed in much the same way as it would have changed had the wire been subjected to a comparatively violent amount of mechanical disturbance, and to a degree much greater than could be explained as due to the slight mechanical disturbance caused by contact with the magnet. The influence of magnetisation, both in this and other respects, has formed the subject of many other experiments which will be described in the sequel.

§ 17. So far, only those results have been stated which were mentioned in the paper, "Part I.," communicated to the Royal Society in 1881. Figs. 1 to 10 inclusive have been copied from that paper. It remains to describe later experiments, made for the most part in 1883. Their principal object has been to examine the relation between the effects of stress on thermoelectric quality, on the one hand, and on magnetisation on the other. A glance at the figures given here and at those given elsewhere,* to illustrate the changes of magnetism produced by loading and unloading an iron wire, will show that there is a very striking likeness between them; enough likeness, in fact, to make the hypothesis very natural that stress acts on the thermoelectric quality of iron by altering its magnetisation, and that the variations of magnetism are the immediate cause of thermoelectric change. In the writer's earlier experiments no precautions were taken to rid the iron of magnetism to begin with, nor to place it in a position where it would escape magnetisation by the terrestrial field, and it was certainly in a more or less magnetic condition throughout the experiments. Sir W. THOMSON has shown† that change of magnetism does give rise

* Phil. Trans., vol. 176 (1885), Plates 62, 63; Proc. Roy. Soc., vol. 34 (1882) pp. 43, 44.

† *Ibid.*, vol. 146 (1856), p. 722, § 134.

to change of thermoelectric quality, and it is well known from the work of THOMSON and others that stress affects the magnetism of iron, whether that be residual or induced. Herr COHN has not overlooked the possibility that the thermoelectric effects of loading may be explained as secondary results of the changes of magnetisation which it causes, but says that, while he has no conclusive evidence either way, he inclines to the view that the observed phenomena are not to be explained as results of changes in magnetism.

The matter is one of much interest, for if the results are due to stress simply, and not to the magnetic changes caused by stress, we have here a novel instance of hysteresis, entirely distinct from the instances which are common in magnetic phenomena.

§ 18. To put the question to a conclusive test the writer, in his second series of experiments, employed independent means to produce and to measure magnetisation in the loaded wire. In some experiments the wire was magnetised to begin with, or was exposed throughout to a constant magnetising force, while the effects of loading and unloading on the thermoelectric quality were examined. In others the wire was completely demagnetised before loading began, and was kept free from magnetising force during the process. In others still the effects of magnetisation alone on thermoelectric quality were measured, by varying the magnetic field while the wire was either without load, or was kept loaded with a constant weight.

The results proved beyond question that the effects of stress on thermoelectric quality (though modified by the presence of magnetisation) are not secondary effects of the changes of magnetisation; and that the hysteresis of thermoelectric quality with regard to stress is a thing distinct from the hysteresis of magnetism, though probably proceeding from the same peculiarity in the molecular mechanism.

§ 19. In these experiments the loading was done by lead weights, instead of by a water-tank. The wire was held in much the same way as before, with the hot junction in a vessel of heated oil, and the cold junction simply exposed to the air. The wire was set in a sloping position at right angles to the lines of terrestrial magnetic force; and over it, covering the whole of the portion under stress, was slipped a tube, on which a magnetising solenoid was wound. Over against the upper end of the stressed portion was a mirror magnetometer, to measure the magnetisation of the part under stress. The continuations of the iron wire on either side of the part under stress were led away so that they scarcely affected the magnetometer, and were as much as possible prevented from becoming magnetised.

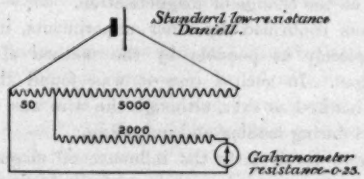
§ 20. An arrangement was provided by which the wire would be demagnetised by reversals, in the manner described in the writer's paper of "Researches in Magnetism," * by passing a continuously and gradually diminishing current through the solenoid, while the direction of the current was rapidly reversed many times.

The magnetisation of the wire, and the magnetising force applied to it, were

* Phil. Trans., vol. 176 (1885), p. 539, § 19.

determined in absolute measure by the same methods as those described in the paper referred to. To reduce the thermoelectric effects to absolute measure, the resistance of the circuit, consisting of the iron wire, leading wires, and galvanometer, was carefully determined, and the "galvanometer constant" was found by passing through it the current produced by exposing the ends of a coil of large and known resistance to a difference of potential amounting to rather less than $\frac{1}{100}$, $(\frac{2}{207})$ of the E.M.F. of a standard DANIELL'S cell, the fractional potential being furnished in the manner sketched in fig. 11. From this was calculated the deflection which, in the actual thermoelectric circuit, would be produced by 1 microvolt ($1 \text{ volt} \times 10^{-6}$), and the thermoelectric effects are expressed in microvolts.

Fig. 11.



§ 21. In all the following experiments pieces of iron wire from the same bundle were used, and the experiments of fig. 12 to fig. 31 inclusive were made with the same specimen. The wire was 0·64 mm. in diameter originally. After annealing it was stretched with a load of 12 kilos., which gave it a considerable permanent set, and reduced its diameter to 0·61 mm. The subsequent loadings were kept within this amount, and consequently caused no further permanent set.

§ 22. In the first experiment, fig. 12, the wire had some residual magnetism, though no magnetising field was in action, in the direction of its length. The cycle of loads 0—11 kilos.—0 was repeated several times, and the thermoelectric and magnetic effects were observed. In this case they have not been reduced to absolute measure; and the scale readings are stated below as they were taken. The magnetometer and galvanometer readings are taken from the true zero.

Load.	Magnetometer.	Galvanometer (thermoelectric current).	Load.	Magnetometer.	Galvanometer (thermoelectric current).
kilos.			kilos.		
0	37	50	11	77	−12
2	45	69	10	78	−23
4	70	12	8	80	−50
6	84	−26	6	84	−85
8	84	−46	4	86	−79
10	79	−24	2	68	−23
11	77	−12	1	56	+18
			0	41	50

Fig. 12, Plate 21, shows the changes of thermoelectric current in relation to load in this experiment, and fig. 12A the (simultaneous) changes of magnetism, also in relation to load. The latter are not so perfectly cyclic as are the former, and in other respects, which will be evident on a close inspection, the two curves present noticeable features of difference, as well as a very striking general similarity.

§ 23. The effect of magnetisation, separately, was then examined. While the wire was kept free from load a current was established in the magnetising solenoid, and strengthened step by step until the magnetisation was as great as the maximum value reached by loading in the previous experiment. This was found to produce very little change of thermoelectric quality—only about one-twentieth as much as occurred during loading. It was clear from this that the thermoelectric effect of stress was not a secondary result of the change of magnetisation.

The same result was confirmed by other experiments, in which the wire was demagnetised as completely as possible by the method of reversals before the process of loading began. In such a case it was found that the thermoelectric effects of load were as marked as ever, although the wire was, and remained, sensibly free from magnetisation during loading and unloading.

§ 24. To examine more particularly the influence of magnetism on the thermoelectric effects of stress, the same wire was used in a group of experiments, whose results are exhibited in figs. 13 to 18, Plate 22. In the experiments in this group the wire received more or less magnetisation to begin with, by the application of a magnetic field which was maintained constant during the process of loading and unloading. The changes of magnetism and the changes of thermoelectric quality, produced by loading and unloading, were then observed, and both are shown graphically in their relation to the loads, and in absolute measure. The temperatures of the junctions were 160° C. and 20° C.

In each case, after the magnetising field had been brought to a constant value, the load was raised from 0 to 12 kilos. (which was the stress originally employed in stretching the specimen, § 21). It was then reduced to 0, then reapplied up to 12 kilos., and finally reduced to 0. The effects of the first application and removal are shown in each figure by full lines, and the effects of each second application and removal by dotted lines.

First, the wire after being set in position at right angles to the earth's lines of magnetic force was demagnetised by the method of reversals, and then loaded, while readings of the thermoelectric effect were taken. These are given, reduced to microvolts, in Curve Iα, fig. 14. The magnetometer was watched during this process, but it remained undeflected, there being neither residual magnetism nor inducing field. The second "off" curve in Iα was so nearly coincident with the first "off" curve that it has not been drawn.

§ 25. Next (the wire being free from load) a current was established in the magnetising solenoid, of such strength as to give an intensity of magnetisation $\mathfrak{I}=160$

c.g.s. units. Then, while the current was kept up, the process of loading and unloading was gone through twice. The resulting changes of magnetism are shown in fig. 13, Curve II, and the simultaneous changes of thermoelectric quality in Curve IIa, fig. 15. In order to make the curves of magnetism and thermoelectric quality have a similar aspect, the former are drawn as they would appear if the sign of magnetisation (which is arbitrary) were reckoned negative.

§ 26. Next, the magnetisation was raised to $\mathfrak{I}=228$ by raising the field to a higher value, which was maintained while loads were applied and removed twice. The magnetic results are shown in Curve III, fig. 13, and the thermoelectric results in Curve IIIa, fig. 16.

§ 27. Next, in the same way, the magnetisation was raised to $\mathfrak{I}=598$, and the same procedure followed. The corresponding curves are IV, fig. 13, and IVa, fig. 17.

§ 28. Finally $\mathfrak{I}$ was further raised to 696, and the wire loaded. The magnetic and thermoelectric curves for this last case are V, fig. 13, and Va, fig. 18, respectively.

§ 29. The magnetic Curves II, III, IV, and V are similar to others published by the writer in a former paper,* and their characteristics need not be dwelt on here. A comparison of the thermoelectric curves with them and with each other shows well the gradual transformation which these undergo in consequence of increased magnetism, and the curious likeness which subsists between the effects of hysteresis in the two cases. In both, the effects of the second loading and unloading differ from the effects of the first in much the same manner, except that there is little in the thermoelectric curves to correspond to the immense changes of magnetism which take place when loading begins. In both, the ascending limb of the "on" curve (corresponding to comparatively high values of load) is a small feature when the magnetisation is weak, and becomes transformed into the principal feature as the magnetisation becomes intense. It is exceedingly interesting to trace the change from the familiar form shown of Ia to that of Va, whose connexion with the other would scarcely be suspected were the intermediate links absent. This transformation, although occurring to some extent in both sets of curves, is much more complete in the thermoelectric than in the magnetic group. One of its results is that, whereas in ordinary conditions the general effect of stress on thermoelectric quality is that a moderate pull makes the wire more negative, this effect is reversed when the wire is strongly magnetised. Another general effect of strong magnetisation is that it reduces the influence of hysteresis, making the "on" and "off" curves less widely asunder than they otherwise are. Another is that it makes the wire, when there is no load, thermo-electrically negative. The positive maximum which occurs at an early stage in the first loading is very conspicuous in Ia, less so in IIa, and absent in the remaining figures. The similar positive maximum which occurs during the second loading is present in Ia, IIa, and IIIa, but has become a mere inflection of the curve in IVa and Va. The

* Phil. Trans., vol. 176 (1885), p. 523.

negative maximum on the "off" curve occurs later and later during the removal of load as the intensity of magnetisation increases.

§ 30. Figs. 19–22, Plate 23, exhibit the results of another experiment, on the same wire, made with the object of comparing the changes of magnetisation and thermoelectric quality caused by changes of magnetising force with those caused by changes of load.

The wire—free from load—having been demagnetised by reversals, was carried through a cycle of magnetisation by gradually applying a magnetising force $\mathfrak{H}$ of nearly 17 c.g.s. units, gradually withdrawing and reversing, and finally re-reversing $\mathfrak{H}$, while the magnetisation and thermoelectric quality were measured in the usual way. The consequent changes of $\mathfrak{I}$ in their relation to $\mathfrak{H}$ are shown in fig. 19. The simultaneous changes of the thermoelectric E.M.F. are shown, in their relation to $\mathfrak{I}$, in fig. 20. The precise form of the curves in fig. 20 is somewhat uncertain, in consequence of irregularities which were perhaps due to the difficulty of keeping the thermal condition absolutely constant during so long an experiment. The effect of the first magnetisation is shown by the full line (in fig. 20), of the first reversal by the lower broken line, and of the second reversal by the upper broken line. There is not much evidence of hysteresis in the relation of thermoelectric quality to intensity of magnetisation.

Then the wire was demagnetised by reversals, the magnetising field was reduced to zero, and the process of loading and unloading (up to 12 kilos.) was performed. Fig. 21 shows the resulting thermoelectric changes to the same scale of E.M.F. as is used in fig. 20. During the process the magnetometer was observed, and a very slight residue of magnetism which had escaped the demagnetising process was seen to go through the minute changes shown in fig. 21, where $\mathfrak{I}$ is drawn to the same scale as in figs. 19 and 20. It is obvious, by comparison of figs. 20, 21, and 22, that these changes of $\mathfrak{I}$ were wholly incompetent to account for the changes of E.M.F. in fig. 21. The changes of E.M.F., caused even by the strongest magnetisation reached in the experiment, fell short of those which occurred during the application and removal of load when the wire was almost wholly free from magnetism.

§ 31. The effect of magnetisation on the thermoelectric quality of iron is that longitudinal magnetisation, in either direction, makes the magnetised part more negative in the conditions examined in fig. 20. In that case there was no load on the wire, and the question suggested itself: How is this effect of magnetisation modified by the presence of a constant stress? To answer this question a group of experiments was carried out, whose results are exhibited in figs. 23, 24, 25, and 26, Plate 23. In each of these the wire, demagnetised by reversals to begin with, was alternately magnetised first in one and then in the other direction, as in figs. 19 and 20. The relation of the thermoelectric E.M.F. to the intensity of magnetisation $\mathfrak{I}$ is shown by the figures. Fig. 23 is for the case of no load, and is virtually a repetition of the previous experiment of fig. 20. Fig. 24 is for the case where a constant load of 3 kilos. was applied to the wire. Fig. 25 is for a constant load of 6 kilos., and fig. 26 for 12 kilos. (Figs. 23, 24, and 25 are drawn about the same axes; fig. 26 is drawn about a separate axis of $\mathfrak{I}$

to avoid confusion.) After each of these loads was applied, the wire was subjected to the process of demagnetising by reversals before the magnetisation, whose effects were to be noted, was begun. The importance of this will appear in the sequel. (§§ 32–34.) In all four figures the greatest magnetising force applied was nearly the same, but the magnetisation in figs. 24 and 25 (and to a less degree in fig. 26) was greater than in fig. 23 on account of the fact that the presence of a moderate amount of load increases the magnetic susceptibility of iron, especially of iron which is treated as the specimen under test was treated.*

The figures show that the presence of load diminishes the general thermoelectric effect of magnetisation, and finally reverses it when the load is great. For the reason already given, the form of these curves is not very precisely defined, and the evidence of hysteresis in the relation of E.M.F. to magnetism is, at the best, inconclusive.

§ 32. It has been said that in figs. 24, 25, and 26, after the load was applied, the wire (although then not at all magnetic) was subjected to the process of demagnetising by reversals before the observations were taken. This was because it was found that the process in question was very effective in wiping out what may be called the historical traces of previous actions that are left in consequence of hysteresis. This is a very interesting part of the subject, which may easily be explained by a few examples. Thus, if in the experiment of fig. 24 the wire had simply been demagnetised by reversals with no load, and then loaded to 3 kilos., before the observations were begun, the results would have been (during the first application of magnetising force) very different from those actually obtained. The difference of conditions would have been that, owing to the accumulated effects of hysteresis during the application of load from 0 to 3 kilos., the wire would have possessed a distinctly altered molecular structure from that which it possessed when (as in the actual conditions of the experiment) this accumulated result of hysteresis was removed by repeating the process of demagnetising by reversals after the load was on. Precisely analogous phenomena have been noticed in the writer's paper on magnetism,† where it has been shown that the magnetic susceptibility of iron depends not only on the load present, but on past values of the load, until by the process of demagnetising by reversals the metal is made indifferent to *how* the actual state of load has been reached.

§ 33. In fig. 27, Plate 23, the wire, previously loaded to 12 kilos. and unloaded many times, was subjected to the process of demagnetising by reversals, while the load was off. The thermoelectric E.M.F. was then +6·2 microvolts (see *a*, fig. 27). Then load was applied, up to 12 kilos. (Curve *ab*) and removed (Curve *bc*). The want of coincidence between *c* and *a* represents a difference of molecular structure, caused by the application and removal of load. If the process of demagnetising by reversals had then been repeated, the E.M.F. would have risen to the value shown by *a*. It must be understood that throughout the loading and unloading there was no magnetic field in action and no magnetism in the wire. Next, loading was resumed (Curve *cd*) and continued up to 3 kilos. That load was then kept constant, while magnetisation of the wire

* Phil. Trans., vol. 176 (1885), p. 609, § 93. † *Ibid.*, p. 612, §§ 96–101.

was begun. The results of magnetising are shown in the upper part of fig. 28, Plate 28, which is a curve of E.M.F. and $\mathfrak{I}$. Starting from *d* (fig. 28)—a state identical with *d* in fig. 27—the wire was magnetised until with a value of $\mathfrak{I}=780$ the point *e* was reached. Then gradual reversal of the magnetisation carried the E.M.F. through the values shown by the dotted line *ef*.

§ 34. Next, the wire was demagnetised by reversals and the whole process of loading and unloading repeated twice, but at the second time the unloading was stopped while 3 kilos. remained on the wire. The E.M.F. was then found to be −2·3 microvolts—a state shown by the point *g* in fig. 28. Then, with the load constant, magnetisation gave the Curve *gh*, and reversal of magnetisation gave the Curve *hi*.

Both parts of fig. 28, therefore, represent the thermoelectric effects of magnetising, while a constant stress of 3 kilos. is maintained. But they differ in this, that in the upper part of the figure that load was reached by increment from 0 to 3 kilos., whereas in the lower part of the figure the load was reached by decrement from 12 kilos. This difference in the history of the loads causes a difference of molecular structure which is nearly, though perhaps not altogether,* obliterated by a single application of a strong magnetising force, and in its obliteration gives rise to the widely different curves *de* and *gh*, both of which represent the first effect of magnetisation on the thermoelectric quality of the same piece of iron in one and the same constant state of stress.

Fig. 28 will make it clear why in the series of experiments shown in figs. 23–26 the process of demagnetising by reversals was performed *after* each of the several loads was applied.

§ 35. The experiment was then extended, in the same manner as before, to determine the thermoelectric effects of magnetising under 6 kilos. of load, starting in one case from the point 6 kilos. on the "on" curve (reached, after demagnetisation, by the loads 0—12—0—6), and in the other case from 6 kilos. on the "off" curve (reached, after demagnetisation, by the loads 0—12—0—12—6). The results are shown in fig. 29, Plate 23, from which it will be seen that the first magnetisation gives widely different curves of E.M.F. and $\mathfrak{I}$ in these two cases, but the subsequent reversal of magnetisation gives effects so nearly similar in the two cases that they have been drawn by a single line (the dotted line of fig. 29).

§ 36. Finally, fig. 30, Plate 23, shows in the same way the effect of a first magnetisation and subsequent reversal of magnetism, under a constant load of 12 kilos., reached, after demagnetisation, by loading to 12, unloading to 0, and loading again to 12 kilos.

§ 37. The experiments of figs. 28, 29, and 30 differ from those of figs. 23–26 in this respect, that in the latter the effects of hysteresis, accumulated during the process of loading or unloading which preceded magnetisation, were wiped out by the process of

* The Curves *ef* and *hi* differ by an amount which may possibly be due to variation of thermal conditions in this protracted experiment.

"demagnetising by reversals" before the magnetisation, whose effects were to be observed, was begun; whereas in figs. 28–30 the molecular instability to which hysteresis in loading has given rise is being overcome by the first magnetisation, and this is causing changes of thermoelectric quality which are superposed on what may be called the normal variation due to magnetism. In the experiments of figs. 23–26 the process of "demagnetising by reversals" was resorted to, after each load was on the wire, not for the purpose of removing magnetism (since there was none to remove), but in order to produce a molecular commotion by which this state of instability should be destroyed, and the reminiscence of previous stresses be dissipated.

§ 38. To illustrate more fully the effect of the process of "demagnetising by reversals," an experiment was made whose results are set forth in fig. 31, Plate 23. After loading in the usual manner (the wire being free from all residual magnetism, and exposed to no magnetic field), up to 2 kilos. (Curve *ab*, fig. 31), the process of demagnetising by reversals was gone through while the load was kept constant. The E.M.F. fell in the process from 11·2 to 5·35 microvolts (*bc*, fig. 31). Then loading was continued, up to 4 kilos., giving the Curve *cd*; and at 4 kilos. the process of reversals was again performed, causing a fall of E.M.F. from 5·75 to −0·5 microvolt. Further loading (up to 6 kilos.) gave the Curve *ef*, and the process of reversals gave the fall *fg*: further loading gave the Curve *gh*, and so on, the process of reversals being applied at 8, 10, and 12 kilos. Similarly, at each step of 2 kilos. on the "off" curve, the process of reversals was applied with results which (along with those of the intermediate unloadings) are shown in the figure by dotted lines. As the figure is somewhat involved, the observed deflections and the values of E.M.F. calculated from them are given numerically below:—

Load.	Galvanometer deflection.	E.M.F.	Load.	Galvanometer deflection.	E.M.F.
kilos.		microvolts.	kilos.		microvolts.
0 (after reversals)	165	9·7	11	4	+0·25
1	191	11·2	10	− 18	−1·05
2	191	11·2	After reversals	− 27	−1·25
After reversals	91	5·35	9	− 55	−3·25
3	109	6·4	8	− 80	−4·7
4	98	5·75	After reversals	− 70	−4·1
After reversals	− 8	−0·5	7	− 83	−4·85
5	+15	+0·9	6	− 90	−5·3
6	+11	+0·65	After reversals	− 56	−3·3
After reversals	−50	−2·95	5	− 78	−4·6
7	−28	−1·65	4	− 75	−4·4
8	−32	−1·9	After reversals	− 22	−1·3
After reversals	−80	−4·7	3	− 17	−1·0
9	−47	−2·8	2	+ 7	+0·4
10	−20	−1·15	After reversals	+ 70	+4·1
After reversals	−34	−2·0	1	+ 61	+3·6
11	− 2	−0·1	0	+ 93	+5·45
12	+34	+2·0	After reversals	+143	+8·4
After reversals	+21	+1·25			

Temperatures, 144° C. and 17° C.

From these results, which have been confirmed by other independent experiments, it will be seen that the process of "demagnetising by reversals" has an effect closely similar to that of mechanical vibration in removing traces of hysteresis. (*Cf.* §§ 14, 15.)

§ 39. One very remarkable feature in fig. 31 is the positive maximum (or negative minimum) which the wire passes through when, after the process of reversals, one continues to add more load. This maximum occurs for all values of the load less than 8 kilos., and is found, generally, when rather more than 1 kilo. has been added to the load present during the process of reversals. Further, during the removal of load a negative maximum (or positive minimum) occurs in each stage, as is shown by the dotted curves. These features are conspicuous in the descending limb of the main curve, that is to say, in that part of the diagram which lies between the zero of load and a load of about 8 kilos. In the right-hand portion, or ascending limb, the effects of hysteresis are comparatively indistinct, and the process of reversals affects the actual and subsequent values of the E.M.F. to a much less degree.

§ 40. Experiments like that of fig. 31 were made, with this difference, that instead of applying the process of "demagnetising by reversals" at the points *b*, *d*, *f*, *h*, and so on, the wire was briskly tapped, so that a merely mechanical in place of a magnetic disturbance was set up. Then, when the vibration had subsided, and the altered value of the E.M.F. had been observed, the process of loading was continued. The results were entirely similar to those of fig. 31. After tapping (under any load up to 6 or 8 kilos.) the resumption of loading caused the E.M.F. to have a positive maximum or negative minimum, and the resumption of unloading gave a positive minimum or negative maximum, as in that figure. The molecular state produced by brisk mechanical vibration is like that produced by applying the process of reversals, at least in these particulars—that the traces of previous stress-changes are more or less completely obliterated, and the next succeeding application or removal of stress causes similar complex variations in thermoelectric quality.

§ 41. Attention has already (§ 13) been drawn to the fact that the characteristic difference between "on" and "off" curves of load and thermoelectric quality is not a mere lagging in the change of thermoelectric quality. If it were, a reversal, say, from unloading to loading would cause in the earliest stages no thermoelectric effect, instead of causing (as it generally does) an effect whose sign is the same as that which was previously occurring during the unloading. It occurred to the writer to inquire whether this might not be accounted for as follows, by the instability of molecular structure which, generally speaking, exists at any point in the process of loading or unloading. To fix the ideas, suppose that, after a considerable load has been on, the process of unloading is stopped at a point at or near the zero of load. Before the stoppage the E.M.F. has been rapidly becoming more positive. But the metal is, owing to hysteresis, in a less positive state than it would reach if we were now to vibrate the wire, or subject it to the process of "demagnetising by reversals." In the circumstances assumed any disturbance will produce a change towards positive.

But suppose that no disturbance takes place, and let the process of loading be begun. The observed effect is, first, a change towards positive, then soon a positive maximum, and then a great and continued change towards negative. It is open to conjecture that the first (positive) effect of loading is due to the mere molecular disturbance associated with change of stress; that though the proper effect of loading is negative, that effect lags, and superposed on it there is a positive effect due to the simple disturbance caused by loading, acting on the unstable molecular structure to which the previous unloading has given rise. This view, were it tenable, would certainly make the fact less extraordinary than it otherwise is, that in general a change from loading to unloading, or *vice versâ*, continues the same kind of thermoelectric change as has been going on before.

But this view appears to be negatived by the experiment of fig. 31, and by similar experiments made with mechanical shaking substituted for magnetic shaking. For we find that at *a*, for instance, in spite of the violent molecular commotion which preceded the loading, nevertheless the first effect of loading is to carry the wire through a positive maximum of thermoelectric quality, before beginning that change towards negative which is the main consequence of any moderately great load. And after every other shaking up at intermediate points in the loading the same curious positive maximum comes in before the main trend of the "on" curve is resumed. It is clear that this puzzling characteristic of the curves cannot be ascribed to any ordinary mechanical agitation which may be assumed to accompany changes of stress; for not only does it appear after the metal has been thoroughly shaken into what may be presumed to be a stable state, but at points, such as *c*, *e*, *g*, &c. (fig. 31), it asserts itself in the direction *opposite* to that in which the thermoelectric quality has been altered by mechanical or magnetic disturbance. And it is not a little remarkable that the phenomenon under discussion has no counterpart in the (otherwise similar) hysteresis which the induced or residual magnetism of a piece of iron exhibits when it is forced to vary by application and variation of stress.

§ 42. Up to this point all the experiments of the new series (from § 21 onwards) were made with the same sample of iron wire, which, as has been stated (§ 21), was a piece that had, after annealing, been moderately stretched beyond its elastic limit. Other experiments with other samples, treated in the same way, confirmed the results which have been described. The examples which have been cited are chosen out of a much larger number of tests, which need not be particularly described, as they exhibited no features which have not been already referred to. In the experiments which have been given (§§ 21 to 38) the hot junction had a temperature of from 130° C. to 160° C., and the cold junction a temperature of from 15° C. to 20° C. Care was taken to preserve the temperatures nearly constant during any one experiment, but it was not considered necessary to give them strictly the same values from day to day, as the aim of the research was to study, not the dependence of thermoelectric effects on temperature and temperature differences, but those peculiar features

in the thermoelectric behaviour of iron, with any assigned junction temperatures, which have now been fully exhibited by means of curves.

§ 43. Experiments were made with other pieces cut from the same bundle of iron wire (diameter=0·64 mm.), but not annealed, to test the changes of thermoelectric quality which are caused by longitudinal pull in iron in the hard-drawn state. The results of two representative experiments of this kind with two distinct pieces of wire are shown in figs. 32 and 33, Plate 23. In both cases the portion of the wire between the hot and cold "junctions" was, for some accidental reason, rather strongly negative with respect to the adjoining portions when there was no load, but became positive while under a strong pull.

In the experiment of fig. 32, the temperatures were 135° C. and 21° C. The wire was loaded with 13 kilos., then unloaded, and during the next loading the observations shown in fig. 32 were taken. The form of the curve is similar to that given by COHN for hard wire, except that the loop which hysteresis causes at the upper end appears to have escaped his notice. The general result is that loading has a *positive* effect; and, further, that a mechanically hard wire behaves very similarly to a strongly magnetised but much softer wire. That it is so will be seen by comparing fig. 32, where the wire is hard-drawn but not magnetised, with figs. 17 and 18, Plate 22, where the wire was only slightly hardened by stretching, but was strongly magnetised. Obviously we should expect that iron in a state of hardness intermediate between these would give results resembling those of figs. 15 and 16, which refer to moderately magnetised wires; and this corollary has been verified by experiment with a piece of the same iron which was annealed, but much less perfectly than in the former experiments, and was then loaded and unloaded without magnetisation. In fact the series of figures 14 to 18, which exhibit the effects of various degrees of magnetisation, serve also to illustrate how under various degrees of mechanical hardness there is continuity in form between the familiar type of fig. 4, or 12, or 14, or 21, and the very different looking curve of fig. 32. The phrase, "mechanical hardness," must be understood here to refer only to the kind of hardness which comes of permanent strain, such as stretching or wire-drawing; for the writer has made no experiments to determine whether a like relation holds between the various degrees of hardness which can be given to steel by variations of temper apart altogether from strains. Hardness of the latter kind may confidently be expected to reduce hysteresis, which has now been shown to be reduced by the presence of load, by the existence of a magnetic condition, and by the existence of a state of non-elastic strain or "set."

§ 44. In fig. 33 the temperatures were 143½° C. and 19° C. The wire—an unannealed specimen—had a load of 14 kilos. applied to it; this was removed and a load of 12 kilos. was applied and removed while the observations were taken whose results are exhibited in the figure. This figure, along with the last, shows that the curves for hard-drawn iron differ from the curves for softer wire chiefly in this, that the negative maximum, familiar in the earlier figures, is now shifted so close to the line of zero

load as to appear as a mere inflection in the "on" curve, though still present in the "off" curve; while, as part of the same action, the outer or ascending limb of the curve becomes the conspicuous portion and occupies nearly the whole figure.

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§ 45. At the conclusion of the above experiments with iron, the effects of stress on the thermoelectric quality of certain other metals were examined in the same way, with the view of testing for hysteresis in changes of thermoelectric quality caused by applying and removing longitudinal pull. Wires of silver, copper, lead, magnesium, and german silver were tried, but in no case did the results of loading and unloading afford any evidence of hysteresis. In curves showing the relation of thermoelectric E.M.F. to load, the "on" and "off" curves did not differ by more than the limits of experimental error. With all the wires named, loading produced in the part loaded a change towards *positive*,* and the change remained nearly proportional to the stress up to the largest value to which the load was increased. In silver and magnesium the E.M.F. rose to about 4 microvolts when the load was increased to as large a value as it was judged safe to apply without risk of breaking the wire; in the other metals the range of thermoelectric change was much smaller.

Another metal examined was aluminium, which, unlike the others, changed to *negative* when loaded. The change was very small, amounting to only half a microvolt under the greatest load. A tin wire showed no measurable change of thermoelectric quality, either positive or negative, when loaded up to its breaking point.

So far there is no evidence that the peculiar behaviour of iron, the study of which has been the object of this paper, is not peculiar to that metal. It will be interesting to see whether similar characteristics are exhibited by the other strongly paramagnetic metals, nickel and cobalt.

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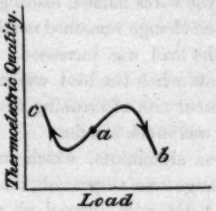
§ 46. The changes of thermoelectric quality through which a piece of iron passes when subjected to varying amounts of stress are so complex that it seems impracticable, until our knowledge of molecular structure is greatly extended, to attempt any mechanical hypothesis by way of explanation. With respect to one very curious feature, however, to which attention has been drawn in §§ 13, and 39–41, it may be worth while to point out an analogy which suggests a connexion between the structure of iron and that of a granular medium. Professor Osborne Reynolds has pointed out† that when a substance composed of granules in contact is subjected to

* *Cf.* Thomson, Phil. Trans., vol. 146 (1856), p. 729, § 144.

† Phil. Mag., ser. 5, vol. 20 (1885), p. 469, "On the Dilatancy of Media composed of Rigid Particles in Contact."

strain, distortion is accompanied by certain complex changes of density, which depend on the arrangement of the particles. Thus, if a bag of shot or sand be shaken into the stable arrangement which corresponds to maximum density, and then be subjected to stress by (say) compression between two planes, the density will at first diminish, and then, as the stress is increased, will pass a minimum and begin to increase. So the thermoelectric quality of a piece of iron which has been shaken into a condition of molecular stability changes, first, towards (say) positive and, later, towards negative (fig. 31), when stress is applied and increased. Here, however, we are dealing with a quality, which, unlike density, is directional; and the changes have, in fact, the reverse signs from those just stated if the direction of the applied stress is reversed. Thus, let a (fig. 34) be a point expressing the relation of thermoelectric E.M.F. to

Fig. 34.



load, which is reached by tapping the metal or by exposing it to the process of demagnetising by reversals. If from this point we go on loading we have the curve $a\,b$. If from the same point we had begun to unload we should have had the curve $a\,c$.

The property of dilatancy, which Professor Reynolds has shown that granular media possess, does not in itself afford any clue to the changes which a directed quality like that now under examination suffers when the condition of stress is varied. But the same considerations (which explain why, in the changes of volume accompanying strain in a granular medium after vibration, there is a point of inflection similar to that which is the most striking feature of these thermoelectric curves) suggest that a granular medium may be the kind of molecular mechanism required to account for the characteristics which have been described in this paper, and that the remarkable feature in the hysteresis of thermoelectric quality with regard to stress commented on in §§ 13 and 41 is to be attributed to a periodic arrangement of granules. The difficulty of any dynamical explanation is not a little increased by the fact that the variations of magnetism which accompany variations of stress, while otherwise not very dissimilar to the variations of thermoelectric quality, exhibit none of this inflection, either when a change takes place from loading to unloading (or from

unloading to loading), or when the process of loading or unloading is resumed after the metal has been subjected to vibrations while supporting a load of any assigned value.

[September 17, 1886. The influence of magnetisation on the thermoelectric quality of iron forms the subject of a chapter in Messrs. BARUS and STROUHAL'S comprehensive memoir "On the Physical Characteristics of the Iron Carburets."* Their experiments, like those of the writer, confirm THOMSON'S observation that longitudinal magnetisation makes iron thermoelectrically negative (in the sense used in this paper). Their absolute measurements of the effect are of the same order of magnitude as those in the examples cited above. Messrs. BARUS and STROUHAL have investigated with much fulness the thermoelectric effects of tempering, in the case of steel, and have pointed out that these are far greater than the effects of magnetisation. It may be added that they are also far greater than the effects which result from hardening by mechanical strain. Iron hardened by stretching is thermoelectrically positive to soft iron: glass-hardened steel is also thermoelectrically positive to soft steel, but to a much greater degree.—J. A. E.]

* Bulletin of the U. S. Geological Survey, No. 14, 1885.

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XI. *On the Sympathetic Vibrations of Jets.*

By Chichester A. Bell, *M.B.*

Communicated by Professor A. W. Williamson, *F.R.S.*

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In the year 1833 Felix Savart published a remarkable memoir* on the structure of liquid jets from circular orifices, in which he demonstrated the following facts:—

A vertical jet of liquid from a circular orifice presents in general two distinct portions—an upper continuous or rod-like portion, and a lower or troubled portion; the relative lengths of these two regions being approximately the same in air as in a vacuum.

The lower or troubled portion consists of separate drops, as proved by the fact that objects may be viewed through it, even when the jet is composed of an opaque liquid like mercury.

The troubled portion of a jet which is not perfectly insulated from accidental vibrations, or from the vibrations produced by its impact against a solid body, presents swellings at regular intervals, these swellings (*ventres*) being the result of certain longitudinal and transverse changes of shape, of regular period, executed by each drop during its fall; but a jet from a carefully insulated orifice does not exhibit these swellings, but only slight irregularities which continually change their position.

In the upper half of the first of these regular swellings the drops are not yet completely detached from the main stream, but only become so in the lower half.

A jet which is allowed to strike against a stretched membrane at a certain distance from the orifice, gives out a musical sound of definite pitch, or some sound related thereto; its continuous portion at the same time becomes shortened, and resolution into drops occurs higher up in the stream.

The pitch of the characteristic sound depends neither on the nature nor the temperature of the liquid, but is determined solely by the velocity of the stream and the diameter of the orifice.

A jet which is exposed to the action of external vibrations may also have its continuous portion shortened, and may reproduce the impressed vibrations by impact against a membrane; but the musical sounds which have this effect must either be related to the characteristic of the jet or not differ from it widely in pitch.

* Annal. de Chimie, vol. 53 (1833), p. 337, and Poggend. Annal., vol. 29 (1833), p. 353, and vol. 33 (1834), pp. 451 and 520.

From his experiments SAVART concluded that periodic movements of the jet exist independently of communicated vibrations, and that these movements are not due to vibrations of the edges of the orifice, nor to friction of the liquid against these edges. He attributed them rather to periodic oscillations of the entire mass of fluid behind the orifice, and concluded that the vibrations excited by impact of the jet against any solid, when conducted back to the orifice, assist and intensify these normal vibrations. External vibrations act in a similar way, but such external vibrations may sometimes alter the normal period of the jet vibrations.

In a later paper* SAVART showed that the jet vibrations are preserved in the "nappe," or thin sheet of liquid formed when the jet strikes normally on a small surface.

In a paper published after his death† SAVART endeavoured to show, by some very beautiful experiments, that periodic oscillations of the mass of fluid behind the orifice did actually take place. Filling large glass tubes with water, and allowing the liquid to escape through circular openings in flat plates which closed their lower ends, he found that the tubes emitted musical sounds, the intensity of which passed alternately through certain maxima and minima as the level of the liquid sank within them. SAVART established the following law:—The pitch of the tone produced is independent of the diameter of the tube-reservoir, but varies directly as the square root of the height of the liquid therein, and inversely as the diameter of the orifice.

In the year 1852 SONDHAUSS ‡ imitated some of SAVART's experiments with jets of air rendered visible by mixture with smoke. Using a circular orifice, and a pressure equal to one inch of water, he demonstrated that an air-jet, like a water-jet, presents a continuous and a troubled portion. In the experiment described the continuous portion was about one inch in length, and beyond this the jet showed protuberances analogous to those observed in liquid jets. SONDHAUSS also showed that an air-jet, projected against a surface, spreads out from the point of impact in the form of a "nappe." He does not appear to have observed the phenomenon of the sensitive air-jet, possibly because the pressure he employed was too high.

In 1853 MASSON § in an elaborate paper described an extended series of experiments on the tones produced by efflux of air. In some of these the jet was allowed to escape from a circular orifice into free air, and for this case he established the following laws:—

The flow of air through an orifice in a metallic plate is not continuous; it varies periodically, and the orifice is the seat of a vibratory movement which gives rise to sound when communicated to the external air:

* Annal. de Chimie, vol. 54 (1833), pp. 55 and 113; POGGEND. Annal., vol. 29 (1833), p. 356.

† POGGEND. Annal., vol. 90 (1853), p. 389; Comptes Rendus, vol. 37 (1853), p. 208; Phil. Mag., ser. 4, vol. 7 (1854), p. 186.

‡ POGGEND. Annal., vol. 85 (1852), p. 58.

§ Annal. de Chemie, ser. 3, vol. 40 (1854), p. 333; Phil. Mag., ser. 4, vol. 6 (1853), p. 449.

The number of vibrations of the air at the orifice varies directly as the square root of the pressure in the reservoir, or as the velocity of efflux; but is independent of the diameter of the orifice.

The first part of the law regulating the number of vibrations holds also for liquids, as pointed out by SAVART.

MASSON's experiments were made with large orifices, and, with few exceptions, at comparatively high pressures. By calculation from his results he determined the rate of vibration corresponding to a pressure of 1 mm. of water to be 626½ per second. In the great majority of his experiments the sounds of the jet were reinforced, as he supposed, by placing over the orifice tubes of various diameters and shapes.

About the same time SONDHAUSS,* apparently having no knowledge of MASSON's work, also published experiments on sounds produced by the efflux of air. He found that a jet of air, at a pressure of 1 or 2 mm. of water, gave no sound; but a "rushing noise" was produced when the pressure was increased. At a pressure of from 5 to 30 mms. of water, a jet, although spontaneously emitting no sound, was capable of being thrown into vibration when an organ pipe was made to speak in its neighbourhood. The tone emitted by the jet was usually, but not always, an octave lower than that of the pipe. The pitch of the tone to which a jet would respond rose in general through the scale as the pressure was increased; but the jet was more powerfully affected by some tones than by others. SONDHAUSS expressly claimed the discovery of these sympathetic tones in air-jets.

He further found that an otherwise silent jet was caused to "sing" by placing in front of it an edge, a point, or an orifice in a plate. The pitch of the tone produced in this way, he concluded, varies directly as the velocity of efflux, and inversely as the distance between the orifice and receiving plate. He thought that the vibrations of the jet were due to friction of the air against the edges of the orifice, and compared the jet to a solid vibrating rod. His experiments were made with a number of orifices of various shapes.

In a note on this paper, MASSON† disputed the supposed analogy between the jet and a vibrating rod; and expressed the opinion that the vibrations of air jets arise at the orifice in the same way as those of liquid jets.

In 1865 SONDHAUSS‡ obtained with water-jets results very similar to those previously obtained with air. He found that such jets could also be set in vibration by allowing them to strike against an edge or a plate. According to his observations the pitch of the tone produced rises with the velocity of efflux, but falls as the distance of the receiving plate from the orifice is increased. Here again, SONDHAUSS regarded the jet as an elastic rod, thrown into vibration by friction against the edges of the orifice.

* POGGEND. Annal., vol. 91 (1854), pp. 126 and 214.

† Annal. de Chimie, ser. 3, vol. 41 (1854), p. 176.

‡ POGGEND. Annal., vol. 124 (1865), pp. 1 and 235.

In 1866 KUNDT* obtained musical tones by the impact of two flames, or of an air-jet against a flame. He considered it a desirable condition that the flame and jet orifices should be of unequal size.

LECONTE in 1858, and BARRETT in 1867, discovered the now well-known "sensitive flame," which was investigated by BARRETT and TYNDALL.† In order to detect possible changes behind the orifice, which might account for the phenomenon, these physicists mixed smoke with the gas previous to ignition. No such changes were discovered; but they were thus led to observe that jets of unignited gas and air are also changed in form by sound, a fact strangely overlooked by SONDHAUSS. The changes occurring in such jets were found to be very similar to those induced by sound in liquid jets. BARRETT and TYNDALL stated that an ignited jet of gas—a flame—is sensitive to higher tones than an unignited jet at the same pressure, and that the effective tones are higher the smaller the orifice. They attributed the phenomenon to the resonance of the gas in the connecting tube.

In the course of these experiments it was noted that a sensitive flame sometimes *reproduces* the tone by which it is affected.

About this time also, TYNDALL‡ produced a new kind of "sensitive flame" by blowing a jet of air through a gas flame: and observed that the ordinary blowpipe flame could be brought into a sensitive condition. TYNDALL considered that the tones which visibly affect a sensitive flame must be of about the same pitch as those produced by friction of the gas against the orifice.

In a subsequent note BARRETT§ compared a flame at different pressures to a series of tuned reeds. He also made the interesting discovery that sensitive flames may be violently affected by vibrations far beyond the limit of audition.

In 1875 DECHARME‖ not only produced tones by directing a jet of air against the surface of mercury, but observed that the latter was thereby thrown into vibration, as proved by the motions of a beam of light reflected from it.

The same experimenter also obtained musical tones by blowing air through a tube against a flame. Carbonic acid and pure oxygen, used instead of air, gave but feeble effects, and nitrogen hardly any. He attributed the action of carbonic acid to its decomposition by the flame.

More recently NEYRENEUF¶ has studied the phenomena of jets and flames. He observed that the "nappe" formed by the meeting of two flames, or of an air-jet and a flame, not only responded to, but also reproduced a limited range of tones uttered

* POGGEND. Annal., vol. 128 (1866), p. 610.

† Phil. Mag., ser. 4, vol. 33 (1867), pp. 92, 211, 287.

‡ Phil. Mag., ser. 4, vol. 33 (1867), p. 375.

§ Nature, vol. 16 (1877), p. 12.

‖ POGGEND. Annal., Ergzgsb. 7 (1876), p. 176; Comptes Rendus, vol. 80 (1875), p. 1602, and vol. 81 (1875), p. 339; Phil. Mag., ser. 4, vol. 50 (1875), p. 496.

¶ Annal. de Chimie, ser. 5, vol. 25 (1882), p. 183.

in their neighbourhood. But very frequently the "nappe" emitted a harmonic of the exciting tone, instead of the tone itself.

In 1884 Lord RAYLEIGH* demonstrated the sensitiveness to sound of jets of coloured water in water, and also published some interesting drawings of the appearances of these and of vibrating sensitive flames when viewed intermittently.

The literature of sensitive flames is extensive, but it is not necessary to catalogue it here.

My attention was directed to this subject in the following way:—Happening to hold the flame of a BUNSEN burner in front of a small orifice in the end of a glass tube, from which a rapidly pulsatory jet of air issued, in such a way that the air-stream impinged on the flame, I noticed that a loud musical sound was produced, the pitch of which corresponded with the rapidity of the jet pulsations. So far there was little requiring explanation. But a feature of the phenomenon which strongly riveted my attention was this: that while the sound was of feeble intensity when the flame was *quite close* to the orifice, it increased in loudness as the lamp was withdrawn along the jet path, and reached its maximum with orifice and flame at a certain distance apart. Beyond this distance the sound rapidly degenerated into an unmusical roar.

In order to study this phenomenon more carefully, a small round hole was pierced in the diaphragm of an ordinary hand telephone, and the chamber behind the diaphragm placed in communication with a reservoir of air under gentle pressure, which previous experiment had shown to give the best result. The telephone being put into circuit with a battery and rheotome in a distant room, each motion of the diaphragm inwards naturally produced compression of the air in the chamber behind.

A small flame being placed in front of the orifice, it was found that not only musical vibrations, but also the more complex vibrations of speech, impressed upon the diaphragm, could be quite loudly reproduced. In fact, when the telephone was put in circuit with another telephone, or better with a battery and microphone in the distant room, speech loud enough to be heard over a small apartment could be readily produced. The words, however, were certainly lacking in distinctness.

A most natural assumption was, that the sounds heard resulted from the impact of a series of puffs of air against the flame. Accordingly I attempted to produce a greater volume of sound by increasing the number of jets. Several small round holes were pierced in the telephone diaphragm, and a separate flame placed in front of each of them. The experiment did not justify my anticipations; for it soon appeared that by far the most effective jets were those that issued near the centre of the diaphragm, where the greatest motion existed; an orifice near the circumference of the diaphragm yielding only feeble sounds. In fact it looked as if *motion of the orifice*, rather than compression of the air, was the chief agent in the phenomenon. This conclusion was

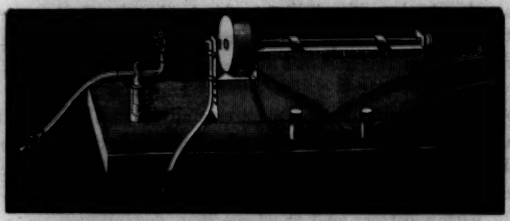
* Phil. Mag., ser. 5, vol. 17 (1884), p. 188.

speedily verified by attaching a light glass jet to a piece of soft iron, soldered to a brass spring, and letting the soft iron take the place of the telephone diaphragm. (Fig. 1.)

The results with this arrangement were still more satisfactory than those previously obtained, and speech as well as musical and other sounds were reproduced of considerable loudness, if not always very distinctly. A singular fact also revealed itself, viz., that the effects were about the same whether the vibratory motions of the jet-tube took place in the direction of its length or transversely. It should be mentioned that certain desirable conditions of air pressure, size of jet orifice, and size and distance of flame had previously been ascertained by experiment.

At various times in the course of these experiments I had noticed that a flame against which an air-jet was playing under suitable conditions, was not only visibly affected by sounds, but that sometimes it seemed to *speak back* words uttered in its

Fig. 1.



neighbourhood, the repeated sounds being comparable to a feeble echo from a near surface. It was also found that a vibrating body, such as a tuning fork, held near the jet orifice, caused the flame to emit the sound corresponding to its vibrations.

A jet of air, then, playing against a flame under proper conditions, formed a combination capable of reproducing, as sound, vibrations communicated to the air at the orifice, either from the air behind or external to the orifice, or from the jet tube or support.

But, very strangely, scarcely any but irregular blowing noises could be heard from a vibrating jet when the ear, or the end of a tube communicating with the ear and large enough to receive the whole mass of air in the jet, was held opposite to it, instead of the flame. Tubes of various sizes, with conical and cylindrical ends, and held at various distances, were tried without result, until finally it became evident that the changes in the jet, effective in producing sound, must be *relative* changes of different portions of it. This view was also suggested by the appearance of the flame in the preceding experiments. At the plane of contact of the jet with the flame, a peculiar ring of blue flame could be seen, and this ring evidently vibrated violently while the flame was emitting sound (fig. 1).

Accordingly, with the object of exploring every part of the jet, I took a hearing-

tube with a *very small* opening in its end (about as large as the jet orifice), which could be applied to different points in the jet-path. With this arrangement the results were strikingly different. When the end of the hearing-tube was moved about in front of, and not too close to, the orifice from which a vibrating jet issued, the vibrations were distinctly audible over a wide area, and at points where the large hearing-tube gave absolutely no sound. When the hearing orifice was held at any point on the *axis* of the jet, the sound heard was much louder than at surrounding points in the same section, occasionally attaining a remarkable degree of intensity; and this intensity invariably reached its maximum at a certain distance from the jet orifice. When the hearing-tube was moved from this position *away* from the jet orifice, the sound first became confused, and then changed to an unmusical roar; but when it was moved towards the orifice the sound gradually died away, and nothing could be heard when the two orifices were close together. The diminution in loudness was also very rapid when the hearing orifice was moved from a point on the jet axis in any direction towards the circumference.

Fig. 2.

It now became of interest to discover the connexion between the point of maximum sound-intensity and the structure of the jet. To this end I made use of a well-known device for rendering visible the motions of air, viz., mixing it with smoke. Professor TYNDALL (*loc. cit.*) showed that when certain sounds were produced in the neighbourhood of a smoked air-jet at suitable pressure, the result was a shortening of the continuous column, the jet appearing to become confused, and to lose its rod-like character sooner than it would if undisturbed.

Experiments with smoked air soon threw some light on the phenomena above described. A jet of air charged with tobacco-smoke was allowed to escape at a pressure of about $\frac{1}{8}$th of an inch of water from an orifice, about $1\frac{1}{2}$ mm. in diameter, in the end of a glass tube. The atmosphere being perfectly still, and strict silence observed, the issuing stream was continuous and rod-like for a length of about 35 centims. This rod-like portion was slightly conical, the apex of the cone being at the jet orifice. Beyond this distance the jet expanded and rapidly became dissipated. Any continuous vibration of moderate pitch, impressed upon the orifice in any way, caused the continuous column to become shorter, and the jet to "break" or become confused at a distance from the orifice varying from 6 centims. to 1, according to the intensity of the vibrations.

On now introducing the small hearing orifice into the path of the jet, it was found that just beyond the breaking point an indistinct or harsh repetition of the disturbing sound was heard; that just at, or a little inside of, this point the repeated sound was clear and of maximum intensity; and that it died away as the hearing orifice was moved from this position, either towards the jet orifice or out of the line of the jet axis (fig. 2). Substituting a small flame for the hearing-tube, the greatest loudness and purity of tone were obtained when the flame was placed just within the breaking point; and on moving the flame the same changes in loudness and distinctness were noted as occurred with the hearing-tube.

In this experiment it was found most convenient to cement the jet-tube to the armature of an electromagnet, thrown into vibration by a battery and rheotome. The attention of the experimenter could then be fixed solely on the sounds proceeding from the jet.

This experiment was repeated many times, the jet being subjected to vibrations of varying pitch and intensity. The result was always the same. Whenever the jet "broke" in front of hearing orifice or flame, under the influence of a pure musical tone, either it gave forth no sound, or the sound heard was harsh and rattling in character. In the latter case the *rhythm* of the impressed vibrations was preserved, but the *quality* was lost.

In repeating these and similar experiments certain precautions should be observed, some of which have been pointed out by other experimenters in this direction.

The edges of the jet orifice, and of the receiving orifice, ought to be perfectly smooth and circular. Any irregularity in them tends to cause whistling or scraping noises in the hearing-tube, and also makes the jet more liable to "sing," in a manner presently to be described.

The air passage, from the reservoir to the orifice, should be wide and unobstructed. It is therefore better to control the outflow by regulating the pressure in the reservoir, rather than by any form of valve.

The air should be free from dust, and from suspended particles of water. Each little particle, as it passes the orifice, gives rise to a disturbance which grows as it travels along with the jet, and causes a crackling noise when it reaches the hearing orifice. Should the air be very dusty there will be a distinct gain by filtering it through a wide tube lightly packed with coarse cotton.

The apparatus actually necessary for these experiments is extremely simple. The air reservoir may take the shape of a bag made from a single sheet of thin vulcanized rubber cloth, five feet long by two feet broad, folded once and joined along its edges with rubber cement. Ordinary gas bags are much too stiff and heavy for the low pressures required, and elastic bags are quite useless. Into one corner is cemented one limb of a three-way stop-cock; of the remaining limbs one is connected to a laboratory blower, and the other leads to the jet by a wide rubber tube

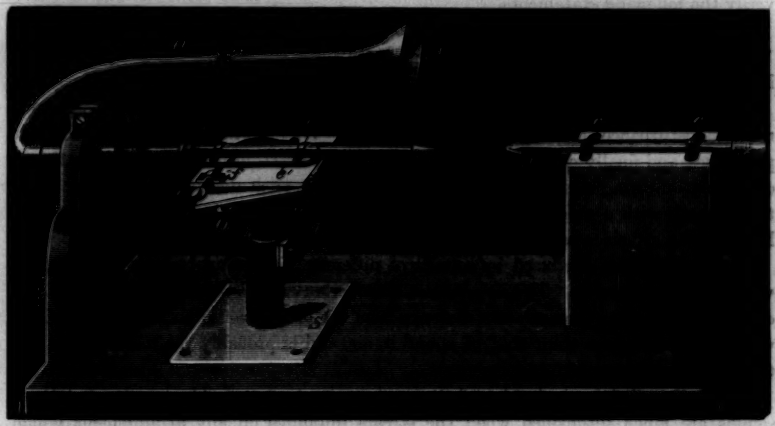
Pressure is best applied to the bag, after it has been inflated by the blower, by means

of a light board, on which weights may be placed. A fairly uniform pressure may thus be maintained for a considerable time when the jet is small.

For the preparation of the jet orifice a wide tube of fusible glass is selected, free from lead and of uniform thickness; this is carefully heated at one point with constant rotation, and drawn out. The constricted portion is then cut with a sharp file where it has the desired internal diameter, the edges of the orifice carefully ground off on a stone and finally slightly fused.

To facilitate the adjustment of the hearing-tube or flame, some form of apparatus like the following is desirable (fig. 3). On a wooden base, A A′, about 12 inches long by 4 inches broad, a vertical block B is mounted. The glass jet-tube J rests in a groove

Fig. 3.



on the top of B, and is held in position by two spring clips, $c\,c'$. The hearing tube, T, is a brass tube about $\frac{1}{4}$ of an inch in internal diameter, on the end of which nozzles, N, having orifices of different sizes, may be screwed. T is held in position by two stout clips, $d\,d'$, on a brass plate, $e\,e'$, about 2 inches long by $\frac{3}{4}$ of an inch wide, which is pivoted at e' on another brass plate, $f\,f'$, of nearly the same length but about 2 inches wide. A stout spring, s, presses the little plate $e\,e'$ firmly against the point of the screw q, which plays in a block on the plate $f\,f'$. The plate $f\,f'$ is hinged at one end to the similar plate $g\,g'$, soldered obliquely on the top of a brass pillar, P, which slides easily in the tube Z, and may be clamped in any position by the screw O. The tube Z is soldered to a brass plate, $s\,s'$, which may be screwed to the base board. A stout screw, Q, plays through the plate $g\,g'$, and the plate $f\,f'$ is firmly drawn down against its point by a spiral spring, not shown in the figure. Thus, by sliding P in the tube Z, the orifice may be brought approximately into position; after which a finer

adjustment may be effected by turning screw Q, by which the orifice is moved up or down, or screw *q*, by which it is moved laterally. The tube T is connected by a piece of rubber-tubing, R, to the stout brass tube M, fixed permanently in the wooden upright, L. The end of the flexible rubber tube, H, provided with the ear-piece, E, is slipped over the tube M. The ear-piece may thus be applied to the ear without fear of disturbing the adjustment of N. The distance of the jet orifice from N may be varied by sliding the jet tube J to and fro in the groove in which it rests. J is connected by rubber-tubing to the reservoir.

For experiments with flames the tube T may be withdrawn, and a tube with burner, like that shown in fig. 1, may be substituted for it.

With this apparatus it is easy to repeat the above-described experiments, and to verify the following statements.

A jet of air at moderate pressure (below 10 mms. of water), from an orifice 1 to 1½ mm. in diameter, forms a continuous column for a certain distance, beyond which it becomes confused and breaks up.

Any impulse, such as a tap on the jet-support or a short and sharp sound, causes a minute disturbance to start from the orifice. This disturbance increases as it progresses, and finally causes the jet to break. By directing the jet against a flame, or against a hearing orifice, it is readily perceived that such disturbances travel along the jet-path with a velocity which is not that of sound in air. In fact, the sound heard in the ear-piece resembles an echo of the disturbing sound.

The disturbances produced by sounds of different pitch travel along the jet-path with the same velocity. This is evident, since otherwise distinct reproduction of the complex vibrations of speech at a distance from the orifice would be impossible. This velocity is much less than that of sound in air, and is probably the mean velocity of the jet stream.

A vibrating air-jet, playing into free air, gives rise to very feeble sounds; but these sounds are much intensified when the stream impinges on any arrangement which serves to divide it into two parts. Of such arrangements, a surface pierced with a small orifice is the best, the orifice being placed in the jet axis.

A jet of air at low pressures responds to and reproduces only sounds of low pitch. Sounds above a certain pitch, which varies with the pressure, either do not affect it or are only faintly reproduced.

At pressures between 10 and 12 mms. of water the jet reproduces all the tones of the speaking voice, and those usually employed in music, with the exception of very shrill or hissing sounds. When the pressure in the reservoir equals about 13 mms. of water, hissing sounds are well reproduced, while sounds of very low pitch become fainter. At higher pressures, up to about 25 mms. of water, shrill or hissing noises produce very intense disturbances, while ordinary speech tones have but little effect. But at these pressures sounds of high pitch frequently cause the jet to emit lower sounds, of which they are harmonics.

In general, a pressure of about 12 mms. of water will be found most suitable for reproducing speech or music. Under this condition the jet is very sensitive to disturbances of all kinds, and will reproduce speech, singing, and irregular sounds included under the term "noises."

It must be understood that the pressures here given are only suitable for jets about 1 mm. in diameter. When the diameter is only a small fraction of a millimeter the above limits may be greatly exceeded, since the velocity of efflux no longer depends solely on the pressure.

A jet escaping from a perfectly circular orifice into free air does not vibrate spontaneously so as to produce a musical sound. But musical vibrations may be excited in it by the passage of the air on its way to the orifice through a resonant cavity or through any irregular constriction.

An air-jet impinging on any obstacle, such as a flame, frequently vibrates spontaneously if the obstacle is at sufficient distance and of such a nature as to diffuse the disturbances produced by impact, or throw them back on the orifice.* This constitutes one of the chief drawbacks to the use of a flame as a means of converting the jet vibrations into sound. The disturbances excited in the surrounding air by the impact of the jet upon it are so intense as to react easily upon the stream as it escapes from the orifice. When, therefore, the jet is thrown into any state of vibration it tends to continue in the same state, even after the exciting sound has ceased.

A jet of air usually responds most energetically to some particular tone or set of related tones. Such a particular tone may be called its fundamental. The practical inconvenience arising from this may be diminished by increasing the air pressure, until the jet fundamental is higher than any of the tones to be reproduced.

The influence of reflected vibrations in disturbing the jet may be well shown as follows:—Let one end of a short piece of rubber-tubing be brought close to the orifice of a silent jet playing against a hearing-tube, but not in such a way as to obstruct it. The jet will remain silent. But if the other end of the tube be held near the hearing orifice the jet will begin to "sing," and will continue singing so long as the tube is held in this position. The tone produced, while depending mainly on the jet, is not independent of the resonance of the tube, since a change in the length or diameter of the latter frequently alters the pitch. It is noteworthy that the arrangements used by Sondhauss and Masson in their experiments were precisely such as would cause disturbances produced by the jet, either by impact or otherwise, to be thrown back on the orifice. This may account for some of the irregular and apparently anomalous results obtained by them.

* Lord Rayleigh (Phil. Mag., ser. 5, vol. 17 (1884), p. 189) suggested that the musical sound, sometimes obtained when an air-jet is directed against a flame, results from the tendency of any regular cycle in the mode of disintegration of the jet to propagate itself, when a connexion exists between the orifice and the body on which it strikes, as in Savart's experiments with water-jets.

In experimenting with flames it is well to observe the following precautions. The flame should be short: the gas should escape at very low pressure from a circular orifice about the size of the jet orifice. If the burner orifice is small, and the gas pressure high, the flame itself may become a source of vibrations; and these vibrations, intensified by the impact of air upon it, may be thrown back on the jet orifice.

The loudest sounds are obtained by directing the jet below the apex of the blue flame-zone. In this region the gas has not yet been mixed with the surrounding air, and the intensity of its combustion will therefore depend on the supply derived from the jet. It is remarkable that vibrations are very feebly reproduced when pure oxygen is substituted for air in the jet. In this case the blue flame ring formed by impact becomes very small, the combustion of the oxygen becomes localized, and the necessary condition of the separation of the jet into two portions is probably not fulfilled.

The flame ought to be of sufficient size to apparently absorb the jet. A complete blue flame ring will then appear, the plane of which intersects the angle between the jet and the flame. The centre of this ring will then appear dark, and very little light will be given out.

The best distance of flame or hearing-tube from the jet orifice must be determined by trial. The adjustment is, however, a matter of a few seconds' manipulation, and may easily be effected with eye and ear as guides. When it is desired to reproduce accurately a series of sounds, the distance should be such that the jet will not "break" in front of the hearing orifice or flame, under the influence of the most intense vibration to which it is to be subjected.

When a flame and an air-jet meet at right angles, sounds may also result from vibrations impressed on the flame orifice. The properties of flames are in general parallel to those of gaseous jets. Without going into details, the chief differences may be summarised as follows:

A flame behaves as a more rigid body than an unignited jet, and, except in the so-called "sensitive" state, *i.e.*, at high pressure, it is not easily affected by aërial sound vibrations.

The pressure necessary to make a flame respond to any given range of tones is higher than that required for an air-jet from the same orifice. In these two relations, therefore, a flame is intermediate between a gaseous and a liquid jet.

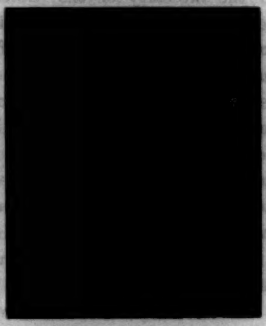
Except at high pressures, when it is approaching the sensitive state, a vibrating flame from a circular orifice does not emit sound. But its vibration may always be rendered audible by blowing air against it. The best result is obtained when a stream of air at gentle pressure is directed from a *wide* tube just below the apex of the blue zone (fig. 4).

The phenomenon of interference is strikingly shown when vibrations of slightly different pitches are impressed on the flame and air-jet orifices respectively.

Having now briefly described the more prominent characteristics of vibrating

gaseous jets, I propose to give a few experiments which may be of value when the nature of the changes occurring within them comes under discussion. This subject is one of some difficulty. No complete theory of jets has been worked out, and I have searched through the literature of the subject without finding anything that would materially assist in explaining their peculiar properties.* We have here to deal with extremely rapid motions of a fluid which is invisible, and the evidence under such circumstances must be mainly indirect. Many obscure points will, however, become clear when the very similar phenomena of liquid jets are considered.

Fig. 4.



Let an orifice in the end of a capillary tube, which is connected with a small water manometer, be moved about in the path of a jet of air. The plane of the orifice being always kept at right angles to the jet, it will be found that the velocity of the air along the axis of the stream diminishes continuously from the jet orifice to the breaking point. This fall in velocity is at first slow, but becomes very rapid when the manometer orifice reaches the point where the jet begins to expand.

* [August 26, 1886.—Since writing the above, my attention has been directed to two papers by Lord Rayleigh (Proc. Lond. Math. Soc., vol. 10 (1878), p. 4, and vol. 11 (1879), p. 1), in which the explanation of the normal tones, at least, of air-jets, is doubtless to be found.

In the first of these papers it is shown mathematically that in the case of a cylindrical jet of fluid moving in still fluid (supposed frictionless) disturbances impressed upon the jet give rise to waves which move forward with the velocity of the jet, and increase in amplitude very rapidly with the time; and that the instability of the jet increases without limit with the pitch of the disturbances.

The second paper deals, amongst other topics, with cases of jets of frictionless fluid under certain hypothetical conditions which approximate to those of the actual jets which come under experiment. It is shown that the instability of the jet, so far from increasing without limit with the pitch of the disturbances, soon reaches a maximum; and that for disturbances of still shorter wave-length than those which are most effective this instability is finally exchanged for actual stability.

It is to be hoped that the distinguished author of these papers may himself discuss their full bearing on the facts here recorded.]

Further, let the manometer orifice be moved from any point on the axis of the jet towards the circumference; a fall in velocity may again be noted which is steep near the jet axis, but becomes slower as the circumference is approached.

A similar change may be noticed just at or even inside the jet orifice; but here the fall is more steep than at external points.

Now let the manometer orifice be fixed at different points along the jet axis, and let the jet be subjected to any continuous musical vibrations by which it is affected. The vibration of the jet will then be attended by a diminution of mean velocity along the axis, which is almost imperceptible close to the jet orifice, but becomes sensible at a distance depending on the intensity of the vibration. If the vibration be gradually increased so as to cause the breaking point to approach the manometer orifice, the rate of fall becomes gradually greater; and if the jet "break" a little in front of the orifice the manometer may indicate no excess of pressure.

Finally, let the manometer orifice be placed at a distance from the axis of the jet, and very near its circumference, and let the jet be continuously vibrated as before. It will then be found that changes in the mean velocity, similarly dependent on the intensity of the vibration and the distance from the orifice, occur along the outer part of the jet, but *opposite* in character to those along its axis, for vibration of the jet now causes *increase* of pressure in the manometer.

It may sometimes happen that greater increase of pressure will be noticed at some points on the circumference of any section of the jet than at others. But by rotating the jet tube it can be seen that this depends on some want of symmetry about the orifice or of the tube behind it, and not on the direction of the sound. A jet which shows this peculiarity may also present in any section *two* (possibly more) points of maximum sound intensity when examined by a hearing-tube, indicating probably a tendency to divide or branch.

The same changes may be shown in a simpler way as follows:—Cut out of thin writing paper a small disc about a quarter of an inch in diameter, on the end of a slender slip of paper, which may serve as a handle, and let this be fixed by the stem at some distance in front of a jet, in such a way that the stream impinges on the centre of the disc. The latter will then naturally appear to be repelled, but will be kept in position by the stem, which acts as a kind of spring. Any sound will now cause the disc to move *towards* the orifice, showing a diminution of pressure against its face.

Now remove the disc and substitute for it a small flat *ring* of paper, on the end of a slender stem, the internal diameter of the ring being about 6 mms., and its external diameter about 12 mms.; and let this be so held at some distance from the orifice that the central portions of the jet pass through the ring, while the extreme outer portions strike its face. The ring will also appear to be slightly repelled while the jet is steady; but any vibration impressed upon the jet will cause it to move further *away* from the orifice, showing an increase of pressure against its face.

The motions of both disc and ring increase with the intensity of the sound, so long as they are within the breaking point.

What relation subsists between the diminution in pressure along the axis of a vibrating jet and the intensity of the disturbing sound is as yet unknown. But it is by no means improbable that, when discovered, it may supply a means of directly comparing the intensities of sounds of different pitches.

The conclusion to be drawn is, I think, that any impulse of the nature of sound, or of any other nature, causes a disturbed transverse layer to start from the orifice of an air-jet; that this layer increases in area as it advances with the jet, the ratio of the pressure at its centre to the pressure at its circumference becoming continually smaller; and that the impact of a succession of these disturbances against a hearing orifice or a flame gives rise to a sound whose constituent vibrations are copies of the disturbing vibrations.

The foregoing experiments supply an explanation of the fact that very little sound results when a vibrating jet plays into free air or into a large hearing-tube. Under these circumstances the diminutions in velocity along the jet axis, due to its vibrations, are balanced by the excesses along its outer portions, an almost complete interference ensuing when the jet breaks. But this interference is prevented when the jet strikes upon an edge or a perforated plate, in fact upon any arrangement which divides the stream into two parts, and sound is the result. A flame probably acts in this way; for inspection shows that while it is emitting sound there is no combustion at the centre of the plane of impact, but very active combustion at the circumference.

This region of active combustion under the best conditions takes the form of a ring, and the explanation of the loud sounds proceeding from it is to be found in rapid changes in the rate of combustion. The occurrence of such changes may be easily proved. Let a fine slip of very thin platinum foil be attached by brazing to two stout copper rods, and let the foil be bent and inserted in the flame in such a way that it coincides with the blue flame-ring, and is raised to incandescence. If now the copper rods be connected in circuit with a battery and telephone, both of low resistance, the changes occurring in the temperature of the flame-ring when the jet is vibrated will be communicated to the little platinum slip, causing corresponding changes in its resistance; hence will necessarily result corresponding changes in the electric circuit, so that the sound or other disturbance acting on the jet will be feebly reproduced in the telephone.

Another simple experiment may suitably find mention here. If a plate of platinum foil be raised to incandescence either by a battery current or by a noiseless blow-pipe flame, and if an air-jet at proper distance be projected against the incandescent surface (of course on the opposite side from the blow-pipe), any vibration or disturbance imparted to the jet orifice will be heard as sound from the plate, and this sound will be louder the higher the temperature to which the plate is raised. This must, of course, be due, either to expansions and contractions of the plate, caused by the

varying impact of the cool jet; or possibly to the ring-shaped waves of air (if they may be so called) leaving the incandescent surface with a higher temperature and of greater volume than they possessed when approaching it.

The foregoing experiments obviously admit of many variations, which it is unnecessary to mention here.

A remarkable fact is that the vibrations set up in a jet by sound seem to be independent of the direction of the sound, and copy only its form. It is quite true that powerful impulses, such as those produced by a vibrating tuning-fork held close to the orifice, will cause the jet to divide or branch, and the direction of this branching will depend on the position of the tuning fork. But, so far as I have observed, *small* disturbances of the gas at the orifice exert the same influence, from whatever direction they may arrive.

Such disturbances may be other than of a mechanical nature. Thus electric strains at the orifice may act as disturbances to the jet. A very effective way to demonstrate this, is to include a battery and a microphone or rheotome in the primary circuit of an induction coil, and to connect either of the secondary terminals with a brass-jet nozzle, mounted on the end of an insulating ebonite or glass tube; while the remaining secondary terminal is either grounded or attached to a conductor held close to the jet orifice. On talking to the microphone the words spoken may be heard from the jet in the usual way. Also by coating a glass jet tube inside and outside with platinum near the orifice, but not *at* the orifice, and connecting these coatings with the induction coil secondary terminals, we can get an arrangement capable of taking the place of the ordinary receiving telephone.

It would not serve the purpose of this paper to describe at length the great number of experiments made with jets of air and other gases. Most of them have only a secondary interest. One other fact, however, is worthy of notice. It is not only at the orifice that jets are easily affected. In fact it is easy, especially by mechanical means, to produce disturbances at a distance from the orifice. But the intensity of the effect on the jet diminishes as the distance of the point of application of the disturbance from the orifice increases.

At a very early stage in this investigation the question naturally presented itself: Would jets of water and other liquids exhibit the same properties as gaseous jets? Referring to the historical notice at the beginning of this paper, it will be seen that much more had been done in the way of reliable experiment with liquid than with gaseous jets; probably because the phenomena of such jets appeal to two senses, sight and hearing, and are therefore more easily studied. A few simple experiments sufficed to show that all the results obtained with gaseous jets may also be obtained with liquid jets, and in a more marked and satisfactory way.

As even the lightest known liquid has a very high density compared with air, we should naturally not expect liquid jets to exhibit the same *sensitiveness* to sound that gaseous jets do. Nor do they in fact; while to purely mechanical disturbances they

are as sensitive, if not more so. To produce powerful effects by sound on liquid jets, it is therefore necessary to supply a lever, by means of which aërial sound-waves may act on the orifice at greater mechanical advantage. Such a lever is afforded by rigidly attaching the jet-tube to a sound-board which offers a large surface to the air. In many cases, indeed, the support or table on which the jet apparatus rests will answer all purposes; but generally it is better to attach the jet tube to a pine-board, about a quarter of an inch thick, and having a surface of about one square foot. A board of this thickness is preferable to a thinner one, since it is not so apt to play the part of a resonator, while it is equally effective in collecting sound impulses from air. Provided its surfaces are free, it is of little consequence how it is supported; but it may be conveniently attached to props at the four corners. It is also a matter of indifference whether the jet tube is parallel or perpendicular to, or forms any angle with, the sound-board.

Fig. 5.

The apparatus shown in fig. 5 is sufficient for all ordinary experiments. The sound-board, S, is screwed to two brackets, projecting from the stout vertical board, V. The latter is supported on leaden or india-rubber feet in such a way that the sound-board projects over a sink or trough.

The jet tube, J, fits in a slotted brass tube, T, which is soldered in the centre of a circular brass plate, Q, about two inches in diameter. This plate is pressed flat against the sound-board by two stout springs, P and P′. A hole in the sound-board, about three-quarters of an inch in diameter, allows the jet tube to project through it, and to be moved a little sideways. The jet tube can thus be easily changed.

The receiving surface is formed by a thin rubber membrane, stretched over the end of a brass tube, R, about $\frac{3}{8}$ths of an inch in internal diameter; this tube is provided with a lateral branch, inclined slightly upwards, to which the hearing tube, H, is attached. The tube, R, slides smoothly up and down within the slightly larger tube, L, fixed on a heavy leaden base, C. This arrangement

allows the distance of the membrane from the jet orifice to be easily varied. The receiving surface ought to have a motion of at least 8 or 10 inches.

The water supply can be drawn from any kind of reservoir, capable of being fixed at different heights, and connected with the jet tube by soft black rubber-tubing. But water may also be drawn from the mains, when the pressure in the latter is tolerably constant. In crowded or noisy neighbourhoods a jet so supplied is exposed to many accidental vibrations; but these vibrations, as well as the disturbances always produced when the pressure is regulated by an ordinary tap, may be deadened by connecting the latter with the jet tube through a coiled rubber tube immersed in a vessel of water. A narrow vertical tube or a mercury gauge connected with the supply-pipe may serve to indicate the pressure.

Glass jet tubes are easily prepared in the way already described for air-jets; but care should be taken to make the drawn-out ends as obtuse as possible. What is practically an orifice in a thin plate may be obtained by heating in a blow-pipe flame the squarely cut end of a tube of uniform thickness, with constant rotation, until it has contracted so as to form an orifice much smaller than required, and then, while the glass is still soft, blowing somewhat forcibly into the opposite end.

Care should be taken to remove air-bubbles as completely as possible from the supply tube. A large bubble will frequently lodge in some bend at a distance from the orifice, and by its oscillation keep the jet violently disturbed.

The fundamental properties of jets may now be illustrated as follows:—Having mounted in the sound-board a jet tube with an orifice about 7 mms. in diameter, let the reservoir be lowered so as to cause water to escape at a pressure of 15 cms. The jet will then form a continuous column for a short distance, beyond which it becomes troubled and breaks into drops. As the pressure is raised the length of the continuous column gradually increases, until the pressure approaches 160 cms., when this portion of the jet gradually loses its rod-like character, and becomes violently disturbed. Before this point is reached, if the orifice is not perfectly circular the surface of the continuous column will appear corrugated or beaded.

Even at a pressure of 15 cms. the jet will sluggishly respond to ordinary voice sounds (which are always complex), or to taps on the sound-board or support; but will remain quite indifferent to simple musical tones of moderate pitch, and to shrill or hissing noises. The visible effect of any disturbance of brief duration will be a momentary shortening of the continuous column, after which the jet resumes its normal appearance. If the source of disturbance be a continuous musical sound of sufficiently low pitch, the shortening of the clear column will persist as long as the sound lasts, and will bear some relation to its intensity.

As the pressure is increased by raising the reservoir the range of tones to which the jet responds, and by which its continuous portion is shortened, is gradually extended, embracing notes of successively higher pitch; while it is still affected by all the lower tones. At a pressure of about 45 cms. of water it will respond to

low whistled tones, and finally at a pressure of about 15 decimeters it will be affected by all the tones occurring in the human voice, and usually employed in music, with the exception of hissing sounds. At a somewhat higher pressure it will also be visibly affected by hissing noises.

These effects are not confined to jets of the size mentioned, but are exhibited by much larger and much smaller ones. But the pressure required to produce any given range of sensitiveness increases with the size of the jet, approximately in accordance with SAVART's law. It is not easy, however, except by very careful construction of the orifice, to get jets of more than 1 to 1½ mm. in diameter to respond to sounds of very high pitch, without, at the same time, a great deal of accidental disturbance. A large jet also requires very careful insulation. On the other hand, jets of as small diameter as ·12 mm. are well adapted to show the influence of pressure on sensitiveness. But for such jets the pressure necessary to produce any given tone does not even approximately follow SAVART's law, being higher than indicated by theory. Small orifices are also useless when sounds are to be produced by impact of the jet, and are very liable to become plugged by little particles of dust, unless the liquid supplied is carefully filtered.

Now, let the hearing-tube, illustrated in fig. 5, be held in the path of the jet, so that the steam impinges on the centre of the membrane, and let the jet be subjected to any continuous vibration by which it is affected, while the ear-piece of the hearing-tube is held to the ear. The following may be noted:—When the membrane is held close under the jet orifice, no sound will be audible in the ear-piece; but as the receiving-tube is gradually withdrawn along the jet path, a sound will be heard corresponding in pitch and quality to the disturbing sound—provided, of course, that the jet is at such pressure as to be capable of responding to all the higher tones to which the disturbing sound may owe its timbre. The intensity of this sound grows as the distance between jet orifice and membrane is increased. Finally, while the jet is still continuous above the membrane, a point of maximum intensity and purity of tone will be reached; and if the membrane be carried beyond this point the sound heard will at first increase in loudness, becoming harsh in character at the same time, and at a still lower point will degenerate into an unmusical roar. In the latter case the jet will be seen to break above the membrane.

Although the jet in this experiment will respond to and reproduce all tones lower than the highest by which it is visibly affected, it will not reproduce all with equal intensity, but will be most powerfully affected by some one tone or set of tones, as pointed out by SAVART. The highest of the proper tones of the jets is usually a fourth or a fifth lower than the highest which it is capable of reproducing. It is worthy of notice that jets from orifices not perfectly circular show this preference for certain tones more than strictly cylindrical jets; and it may possibly be that this preference is invariably the result of some irregularity in the orifice. The inconvenience arising from it when speech or music is to be reproduced, is practically

entirely removed by increasing the pressure until the jet responds to a note of 4000 double vibrations per second, or to the highest tone easily whistled. In speech at least tones of such extreme pitch are always very feeble, and even if the jet break above the membrane when such a note is loudly sounded, it will probably not do so under the influence of spoken words.

Between the musical tones of the ordinary speaking voice, which correspond roughly with the tones of the pianoforte, and sibillant or hissing sounds, there appears to be a considerable interval. And if a jet be at such pressure as to reproduce all the tones of the piano, a much higher pressure will be required to make it respond to hissing noises. But when this point is just reached *small* jets are still sensitive to all lower sounds. Accordingly with small jets, the vibrations of which may be rendered audible in a manner presently to be described, this is the pressure best adapted for reproducing speech, music, and other sounds of all kinds. Moreover, at such pressure the tendency to reinforce particular tones is entirely lost.

At much higher pressures the sensitiveness of small jets to hissing noises becomes pronounced, but the capacity to reproduce lower tones gradually disappears, the deeper tones being the first to lose effect.

The connexion between the spontaneous vibration of a jet and its impact on some solid, is well seen as follows. While the jet is undisturbed by external sounds let the hearing-tube, or a branch from it, be brought near to the sound-board, the open, trumpet-shaped end being held quite close to it. The receiving-tube being introduced into the path of the jet near the orifice, no effect will be produced. But if the membrane be gradually drawn away from the orifice a point will be reached at which the jet will begin to "sing," at first faintly, afterwards more loudly. The singing will continue so long as the same state of affairs is maintained, but will cease when the ear-piece is withdrawn from the sound-board. Even when no special means are used to make the jet self-disturbing, the impact vibrations communicated from the membrane to the sound-board through the intervening air may be sufficient to keep the jet in vibration. If the locality be very quiet the point at which this occurs may even be reached, and the jet remain silent; but any accidental noise or vibration may disturb this state of quietude, and cause the jet to sing.

It is evidently true for liquid as for gaseous jets that all sound disturbances are propagated along the jet-path with the same velocity.

Only a highly practised ear can distinguish between the disturbing sounds and the echo-like reproduction by the jet when both are within range of hearing. In order, therefore, to form an adequate idea of the intensity of sound that may arise from the jet impact, and the marvellous accuracy with which impressed vibrations are amplified and reproduced, it is absolutely necessary that the source of these impressed vibrations should be at a distance from the experimenter. Vibrations may be very simply conveyed to a jet along the wire or cord of a "lover's telegraph." A thin cord is attached at one end directly to the support of the jet, or to the tube itself, and at the other to

the centre of a parchment drum, which may be carried to a considerable distance. The connecting cord being stretched, an assistant may be directed to speak or sing to the distant drum. Or for the air-jet attached to the armature of a telephone magnet, shown in fig. 1, we may substitute a water jet, directed against a suitably placed membrane.

When spoken words are reproduced in these and similar ways, if the pressure in the reservoir be allowed to fall below that required for distinct speech, the characteristic quality of the speaker's voice, which is largely due to overtones, will first be lost; at still lower pressure, certain sounds and syllables will be missed, and the words will become unintelligible. On the other hand, if the pressure be somewhat above the necessary limit, a peculiar nasal quality will be added to the voice, due probably to the greater reinforcement of vibrations of high pitch. At still higher pressure the reproduction will become feeble, and crackling noises, due to minute air-bubbles in the liquid and other accidental causes, will become painfully loud in the ear-piece.

Fig. 6.

When the jet liquid is a non-conductor vibration may also be excited in it by electrical disturbances at the orifice. The jet-tube (fig. 6) is best made of glass. The orifice having been formed, the tube is thoroughly cleansed, and coated outside near the orifice with bichloride of platinum solution mixed with mucilage, dried, and heated to redness in the flame of a lamp, until metallic platinum is burnt into its surface, which can be effected without destroying the shape of the orifice. A fine platinum wire wound round the platinized portion of the tube forms one terminal of a secondary telephonic circuit, and a slip of platinum foil within the tube forms the other. The

primary circuit may include a microphone or a rheotome, and a battery. The vibratory changes in the potential of the two secondary terminals of the circuit, caused by vibrations impressed on the microphone or rheotome, excite disturbances in the jet, which may be reproduced as sound. When, for example, words are spoken to the microphone, these words are accurately repeated in the ear-piece connected with the hearing membrane on which the jet plays.

The action in this case may depend on changes in the surface tension of the liquid at the orifice, accompanying the electrical changes. Distilled water, alcohol, ether, petroleum, &c., may be used as the jet liquid. Small quantities of saline substances in solution, and especially any slight trace of mineral acid, effectively destroy the sensitiveness of water, by increasing its conductivity.

This method may be variously modified. For example either the outer or the inner electrode may be dispensed with, and the corresponding secondary terminal joined to earth.

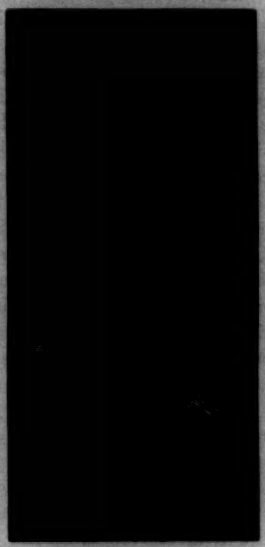
A striking difference must have been noticed between the methods by which gaseous and liquid jets are made to excite vibrations in air. Whereas very little sound results from the impact of an air-jet on a membrane, very loud sounds are produced by the impact of a liquid jet. This might seem to indicate a difference in the character of the motions excited by vibration in jets of the two kinds of fluid. Nevertheless, it can be shown that changes occur in the normal velocity of the central portions of a vibrating liquid jet, and that these are sufficient to excite sound. Let the jet strike upon a stout brass plate, A (fig. 7), pierced with a small orifice, B, which leads to a small flat chamber, of which the brass plate forms one side. The opposite side of this chamber is formed of a thin metal diaphragm, D, through which the pulses generated in the chamber are communicated to the hearing-tube. Just as in the case of the air-jet, the loudest sounds are heard when this orifice receives the central portion of the liquid jet. Feebler sounds are produced from the compact of surrounding portions of the stream, or the jet-nappe. The chamber between A and D should of course be entirely filled with liquid; a second aperture, B′, allows the contained air to escape.

One of the most effective and most interesting methods of studying the vibrations of a liquid jet consists in including a portion either of the continuous column, or of the jet-nappe, in circuit with a battery and a telephone, by which means its vibrations become audible in the telephone. This method may be used with jets of all sizes, but most advantageously with very small ones; and as by its means something may be learnt of the changes occurring within the stream, I purpose to describe it at some length. By making use of small jets we get rid of the necessity for "insulating" the jet orifice; for the disturbances produced by impact are then so feeble as to be practically without effect. The jet is received on a smooth hard surface, which is scarcely capable of communicating disturbance to the surrounding air.

Before describing the method it is well to recall one of Savart's observations. In

one of his papers (*loc. cit.*) he describes the phenomenon of the "nappe." When a jet of water strikes normally on a small flat surface it spreads out as an exceedingly thin film in free air, in shape somewhat resembling an umbrella. This film is bounded by a kind of fringe, where it resolves itself into drops. It was regarded by SAVART as a modification of the continuous column of the jet, and he showed that the jet vibrations are preserved in it. The fringe by which it is surrounded he regarded as analogous to the troubled region.

Fig. 7.



When the jet is allowed to strike upon a large flat surface the appearances are slightly different. The liquid radiates from the point of impact, forming a thin circular film, having the jet at its centre. The film adheres closely to the surface, when this is capable of being wetted by it, and is bounded by a region in which the liquid is piled up, forming a layer many times thicker than the film. In this adherent film also the vibrations impressed upon the jet are preserved.

The electric current may be passed through any portion of the continuous column or of the nappe. It is obviously necessary that the fluid constituting the jet should have a certain degree of electrical conductivity. Many saline substances in solution will confer the requisite conductivity on water; but those substances are to be preferred which do not yield solid insoluble products by electrolysis. The most suitable liquid is distilled water acidified with $\frac{1}{100}$th of its volume of pure sulphuric

acid, especially free from lead. This amount of sulphuric acid cannot be much exceeded without impairing the sensitiveness of the jet, probably by increasing the viscosity of the liquid.

By means of a battery, a telephone, and a couple of platinum electrodes, we can examine every part of a jet of this acidulated water; and the method is so delicate that changes can be detected in every point from the orifice to the discontinuous portion.

That vibratory changes are initiated in the jet stream at its origin, and during its passage through the orifice, may be proved by connecting the inner and outer electrodes of a jet tube similar to that shown in fig. 6 with the terminals of a circuit including a telephone and a battery of a few elements. Vibrations impressed on the jet orifice will then be transmitted to the telephone; and if the jet tube be mounted in a proper sound-board the most complex sounds may be reproduced. But the reproduction, although admirably distinct, is always feeble, as might naturally be expected. The pressure necessary for distinct articulation is not so high as that required when the electrodes are placed at a distance from the orifice. From this we may infer that very rapid vibrations may exist in the liquid at the orifice, which are only propagated along the jet when the stream attains a certain velocity.

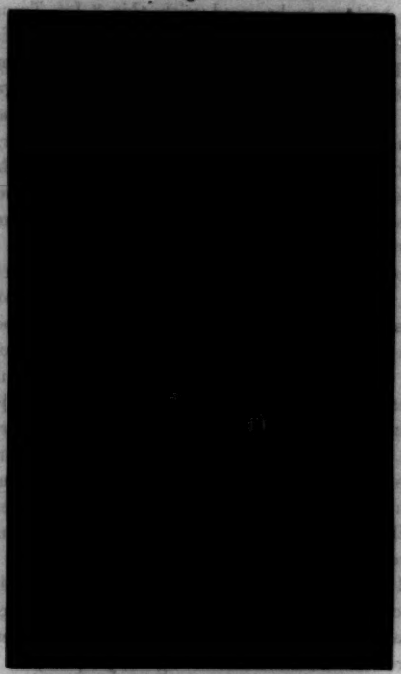
Very similar results are obtained if the jet nozzle be made of platinum, and the jet be allowed to strike upon the rounded end of a platinum rod placed at a little distance from the orifice. The current may be passed in either direction between nozzle and rod. The best sounds are obtained in the telephone when the end of the rod is a little beyond the "vena contracta."

It is evidently impossible to include any length of the jet at a distance from the orifice in a circuit, without transforming it into a nappe; and I have therefore taken advantage of the fact that so far as vibrations are concerned the nappe has all the properties of the jet stream itself. The nappe formed by the impact of a steady jet against an extended flat surface is of about the same diameter, whatever may be the distance of the surface from the orifice, so long as it is formed from the continuous portion; but the intensity of the disturbances transmitted to it from the orifice increases with the distance, as for the jet itself. The simplest way of passing a current through it consists in allowing the jet to strike normally upon the exposed end of a platinum wire imbedded in an insulator, such as ebonite, which is impervious to, and unaffected by, the liquid employed. The jet spreads out from the point of impact as a nappe, which comes in contact with a platinum ring imbedded in the same surface. The wire and ring are made to serve as electrodes.

A very simple form of apparatus is shown in section in fig. 8. The jet tube, J, is mounted in the sound-board, S. The receiving surface is formed by the end of an ebonite plug, P. A platinum wire, E, passes water-tight through this plug, its exposed upper end forming the inner electrode of the "Transmitter." The outer electrode, E′, consists of a little tube of platinum foil which surrounds the wire, E,

but is insulated from it by the ebonite. After it has been fitted on, the top of the ebonite plug is turned off so as to present a smooth continuous surface, slightly convex. Fine platinum wires welded to E and E′, serve to connect them with the terminals of the circuit.

Fig. 8.



The plug, P, is fitted into a small ebonite cup, C, by which it is supported, and into which the jet liquid flows; the liquid is thence carried off by the tube, T, to a vessel in which it collects and from which it may be returned to the supply reservoir. The cup, C, may be supported on any kind of adjustable stand.

An indispensable adjunct to any apparatus for experimenting with small jets is the filter, F. This may consist of an ebonite cylinder, of about the size shown, closed above and below by the screw caps, K and K′. Through the upper cap pass two tubes, X and Y. The tube Y (from the bottom of the filter) is connected to the jet tube by a piece of *black* india-rubber tubing, and the tube X, is similarly

connected with the reservoir. Two perforated discs of ebonite are fitted within the cylinder, and the space between them lightly packed with coarse cotton, which has been previously treated with caustic potash, and thoroughly washed with dilute sulphuric acid and water. This filter serves to hold back bubbles of air and little particles of dirt which might stop up the orifice.

The strength of battery to be used with this "Jet Transmitter" will depend chiefly on the nature of the jet fluid. With the solution of sulphuric acid above recommended, and platinum electrodes, the resistance of the film, and the polarization resistance at the contact of the liquid and electrodes are so great that a battery of high electromotive force is required. But for the same reason the resistance of battery and line is of little consequence, and a large number of elements of small surface may be used. About twenty little zinc-carbon cells charged with a solution of sal ammoniac answer well, but this number may be exceeded with advantage. Such a battery does not sensibly polarize when connected through the film. The strength of battery cannot however be indefinitely increased; for a point is soon reached at which noises in the telephone, due to the escape of electrolytic gas-bubbles from the electrodes, completely drown the sounds due to vibratory changes of the jet. But long before this occurs, sounds of all kinds are loudly reproduced.

As I have before stated, this instrument may be adapted to jets of various sizes, the dimensions and distance apart of the electrodes being regulated by the diameter of the nappe. I have had them constructed with jets only $\frac{1}{7}$th of a millimeter in diameter, discharging about 6 c.c. of liquid per minute under a pressure of about 45 centims. Even such small jets are capable of transmitting speech loudly and distinctly; but it is difficult to get from them a good nappe, the liquid having a great tendency to accumulate on the electrodes instead of flowing evenly away. When the nappe is very small, the electrodes may therefore advantageously be formed by the exposed ends of two fine platinum wires.

With these small instruments it is easy to demonstrate all the principal properties of jets; the absence of spontaneous vibrations, the dependence of the pitch of the fundamental tone and of range of sensitiveness on the diameter of the orifice and the pressure, the growth of disturbances along the jet path, &c.

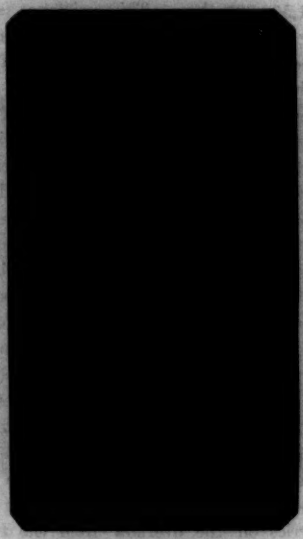
There are two ways in which the jet may be supposed to act on the electric current. It is a fact of observation that the diameter of a vibrating jet at any point is constantly varying; and we may suppose the successive protuberances and constrictions which it exhibits, and which appear in the nappe as waves radiating from the point of impact, to interpose a continually changing resistance in the circuit; or again, the interruptions to the electric flow may take place at the contact of the liquid with the electrodes, being due to changes in the character of the motion, rectilinear or rotatory, of the liquid particles. Probably, as I hope to show, the effect is a compound one, due to both causes.

That the first of these hypotheses is amply sufficient to account for the observed

phenomena is readily proved. It is not difficult to arrange a jet-transmitter so that no part of the jet or the jet film comes in contact with the electrodes.

Thus the jet, or the nappe, may make contact between two masses of liquid in which the electrodes are immersed. This is easily effected by allowing it to strike upon the sharp edge of an insulating (ebonite) partition, P (fig. 9), which separates the two masses of liquid, M and M′. At the edge of the partition the jet divides into two streams through which the electric circuit is completed. The electrodes E and E′ are small platinum plates immersed in M and M′.

Fig. 9.



Making due allowance for the increased resistance necessarily thrown into the circuit, this form of transmitter is very effective. Many other forms of this apparatus have been tried, but it is not necessary to describe them here.

If any further proof be needed, I may give the following. Du Bois Raymond has shown that the polarization at the contact of amalgamated zinc with a saturated zinc-sulphate solution is practically nil. I have therefore constructed a transmitter, having amalgamated zinc electrodes, and a jet of saturated zinc-sulphate solution. This transmitter was perfectly effective, the most striking feature about it being that very small battery power was required, a single Leclanché element being sufficient.

Nevertheless it is highly probable that "polarization" plays some part in the

action. In the ordinary jet transmitter, this polarization of the electrodes, when they are small, is exceedingly intense. If, while the jet is playing, the current be sent between them for a few seconds, and the battery be then suddenly cut out of the circuit, the jet will continue to transmit vibrations to the telephone with gradually increasing feebleness for a minute or even longer. No such effect can be obtained of course with the transmitter having zinc electrodes and a jet of zinc sulphate solution.

Owing to this intense polarization it is quite impossible to measure the resistance of the nappe by any of the usual methods, which might be interesting as enabling us to estimate the thickness of the film. A small transmitter inserted in a WHEATSTONE Bridge will show an apparent resistance, varying with its dimensions and the strength of current, from 1,000 to 1,500,000 ohms. This resistance is also extremely variable, even for the same battery and transmitter: it seems to be influenced by the direction of the current between the electrodes only when these are of unequal size.

Now it is well known that part at least of the resistance included under the term "polarization," is dependent on the state of rest or motion of the liquid at the surfaces of the electrodes; and if changes occur in the motion, rectilinear or rotatory, of the jet particles, such changes might certainly influence this mechanical polarization, and thereby affect the strength of the current. I have already given one experiment which shows that vibrations induce changes in the relative velocity of the particles along the axis of the jet. The following experiment furnishes an additional proof. Just as the impact of a vibrating air-jet, playing in air against an orifice, may excite sound, so the impact of a water jet playing against electrodes immersed in water, may excite electrical undulations. In fact if both jet tube and electrodes of a polarizing transmitter be immersed in a vessel containing the same fluid with which the jet is supplied, the transmitter will not cease to operate, provided that the electrodes are brought much closer to the orifice: for the continuous portion of the jet now becomes much shortened, owing to friction against the surrounding fluid. The diagram (fig. 10) represents the most striking form of the experiment.

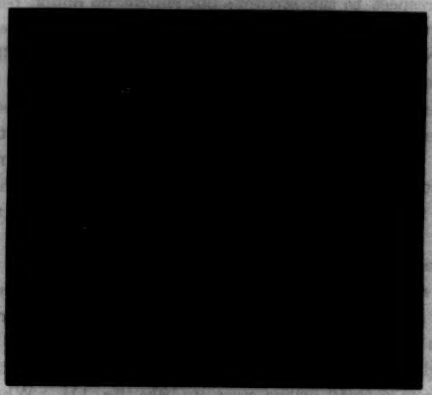
The jet plays against the exposed end of a fine platinum wire, P, firmly embedded in the centre of a small ebonite rod, R, which passes through the bottom of the trough T. A side tube, L, carries off the overflow from the trough. The end of the wire P, and a small platinum plate Q immersed in the liquid of the trough, form the electrodes between which the current is passed (in either direction). The telephonic effects with this arrangement are exactly similar in kind to those obtained from an air-jet with a small hearing orifice. The loudest sounds are heard when P is placed in the axis of the jet at a certain distance from the orifice; and the reproduction grows fainter as the electrode P is moved from this position, either towards the orifice or laterally.

The telephonic effects are feeble, but such as they are they cannot be attributed to the interposition of varying liquid resistances between the conductors.

Under these conditions, in fact, a liquid jet becomes comparable to a jet of air.

If the jet liquid be coloured the analogy becomes strikingly evident. At low pressures the continuous part of the stream of coloured liquid is elongated and unsteady, being easily swayed about by currents in the surrounding liquid: and where it becomes discontinuous it either branches or forms a divergent cone. In this condition it is not sensitive to sounds of high pitch; but a musical tone of low pitch, or a gentle tap on the supporting sound-board, causes it to break quite close to the orifice. As the pressure is increased the breaking point first recedes from and then approaches the orifice; and when the velocity is so great that the jet visibly responds to high tones, the continuous portion, even of a jet 1 mm. in diameter, is little more than a centimeter in length. At the breaking point the liquid then spreads out as a divergent cone, quite similar to the conical brush seen on the end of a smoked air-jet.

Fig. 10.



We have seen that very little sound results from the impact of a vibrating air-jet against a membrane: and precisely the same is true for a jet of water when both jet and membrane are submerged. But the telephonic experiment last described shows that increased effects result when the receiving surface is so small as only to be affected by changes occurring in the neighbourhood of the jet axis.

The analogy may be pushed further. If the velocity of the stream at different points be measured by introducing into its path a capillary tube connected with a vertical tube containing either mercury or water, which may serve as a manometer, a progressive fall in velocity will be indicated as the manometer orifice is moved away from the jet orifice and along the axis; and a similar fall in velocity as the manometer orifice is moved from any point on the jet axis radially towards the circumference. The diminution in the latter case is steeper near the orifice than at a distance from it.

Again, if the manometer orifice be placed at any point in the jet axis, and if the jet be thrown into vibration, a diminution of mean velocity will be indicated, which increases alike with the intensity of the impressed vibrations and the distance from the jet orifice; and becomes most pronounced when the disturbances are sufficient to make the jet break in front of the manometer orifice. These changes are precisely similar to those noticed in vibrating air-jets.

It is interesting to contrast with these the conditions of motion in a water jet projected into air. Using a manometer with a very small orifice in the manner already described, it is easy to show that the velocity of the jet particles at the orifice, and for some distance outside of it, is greatest along the axis, and diminishes from this outwards. At a distance from the orifice, however, a similar change is by no means so evident, partly because most large jets, which can alone be studied in this way, are usually so unsteady that it is not easy to keep the manometer orifice constantly in one position with respect to the stream; but chiefly, no doubt, because, owing to the viscosity of the liquid, the motion of the particles tends to become uniform low down in the stream. Nevertheless I have little doubt that such a fall in velocity occurs throughout the continuous portion of the jet.

In order to detect changes in velocity along the axis of the jet, and to get rid of the acceleration due to gravity, the jet must be projected horizontally: and here the curvature of the path described introduces a new element of difficulty in the observations. But as the result of many experiments it may, I think, be stated that there is a distinct retardation of the axial portions of the jet at a distance from the orifice.

Finally, it is quite impossible to say definitely that vibration of the jet causes a diminution in the mean velocity along the axis, within its continuous portion, or that it does not. Any such change must be slight, and would of course be mainly attributable to friction of the jet against the surrounding air. That this friction is not wholly to be neglected may be shown by allowing a water jet to fall through a horizontal flame from a small orifice, near its base (fig. 11). When the flame is at a sufficient distance from the jet orifice, vibrations impressed upon the jet will cause it to emit sounds even when it is still within the continuous portion. These sounds are no doubt due to the jet carrying along with it a layer of air, which is projected against the flame.

I have now given a brief description of the chief properties of jets as observed by myself and others. Nothing like a complete account of the experiments made has been attempted; simply those have been selected which throw any new light on the subject.

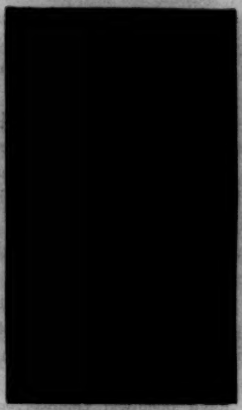
It does not appear, so far, that the phenomena admit of any obvious explanation, properly so called. At the very outset we are met by the difficulty that the motions taking place within a fluid jet, even when it is not subjected to vibration, have been only partially investigated. In what follows, therefore, I shall simply indicate a few

points that seem to me to have been overlooked or not sufficiently dwelt upon in treating of these phenomena hitherto, and point out certain analogies which may at least serve as guides to further experiment, even if they answer no other purpose.

Briefly stated, the facts to be accounted for are the following :—

An impulse given to the fluid at a jet orifice, or to the orifice itself, gives rise to some kind of disturbance which probably moves forward with the mean velocity of the stream, and therefore involves always the originally disturbed particles, or those immediately adjoining them. If the velocity were different for different disturbances, accurate reproduction of complex sounds at a distance from the orifice would be impossible.

Fig. 11.



The disturbance produced by an impulse in the direction of the jet, or by a sound impulse, is circular and symmetrical with respect to the jet axis, and retains its symmetrical form throughout. This is not the case when the impulse is mechanical, and in a direction inclined to the jet axis, or when the orifice is not perfectly circular. These conclusions are arrived at by inspection of instantaneous photographs of jets disturbed by sounds and by vibrating tuning-forks, &c.

A sound vibration of a simple nature conveyed to the jet orifice causes disturbance of a certain length of the column, which disturbance takes the form of a swelling and a contraction. Both swelling and contraction become more pronounced as the fluid travels away from the orifice ; but at any given distance within the continuous portion successive sections of the jet represent the form of the impressed vibration. This statement, however, requires qualification ; for a jet of given size and at any particular pressure, can reproduce at a distance from the orifice only a certain range of tones. But the upper limit of this range is not sharply defined, the tones lying in its

neighbourhood losing their effect gradually; and it is by no means certain that, other conditions being equal, it is the same for different gases, or for different liquids.

At a distance from the orifice, depending on the intensity of the original disturbance, the constricted portion of the jet gives way, a break in the stream being the result. A succession of vibrations impressed upon the orifice therefore cause the stream to resolve itself into drops.

When the impressed vibrations are complex the length of the column corresponding to each presents minor constrictions and expansions, sometimes of very complicated outline. Breaks in the stream occur first at intervals corresponding to these lengths: the detached portions tending further to resolve themselves at a greater distance from the orifice.

The formation and growth of the swellings and contractions are attended by corresponding changes in the velocities of the axial particles of the jet. These changes may be characterised as accelerations and retardations of the velocity measured along the jet path.

When a jet escapes into a fluid which offers a sensible resistance to its motion, its axial particles suffer a gradual retardation. But this retardation is greater when the jet is disturbed by vibrations than when it is in its normal condition.

Such are the phenomena; how far are they explained by theories hitherto accepted? In view of the striking parallelism of the properties of gaseous and of liquid jets, it will hardly be considered inadmissible to assume that the most satisfactory explanation will be that which refers both to the same origin. In the two kinds of jets we have the same tendency to vibrate at some definite rate; the same growth of vibratory changes along the jet path; similar phenomena of discontinuity manifested at a certain distance from the orifice; and similar changes in the motion of the particles of fluid along the axis.

Savart and Sondhauss have occupied themselves chiefly with the origin of normal vibrations: Plateau alone explained not only their origin but also their growth. Now, simple inspection of the table given by Savart of tones generated in a tube reservoir by efflux of water from an orifice in its lower end, shows that these tones are much higher in pitch than those which would result from the impact of the jet against a stretched membrane. Vibrations originating in this way cannot therefore account for the normal jet vibrations, which are invariably connected with impact.

With regard to Sondhauss's idea, viz., that the jet is comparable to a solid rod thrown into vibration by friction against the orifice, it must be admitted that the analogy was not wholly without foundation. If spontaneous vibrations be excited in an air-jet in the manner I have described, it will usually be found that the pitch of the tone rises as the hearing orifice is moved towards the jet orifice. But this does not go on indefinitely; for after the tone has risen through an interval of about a third, it suddenly returns to its original pitch, again rises, again falls, and so on, until the impact disturbances become too feeble to re-act upon the fluid at the orifice. The

analogy, therefore, is incomplete; and, moreover, does not take into account the peculiar changes in the motion of the jet fluid.

So peculiar are these changes, that the term "vibration" as applied to them seems to me entirely misleading. By this term is, commonly at least, signified a kind of motion, in which each particle of the vibrating body executes oscillations about a mean position, whether of rest or of steady motion, to which it tends to return when its motion is interfered with. But in a sounding jet the motions are not of this kind. Any vibration impressed on the orifice, produces a disturbance which tends to grow always in a definite direction; and when the motions are interfered with, the particles do not return to their original positions.

PLATEAU * on the other hand referred the normal vibrations of a jet to quite a different origin. Regarding the jet as a stationary cylinder of liquid, he proved experimentally that under the influence of the "forces of figure" alone, it would break up into shorter cylinders, the length and diameter of which would bear to each other a constant ratio. Each cylinder would further tend to assume a spherical form, by diminishing in length and increasing in diameter. Such changes must necessarily be attended by growing constrictions between successive cylinders; and, finally, disruption must occur at these points, a succession of drops being detached from the stream. At the moment of its liberation, each drop would be in a state of tension or elongation, and thereafter would execute oscillations about its form of stable equilibrium—the sphere. These oscillations always reaching the same phase at a definite distance from the orifice, would give rise to the appearance of the "ventres" or swellings, noticed by SAVART in the discontinuous part of a liquid jet. The tendency to divide would of course exist in the liquid cylinder from the moment of its formation; but when the impulses due to the impact of the successive drops upon an obstacle are conveyed back to the orifice, they materially assist this tendency, and the stream consequently resolves itself into drops nearer to the orifice.

The normal vibrations of a liquid jet are thus completely and satisfactorily accounted for; and PLATEAU by an ingenious train of reasoning, showed also that division might even occur at other than the normal points, thus explaining SAVART'S observation, that a jet could vibrate under the influence of tones about one-fifth higher, and an octave lower, than its normal. Moreover, Lord RAYLEIGH in a later paper † has pointed out that theoretically the lower limit of effective tones is not well defined, thus bringing PLATEAU'S theory into closer agreement with the facts described in this paper.

Now, while it is impossible to deny to this theory the merits of simplicity, of firm experimental foundation, and of being in harmony with facts previously known, there are, I think, cogent reasons why it cannot be held to account for the new phenomena described above. In the first place it is a fact of experience, already noticed by

* Phil. Mag. ser. 4, vol. 12 (1856), p. 286, and vol. 14 (1857), pp. 1 and 431.

† Proc. Roy. Soc., vol. 29 (1879), p. 71, and vol. 34 (1882), p. 130.

SAVART and MAGNUS,* and easily verified by means of the "Jet Transmitter," that a jet from a perfectly circular orifice, carefully protected from vibrations, does not spontaneously divide in a regular manner. On the contrary, rhythmical division only takes place under the influence of external vibrations, or of the disturbances produced by impact, or when the orifice is irregular. It was quite natural that this should have been denied by PLATEAU; for it can be readily understood from the experiments given above, that perfect insulation of a large jet from the vibrations of any solid body on which it strikes, by means of which solid body the musical character of its vibrations could alone be ascertained, is extremely difficult. With small jets, however, such as can be used in the transmitter, this insulation is comparatively easy. Now, experiment shows that the electrodes may be placed even in the *discontinuous* part of a small well-insulated jet, and yet no musical vibrations be excited in the telephone. When we consider that the electrical method is so delicate, that the effects of communicated vibrations are by it perceptible even at the orifice, it is difficult to believe that spontaneous rhythmical divisions could exist without manifesting themselves lower down.

In the second place, the "forces of figure" which cause the resolution of a liquid cylinder at rest, cannot be considered as solely operative in bringing about the changes which take place in a jet vibrating under the influence of sounds lower in pitch than its normal. Experience shows that whatever may be the rate of the impressed vibrations, within the prescribed limits, and however complex their character, all produce disturbances which grow equally along the jet. As I understand PLATEAU's theory, developed by Lord RAYLEIGH, this would not be the case with disturbances growing under the influence of surface tension alone. The forces of figure, then, either must act in some definite way in both cases; or, if in the latter, they are entirely subordinate to, and controlled by, the impressed forces, some further condition than the mere form of the jet must be taken into account to explain the difference. In fact, their sole action seems to be that they fix an upper and ill-defined limit to the rate of vibration of the jet. With all vibrations below this limit they do not in any way interfere.

The conclusion to be drawn is rather that a jet, at least in its upper part, is *not* comparable to a liquid cylinder at rest. I shall show reason for believing that it may become so at a distance from the orifice; and also that the resemblance may be more pronounced in large jets than in small ones.

It is at once obvious that PLATEAU's theory is quite inapplicable to gaseous jets,† the phenomena of which are strictly parallel to those of liquid jets, and identical with those of a liquid jet playing within a mass of similar liquid, in which the "forces of figure" are necessarily entirely abolished. An examination of some instantaneous shadow-photographs of vibrating jets of air, rendered slightly opaque by smoke, has proved to me that these jets also present swellings and contractions precisely

* POGGEN. Annal., vol. 95 (1885), p. 1, and vol. 106 (1859), p. 1.

† See note on Lord RAYLEIGH's mathematical papers.

similar to those seen in liquid jets, and that these changes grow along the jet-path. I have only been able so far to observe this drop-like appearance in jets at low pressure, and even then the outline is somewhat obscured by trails of smoke detached from the stream. I hope, however, to succeed in getting good photographs of jets at high velocities. The practical difficulties in the way can no doubt be overcome.

It seems to me that the true key to an explanation of the vibratory phenomena of both liquid and gaseous jets is to be found in the fact that the velocity of the stream issuing from a circular orifice is not equal at all points, but diminishes from the centre, or jet axis, outwards. When the efflux takes place from a straight circular tube (and even such a jet is "sensitive") this may be caused by adhesion of the fluid to the sides of the tube, or by friction; but when the jet issues from a hole in a thin plate it is no doubt chiefly due to the convergence of the stream lines towards the orifice. This will tend to cause retardation of the normal velocity of the outer layers of fluid: indeed *at* the orifice the outermost layer of a liquid must be in a state of absolute rest. The fall in velocity from the centre of the stream outwards must evidently be more steep in small jets than in large ones; and as I have pointed out, it is precisely the former which give the most accurate reproduction of impressed vibrations.

The normal jet, then, as it leaves the orifice is composed of an infinite number of cylindrical streams, the velocity of each being uniform in any section of it, but varying in some inverse ratio with its diameter. We thus have in the jet the characteristics of the forward motion of a vortex ring; and the motions of a jet may be considered to be the resultants of the motions of an infinite number of parallel and coaxial vortex rings, moving forward with the same velocity.

This comparison of the jet with a series of vortex rings is not entirely fanciful, for experiment shows that when a jet and a vortex ring are projected from similar orifices, the outline of the jet at successive points along its path shows precisely the same shapes that are successively assumed by the vortex ring. For example, all sections of a jet from a circular orifice are circular, provided that there is nothing to interfere with the free motion of the fluid behind the orifice. With the same proviso, the smoke ring from a circular aperture remains always circular. If a jet orifice be elliptical, and the pressure not too high, successive sections of the jet as we proceed from the orifice will be alternately circular and elliptical, and the major axes of successive ellipses will be at right angles to each other. Now these succeeding shapes also characterize the well known vibrations of a smoke ring projected from an elliptical aperture. These analogies might easily be multiplied. Again, if the aperture in a smoke ring box be an elongated rectangle, a suitable tap on the diaphragm will frequently cause the projected ring to break up into two divergent rings, thus imitating the division of a vibrating jet into two streams; a phenomenon which, when occurring under the influence of sound vibrations, I believe to be invariably due to some irregularity at or behind the orifice.

Now, in a perfect fluid the system of rings composing the jet would continue

indefinitely: in the actual jet two forces tend to their destruction, viz., internal friction, or the viscosity of the fluid, and friction against the surrounding medium. In liquid jets in air the first of these is mainly operative; but when the jet is surrounded by a fluid of its own kind, the second is most effective.

In the first case (of a liquid jet in air) internal friction must bring about a gradual equalization of the velocities of the liquid particles, and at a distance from the orifice the jet will therefore approach the condition of a liquid cylinder at rest. As it does so it will gradually come under the operation of the "forces of figure," will divide, and ultimately resolve itself into drops. Owing to minute variations in the motions of the jet fluid the divisions will be somewhat irregular, but in the main they must tend to assume a rhythmical character, as pointed out by PLATEAU.

The point at which discontinuity becomes apparent cannot, however, depend solely on the viscosity of the liquid, but must also depend on its superficial tension, which determines the *rapidity* of drop formation. If this theory is correct, it should be found then, that those liquids which are most mobile, and have the lowest surface tension, yield the longest continuous jets. Now experiments made by SAVART have already shown this to be the case, although he connected the continuity of a jet with the compressibility of the liquid. Causing jets of ether, alcohol, and water to escape from a circular orifice 3 mms. in diameter, and under a pressure equal to 50 cms. of the liquid, he found the continuous columns to be 90, 85, and 70 cms. in length respectively. The compressibility of the liquids are to each other as 131·35 : 94·95 : 47·85, in accordance with SAVART'S theory. On the other hand, the coefficients of internal friction are ·59213; 1·3754 and 1·1858 for these liquids respectively (HELMHOLTZ and PITROWSKI), and the superficial tensions in contact with air 1·88, 2·5 and 7·3 milligrams per millimetre. SAVART'S observations are, therefore, equally in harmony with the theory here proposed. Thus, while water and alcohol do not differ widely in viscosity, drops are sooner detached from a jet of the former liquid since it has the greater superficial tension.

Both superficial tension and viscosity diminish with rise of temperature; we should, therefore, expect to find that, other conditions being equal, a jet of hot liquid preserves its continuity through a greater distance than a jet of the same liquid cold. This I have found to be the case. In one experiment the continuous part of a jet of distilled water at 10° C. was about 17 cms. in length; but increased to about 20 cms. when the water was heated to 70° C.*

When a gaseous or liquid jet plays within a mass of fluid of the same kind, and when accordingly "forces of figure" do not exist, the analogy to a system of smoke-rings in air is much more evident. The action of internal friction in equalizing the velocities at different points in any section is now insignificant; but the jet rapidly

* Lord RAYLEIGH (Phil. Mag., ser. 5, vol. 17 (1884), p. 192) has already pointed out the remarkable influence of viscosity, modified by changes in temperature, on the "flaring" pressure and sensitiveness of a liquid jet in liquid.

loses energy by setting up rotatory or other motions in the surrounding medium. The rate of this loss depends on the velocity of the jet; and since the outer layers are most retarded, it is attended by increased vorticity. Now, at very low pressures, the fall in velocity from the axis outward is not steep, and the jet, therefore, travels as a nearly uniform stream to a considerable distance. Up to a certain point the length of the continuous stream increases with the pressure; and where discontinuity occurs the jet appears to dissipate rather than to break up in any regular manner. The continuous column of a smoke jet in air may thus extend to 12 inches, and of a coloured water jet in water to 4 or 5 inches, when the orifice is from 1 to 1½ mm. in diameter. If the jet in this condition be disturbed by a gentle tap on the supports, a break appears close to the orifice, and behind the break a portion of the fluid appears to be thrown off from the main stream. In the mass thus detached distinct rotatory motions may be seen similar to those observed in a smoke ring when it has nearly reached the end of its career, and is almost stationary.

The production of vortex rings by interrupting the flow of a water jet in water, has already been noticed by Oberbeck.*

When the pressure is increased to a point at which the jet answers to tones of high pitch, the appearance changes. The nearly cylindrical part becomes much shorter, and beyond this the jet expands rapidly in the form of an aigrette, or divergent brush. The motion may then be compared to that of a smoke ring which travels with nearly uniform diameter for a certain distance, but which expands rapidly when its velocity has been considerably reduced by the surrounding air.

I have mentioned above that when a vibrating jet of air is directed into the trumpet-shaped end of a hearing-tube, its vibrations are almost quite inaudible; but that relatively loud sounds are produced when the hearing orifice is very small and placed in the jet axis. A phenomenon strictly parallel to this may be noticed with vortex rings in air. Let a ring, for convenience rendered visible by smoke, be projected into a large funnel connected by a tube with the ear, and the only sound resulting from its impact within the funnel will be a kind of rushing noise. But let it be projected against a *small* orifice in the end of the hearing-tube, and a sharp click will be heard when the ring strikes the orifice. This click will be loudest when produced from the centre of the ring, and faintest, although still sharp and clear, when produced from its circumference. The analogy is thus complete. These effects may be obtained from the ring so long as it continues to move.

Now, assuming that a jet is composed of an infinite number of circular vortex filaments, vibrations impressed upon the orifice may be supposed to act somewhat in the following way. The vorticity of the jet, as we have seen, is the result of the unequal velocities of the layers composing it. Any disturbance, therefore, which alters the rate or distribution of velocities at the orifice, must also change the vorticity of the jet. To take a simple case, let the vibratory motions of the orifice

* Annal Phys. Chem., ser. 2, vol. 2, pp. 1-17.

take place in the direction of the jet, and let the (liquid) jet play into free air. The motions of the orifice being rapid, and the liquid mobile, the velocity of the stream through the centre may be supposed to remain constant. During a forward motion of the orifice we shall then have acceleration, and during a backward motion retardation, of the velocities of the outer layers of the jet. The first change necessarily implies expansion and diminished vorticity, the second contraction and increased vorticity. If we imagine the motions of the orifice to be produced by two rapidly succeeding impulses in opposite directions, the result will be the formation of two vortex rings of unequal size and strength, which will accordingly begin to act upon each other in a known manner. The foremost ring will go on expanding, its velocity at the same time diminishing, while the hindmost ring will contract, its velocity at the same time increasing. This action will continue until either the constricted portion of the jet gives way, or the vortical motions are destroyed by internal friction, and the forces of figure gain the upper hand.

The growth of an expansion and contraction are thus accounted for, if we suppose that the action of neighbouring parts of the jet on the two vortices is small in comparison with their mutual action. Moreover, as this mutual action of two rings varies inversely as some power of the distance between them higher than the second, changes once started at the orifice must increase very rapidly with the time.

Now it is evident that the portion of the jet which has passed through the orifice during the time that this has taken to execute a complex vibration, must present swellings and contractions at successive points proportional to the varying velocity of the orifice; and in this case, for the sake of simplicity, we may imagine as many pairs of unequal vortex rings to be developed in the jet as there are simple vibrations in the impressed complex vibration. The larger rings will then grow at the expense of all smaller rings; and the result will be that the changes in the jet will go on increasing until disruption occurs at a distance from the orifice.

Since the expansions and contractions of the rings are attended respectively by diminution and increase of their velocites, both rotatory and translatory, there must be corresponding variations in the velocities, measured along the jet-path, of the particles of fluid involved in them. Leaving out of consideration the motion of translation of each ring, the motions of the central and circumferential particles are in opposite directions, and the sum of the velocities in one direction must be equal to the sum of the velocities in the opposite direction. The changes produced by vibration must therefore be apparent at every point of a jet; but since the stream lines of a vortex ring are crowded together near its centre, these changes must be most intense along the jet axis, and most feeble along its outer portions. Again, when the rings are destroyed, either by friction or by directing the jet into a trumpet-shaped tube, the opposing external and internal velocities neutralise each other in the main. On the other hand, when the jet strikes upon any object, such

as a perforated plate, which divides each ring into two parts, this interference must be more or less incomplete.

When a jet of liquid escapes into a medium which offers little resistance to its motion, no change, or at any rate only a slight change, can be produced by vibration in the *mean* velocity along its axis. For here the accelerations are balanced by the retardations; and as these changes succeed each other very rapidly, they are incapable of affecting the sluggishly moving manometer, although quite capable of acting on the highly sensitive mechanism of the ear. When, however, the jet plays within a medium which sensibly resists it, the case is different. Even in its normal state the stream gradually parts with its energy to the surrounding fluid, and therefore expands while its velocity diminishes. But when it is thrown into vibration, owing to the greater surface exposed to the medium by the swellings and contractions, the loss of energy and consequent expansion become more rapid; and hence results a diminution of mean volocity. For the same reason, we have in the outer layers, or in the medium immediately adjoining, an increase of mean velocity when the jet is thrown into vibration.

It has been assumed so far that the disturbances communicated to the jet are of the nature of impulses. But the same reasoning can be extended to cases in which the motions of the orifice are undulations, even of such long period that one whole vibration cannot be included in the continuous part of the jet. For here each infinitely thin section of the jet can be regarded as an independent ring or system of rings, which grows or contracts at the expense of adjacent sections.

I have also assumed that the motions of the orifice take place in the direction of the jet. But it is evident that lateral impulses may also alter the vorticity of the stream, and the disturbances will then not be symmetrical. Want of symmetry is not apparent when a jet is disturbed by sounds produced at some distance, whatever their direction; but it is very evident in the photographs of liquid jets thrown into vibration by tuning forks applied to the support. The planes of the rings in the jets, and the transverse diameters of the drops into which it is resolved, are seen not to be always perpendicular to the axis.

It hardly needs to be pointed out that impulses communicated to the fluid either behind or external to the orifice, may cause disturbances precisely similar to those resulting from motions of the orifice itself. Experimentally we know that vortex rings must be produced in this way.

The properties of a jet may undoubtedly be profoundly modified by the viscosity or surface tension of the fluid. When the fluid is highly viscous an impulse communicated to the orifice will tend to produce acceleration or retardation of the whole layer of fluid included within it; and even if the fluid is only moderately viscous, the inequalities of motion constituting vorticity, in virtue of which the initial disturbances tend to grow, must rapidly disappear. On the other hand, the surface tension of a liquid may be conceived to accelerate, to a certain extent, the growth

of the vibratory changes. We have an extreme case of this kind of action in jets of mercury, which are not only highly sensitive, but break very easily under the influence of sound. This may be seen by letting the jet fall into a basin of mercury, and including it in circuit with a battery and coil. Viewing it in a dark room, sparks will then appear wherever breaks occur, and may frequently be seen quite close to the orifice when the jet is disturbed by moderately loud sounds. The vibrations of a mercury jet become much more regular when it is surrounded by water.

In the foregoing I have given the outlines of a new theory of jet vibrations. Treading on somewhat unfamiliar ground, I have endeavoured to base it, as far as possible, on purely experimental considerations. It would be useless to deny that there are many points in which the theory, taken as a whole, is deficient; and my hope is that others, who have made a special study of vortex motion, may be able to supply what is wanting. The strongest claim to attention that can be urged for it is certainly that it refers the otherwise very similar phenomena of gaseous and liquid jets to conditions of motion which may be experimentally demonstrated in fluids of both kinds. Whether or not the theory can stand the test of mathematical treatment remains to be seen. But even should it ultimately prove to have no more solid foundation than analogy, the experimental part of the inquiry which has led to it may still have some value as a contribution to our knowledge of an exceedingly interesting, if somewhat obscure, class of phenomena. Many lines of thought and experiment are already suggested, and on some of these I hope to report at an early date.

XII. The Bakerian Lecture.—*Colour Photometry.*

By Captain Abney, *R.E., F.R.S., and Major-General* Festing, *R.E.*

Received February 18,—Read March 4, 1886.

[Plates 24, 25.]

We think it may possibly be of some interest to the Royal Society if we lay before them the account of the method which we have used, and the results of some experiments which we have made, in measuring the relative illuminating intensities of different parts of the spectrum, as seen by ourselves and by others, and those of different parts of spectra produced under varying circumstances, more particularly as these results have considerable bearing on practical photometry.

§ I. *Formation of a Patch of Monochromatic Light.*

In the Phil. Mag. (ser. 5, vol. 20 (1885), p. 172), one of us described a method which he had previously demonstrated to the Physical Society of forming patches of monochromatic light upon a white screen, and alluded to the possibility of adapting it to researches on colour. The method, briefly described, is to pass light through a spectroscope, using a photographic camera in place of the observing telescope, then to isolate different portions of the spectrum and make them fall in a patch on a white screen by means of a lens as shown in fig. 1. Any variations in the apparatus are described further on.

This forms the basis of the method on which we carried on our experiments.

Having paid much attention to the determination of the electrical energy which is required in carbon filaments, and platinum, and other wires, to give a certain total radiation per unit of area, we have endeavoured to extend these researches further by ascertaining not only the energy evolved in the whole visible portion of the spectrum, but also the intensity of radiation in various parts of it.

§ II. *Photometric Methods Used.*

To pursue this investigation it was necessary to elaborate some photometric method which should enable us to compare the visual intensity of one ray with that of another of different colour. Our first idea was to place white paper with black lines closely ruled on it in the colour patches of different parts of the spectrum, and then to ascertain the distance at which the lines would just not be separated by the naked eye. Crova

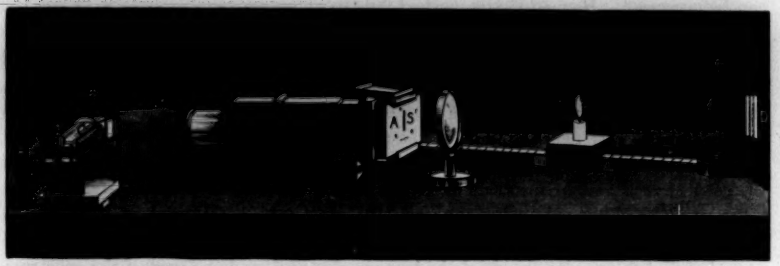
('Paris, Comptes Rendus,' vol. 90 (1880), p. 252) has applied this plan to a spectrum photometer, and gave results which do not quite agree with ours. A great many measures of the solar spectrum were made by this plan, but it was felt to be not as uniformly accurate as could be desired, and to be very fatiguing to the observer. In fact, though feasible in making experiments with our own eyes, it was a plan which we could hardly expect others, whose eyes we might wish to test, to try, seeing that their interest in the results might hardly be even personal and not general. One cause of inaccuracy in this method is the liability of the eyes to astigmatism—a difficulty which one of us had to encounter, and found very hard to overcome, the slightest difference in the inclination of the axis of the eyes rendering readings discordant. By this method it would be necessary, therefore, that any casual observer who might be called in should first have undergone an examination for this defect before any idea could be formed of the value of his readings. Another drawback to it is the fact that from the blue to the violet the accommodation of the eyes is insufficient to permit of the lines being seen sharply, the black lines invariably appearing to both of us hazy and covered with a blue mist. This latter difficulty was, however, partially avoided by the use of a small observing telescope, the focus being altered to suit each ray. But the trouble and uncertainty of this plan caused us to seek for some other method which would be more generally useful, and more easily applicable to others as well as to ourselves.

Our next plan was to cut an aperture in a mask of black paper fastened to ground glass, which was made of such a size that the image of it at a fixed distance when thrown on the screen should be of the same size and shape as the coloured patch of light. Behind this ground glass was placed a candle or incandescence lamp, which illuminated it, and thus a patch of yellowish light was thrown on the screen in juxtaposition to the patch whose intensity was to be determined. By moving the light towards or away from the translucent screen the patch of light became brighter or dimmer, and when the two patches appeared of equal luminosity the distance of the candle from the ground glass was noted, the inverse square of this distance being a measure of the brightness of the patch. This method gave fairly concordant results; but it occurred to us that the close juxtaposition of small patches lighted from different sources, as in the Rumford system of photometry, would be a more convenient method, and this was accordingly tried.

The light was focussed on the slit, S, of the collimator, C, fig. 1, by a condenser filling the collimating lens, and after passing through two flint prisms, P, of medium density, each of an angle of 62°, fell on the lens, L, of a camera, B, which brought the spectrum to a focus on the ground-glass screen. As before described in the paper quoted, a sliding slit, S′, formed in a card, A, was substituted for the dark slide, and the patch of light was formed by being collected by L on a white card, D, placed about 4 feet from the sliding slit. In front of the card was placed an upright rod, E, about half an inch in diameter, to intercept the light from the two sources. A slide, R, carrying

the candle or lamp, T, for the comparison-light, moved on a lath of wood, to which was attached a scale commencing at the screen. When a patch of light of any desired colour was thrown on the white screen the comparison light could be moved along the scale till the luminosities appeared to be equal. Very fair results were obtained by this plan, but the matter of reading was still somewhat difficult until what we have called the "oscillation" plan was hit upon. This plan we will now describe.

Fig. 1.



The illuminating value of the spectrum varies greatly in its different parts; the maximum is usually placed in the orange or yellow, and there is a gradation towards each end of the visible spectrum, the rapidity of which varies according to the kind of light used and the part of the spectrum examined. Now suppose we find that with a source of light at 24 inches from the screen it is approximately of the same intensity as the yellow ray of the spectrum, it is manifest that with the source at, say, 30 inches from the screen, its light will be balanced by that of either of two portions of the spectrum, one on the red the other on the blue side of the yellow. To ascertain which those portions are, the slit A is first moved gently from the yellow to the green-blue. When in the yellow the shadow of the rod illuminated by the source of white light will be palpably darker than the other, and when the slit has passed into the green-blue it will be palpably lighter. We find that the best way of determining the intermediate point where the shadows balance is by oscillating the slide gently between points where first one shadow and then the other is palpably too dark; the oscillations become shorter and shorter until the point of balance is determined. The slide is then moved from the yellow through the red and the same process repeated. We thus get the two points in the spectrum whose illumination corresponds to that of the source of light at its then distance from the screen. By successive alterations in the distance of the comparison-light other pairs of points in the spectrum are determined until the limits of the visible spectrum are reached. The curve of intensities of different parts of the spectrum plotted from these observations will be found to be fairly smooth. This curve we call the "luminosity curve."

The most difficult part of the spectrum to measure is close to the maximum, as in this case at each end of the oscillations the shadow illumined by the candle is too light. It is well at this part to oscillate the comparison-light gently backwards and forwards on the scale (the slit slide being stationary) until a balance is obtained, and to compare the result thus obtained with that by oscillating the slit slide.

By commencing with a narrow slit in the collimator the above method answers admirably as far as $F\frac{3}{4}G$ in the spectrum of a brilliant source of light, and by opening the slit to a moderate degree the intensity of the spectrum as far as G in the violet can be measured. Beyond G the method fails to give very reliable results unless the patch of light be diminished very considerably by substituting a lens of short focus for the re-combining lens, and using a much thinner rod to cast the shadows. It will be seen further on that the slope of the curve in the violet is very gentle. It is, therefore, practicable to use for this part of the spectrum a method which is not well adapted for the part near the limit of the red for reasons which will presently appear. The spectrum itself is thrown on a white screen by a lens of long focus—that which we used gave a visible spectrum 8 inches long. The violet portion is sufficiently brilliant to be measured with accuracy. By oscillating a thin rod in front of the screen at different parts of the spectrum the same alternations of "too light" and "too dark" of the candle-illumined shadow are observed, and the part of the spectrum where there is a balance for each position of the candle is determined with very close approximation down to the limit of visibility of the spectrum.

It might be feared that the white light which illuminates the prism might vitiate the results; but this is not so to any measurable extent, as was proved by using a slit and a second prism with which to disperse the white light. This last plan answers, therefore, perfectly for the whole of the spectrum on the green side of the maximum On the red side the case is different. As a reference to our figures will show, the curve here is very steep, consequently there is a perceptible gradation of luminosity in the length of the spectrum covered by the shadow thrown by the candle, and the comparison is therefore liable to error. By changing the relative positions of the candle, rod, and spectrum, so as to reverse the relative positions of the shadows, we have found that a slightly different value is obtained, but the mean of the two agrees very closely with the value obtained by the other method of comparison.

§ III. *Finding the Position of the Spectrum.*

We fixed our position in the spectrum by means of the bright lines produced by burning different salts in the arc. Those of magnesium and lithium we found most convenient; both of these which we used give copious indications of sodium, and we were thus able to fix the positions on the scale of the red and blue lithium lines, the green magnesium line, and the orange and yellow sodium lines. When necessary, calcium chloride gave us a means of fixing H. A piece of ground glass being placed

against the slit in the focus of the spectrum, it was easy to see when the position of the slit corresponded with that of any line; the reading of the scale on the slide of course gave the position of the line. The positions of these lines being obtained, wave-lengths for different parts of the spectrum could be interpolated. When sources of light other than the sun or the electric arc had to be employed, a magnesium wire on which a small quantity of chloride of lithium in powder had been placed was burnt in front of the slit. This gave clearly the lines of magnesium, sodium (D), lithium (red), with sometimes a glimpse of the blue lithium line. The position in the blue could be fixed by means of the lines in the base of a candle-flame. But it was rarely necessary to resort to these, since the scale of the spectroscope was sufficiently well known to enable any position to be calculated when one or two lines were fixed.

The following table shows the wave-lengths of the scale used in all subsequent tables :—

TABLE of Wave-lengths.

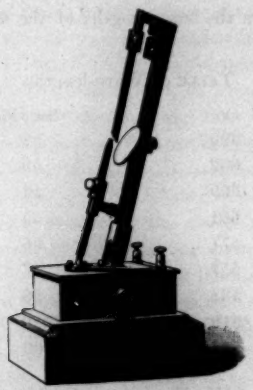
Scale No.		λ	Scale No.		λ
43	=	699	52	=	487
44	=	662	53	=	474
45	=	629	54	=	464
46	=	601	55	=	454
47	=	577	56	=	445
48	=	557	57	=	436
49	=	538	58	=	428
50	=	519	59	=	420
51	=	502	60	=	412

§ IV. *The Source of Light.*

It is unnecessary to call attention to the great importance of selecting a suitable source of light. For the sake of convenience and comfort the light should be bright, for accuracy of reading it should be steady, for comparison it should be readily obtainable, and at the same time have a constant total illuminating value. The natural inclination is to turn to sunlight for such a source; but it is neither steady from minute to minute, except on rare occasions, nor in this climate of ours readily obtainable. The flame of a candle or of an oil lamp is too feeble, and the same applies to the light from an electrical glow lamp, except under favourable circumstances. Such being the case, we turned to the electric arc light, and found, as we have found in so many of our experiments, the results of which we have had the honour of laying before the Royal Society, that with proper management it gives a steady light and has a constant temperature, as our many measurements of the spectrum illumination showed. We used a hand lamp with inclined carbons (fig. 2), which we had for convenience sake slightly altered from SAUTTER LEMONIER'S pattern. The top carbon, which was the positive pole,

was placed slightly in rear of the bottom carbon, so that the crater, which was used as the source of light, might as nearly as possible face the apparatus. We used the cored carbons, as introduced by SIEMENS BROTHERS for use in their lamps. A Gramme M dynamo which one of us has had in constant use in his laboratory for the last five years, and which is driven by a three-cylinder BROTHERHOOD engine coupled to the axle, supplied the current. This machine has worked without a hitch since it was first set up, and for such work cannot be too highly commended. The current is remarkably constant, its strength being 11 ampères, with an electromotive force of 42½ volts at the terminals of the lamp, when the armature revolves at 1400 turns a minute.

Fig. 2.



§ V. *The Spectroscopic Apparatus.*

The spectroscopic apparatus has already been described. It may, however, be remarked, that the collimating lens was filled with the light by means of a white flint-glass condenser, which formed the image of the centre of the crater on the slit. Of the parallel rays from the collimator a certain proportion are necessarily reflected from the first surface of the prism. These being received on a screen about three feet distant, and visible to the operator of the lamp, enabled him to see at a glance if the collimator lens was properly filled with light from the crater, and to readjust the lamp when necessary. After having passed through the second prism and the camera lens, the spectrum was accurately focussed on the focussing screen, and a card, as before stated, inserted in the slide grooves. The slit in this card was ·05 inch wide, and on the edge of the card was a scale of 25 divisions to an inch, and read to quarters of a division. [It may be noted that the scale shown in the subsequent pages is converted

into one of a more convenient range for plotting.] The collecting lens was then so placed that the whole spectrum formed a patch about 4 inches square on the screen.

§ VI. *The Receiving Screen.*

In our earlier experiments we used white cardboard as a receiving screen and for ordinary purposes this answered admirably; but the question arose whether card of the same whiteness could under all circumstances be obtained. On consideration we came to the conclusion that zinc oxide could be used with advantage as a standard white. It is to be found in every laboratory, and can be used in distemper. The medium with which we mixed the zinc oxide was a very pure white gelatine dissolved in water, which we used as sparingly as possible. On comparing the illuminating values of different parts of the spectrum on the cardboard with those of a spectrum on a surface coated with zinc oxide, we found a trace of deficiency in the violet, and a smaller one in the green in the former. This was a matter of some concern, as it rendered necessary a repetition with the zinc oxide screen of the series of experiments which we had made with the cardboard. Other white surfaces were tried, but on the whole the zinc oxide answered best.

A card, or board, was brushed over with a cream of the substance, and when dry a second coat was applied and then flatted down with a brush. An ordinary white card placed alongside it appeared yellowish, while the coated card seemed bluish by contrast.

The portion of the screen used for receiving the lights to be compared was limited to a space about 2 inches square by a sheet of black paper, with a hole of that size cut in it placed close in front of and in contact with the screen.

§ VII. *The Comparison-Light.*

In selecting a comparison-light it should be remembered that it has to be moved and is not fixed in position. An electric glow lamp, with a constant current through it, would no doubt be very convenient were it not for the fact that it does not give out light symmetrically in all directions, owing to the different surface of carbon exposed; and the slightest deviation in angular position makes a considerable difference in the illuminating value of such a source at the screen. The flexible connexion necessary for a gas flame is a source of error, as it is liable to get pinched, and the supply of gas to be thus made irregular, and although many readings were taken with a gas flame we found it inferior, with the appliances at our command, to an ordinary candle, or to the SIEMENS's standard unit lamp burning amyl acetate, which has the value of about ·9 of such a candle.

In the experiments recorded in the immediately following pages a candle was used as the comparison light. Burnt as we burnt it, open to the air but not exposed to

draughts, the light emitted was remarkably uniform. The mere fact that we could place curves taken on different days one over the other, and that they should practically coincide, shows that the errors due to variations of the candle must be very slight. With the SIEMENS's unit lamp similar results were obtained. It may be concluded that the figures given are reliable within a very small percentage. (See Addendum, p. 453.)

§ VIII. *Determination of the Illuminating Value of Different Parts of the Spectrum by the Two Authors.*

When experiments of this nature are undertaken by two people working together, it is clearly necessary that there should be a careful comparison of the effects on their eyes of the same quantity and quality of light. It was a matter of great good fortune that in our case the readings of intensity for each part of the spectrum were identical. As we have not so far found anyone who had a keener perception of the intensity of the whole range of colours than ourselves, we have ventured to call the curve of luminosity, shown in fig. 3, Plate 24, the "normal" curve, as it gives the illuminating value, as determined by us, of the spectrum of the crater of the positive pole. That is to say, the illuminating power of any part of the spectrum is expressed by the ordinate of this curve at the part in question.

We assumed that the light from the arc itself (a part of which must be in front of the crater of the positive pole) would have no appreciable value. That we were justified in our assumption the following experiments will show. With the light we used the strongest part of the spectrum of the arc itself is the green band which commences about "*b*." If the total intensity of the light of the arc and points together at this part of the spectrum be taken as 100, that due to a whole section of the arc alone is only ·42. Now the image of not more than one-half of the total section of the arc is thrown on the slit with that of the crater, we may therefore safely say that the results are affected to a not greater extent than ·2 per cent., a quantity inappreciable compared with errors of observation.

In the violet band, which lies about "G" and on the side of "H," the proportion is somewhat larger,—if the intensity of crater and arc be taken at 100, that of arc alone is 2·5, and not more than half that amount would have to be taken into account; but at this part of the spectrum this difference is practically immeasurable.

It should be noted that the curve which we have given is the mean of a great many, of which none differ 2 per cent. from it and most much less.

To make the diagram more complete we have attached to it the curve of comparative energy as found by the thermopile.

NORMAL LUMINOSITY CURVE.

Scale reading.	Intensity.	Scale reading.	Intensity.
60·0	·35	48·9	80·0
59·25	·43	48·7	86·5
58·31	·54	48·3	91·0
58·0	·60	47·8	97·0
57·4	·70	47·5	99·0
56·4	1·2	47·1	100·0
56·1	1·3	46·9	100·0
55·8	1·4	46·6	99·0
55·5	1·6	46·2	96·0
55·2	1·9	46·0	94·0
55·0	2·1	45·5	84·5
54·2	3·1	45·4	82·0
53·6	4·2	45·2	74·0
53·0	5·8	45·0	65·0
52·0	9·6	44·9	59·0
51·3	15·4	44·8	53·0
51·0	19·4	44·7	45·0
50·6	26·7	44·4	35·5
50·3	36·5	44·2	23·5
50·0	43·5	44·0	16·5
49·8	52·0	43·7	9·3
49·7	56·0	43·5	5·5
49·5	67·0	43·0	2·5
49·2	73·0	42·4	0

ENERGY CURVE of visible Prismatic Spectrum.

Scale reading.	Intensity.	Scale reading.	Intensity.
56·85	3½	49·20	22
55·90	5	48·25	26
54·95	7½	47·30	32
54·00	9	46·35	38
53·05	10	45·40	50
52·10	12½	44·45	64
51·15	15½	43·50	78
50·20	18	42·55	97

§ IX. *General Remarks on ascertaining the Value of Colour Illumination.*

We may remark that the colour whose illumination we found to be most difficult to estimate is green; as, however, any small difference in the position in the spectrum at this part causes but a very small difference in the intensity of the light, the measurement can be made with great accuracy. Blue-green is more easily compared. We found no difficulty in measuring the red, nor the violet when the brightness is fairly powerful. We have already drawn attention to the difficulty of measuring near the maximum, and have explained how measurements here can be checked.

§ X. *Contrast Colours.*

The change of apparent colour by contrast was remarkably shown in the course of our measurements. It must be recollected that the true colour of the shadow illuminated by the candle is yellowish as compared with daylight. At only one part of the spectrum, viz., near D, does it appear to us of its true colour. The apparent change in colour in passing along the spectrum is very significant. In the red part of the spectrum it assumes a bluish tint, and in the green part a ruddy tint. It must be remembered that the field on which these colours lie is compounded of the light of the candle and of that of the particular part of the spectrum employed.

We have endeavoured to describe, as they appeared to us at different parts of the spectrum, the colours of the two lights :—

	Candlelight.	Spectrum.
At 55 of the spectrum scale	Pale oak stain	Deep violet.
53 " "	Dirty orange	Deep cobalt.
52½ " "	Seville orange	Artificial ultramarine.
52 " "	Indian yellow and salmon mixed	Prussian blue.
50½ " "	Dark yellow ochre	Green-blue.
49½ " "	Raw sienna	Laurel leaf.
49 " "	Burnt sienna	Emerald-green.
46½ " "	Yellow (of candle flame)	Yellow (of candle flame).
45 " "	Grey	Scarlet in sunshine.
44½ " "	Slate grey	Cochineal.
44 " "	Payne's grey	Ruby velvet.
43¾ " "	Slightly blue-grey	Cherry-red.
43½ " "	Blue-grey	Morella cherry.
43¼ " "	Dark blue-grey	Dirty red.

§ XI. *Has the Colour of the Comparison-Light any Effect on the Results?*

This is one of the first questions which presented itself to our minds, and one to which we got a satisfactory answer. A cell containing a solution of fuchsine was placed in front of the comparison-candle, the light of which was thus rendered quite red. Although the individual readings taken under these circumstances differed more from the mean than when the naked candle was used, the general result was the same, as it was when green or blue glass or a blue solution in the cell were used to mitigate the colour of the comparison-light. These measurements alone would be sufficient to arrive at the conclusion that the colour of the comparison-light has no effect on the results; but an additional proof will be seen further on.

§ XII. *Does the Quantity of Light Admitted to Form the Spectrum Cause a Difference in the Results?*

This question is one of supreme importance, and we have experimented largely to obtain an answer to it. Observations were made with variations in the width of the opening of the spectroscope slit between $\frac{1}{1000}$th and $\frac{1}{10}$th of an inch, yet we have

been unable to trace any variation from the curve referred to above. We are aware that statements have been made regarding the luminosity of the red of the spectrum, to the effect that when very dim it loses proportionally more luminosity than do the other colours; but in the experiments which we have carried out we have not detected this difference.

§ XIII. *The Illuminating Values of Colour Mixtures compared with those of their Components.*

It has been assumed, but, as far as we know, never been experimentally proved, that the impression on the eyes of a mixed light is equal to the sum of the impressions of each of the components of the light.

If this law be correct, and our observations did not confirm it, it is evident that our method must be untrustworthy. If, on the other hand, our observations, made under varying circumstances of intensity of illumination, obey this law, then the probability is that our method is sound, and the law correct. To test the illuminating value of colour mixtures, instead of the slide with the single slit, one with three slits, of different widths, any of which could be closed at pleasure, was used, so that three slices of the spectrum could be dealt with simultaneously. The collecting lens being used as before, the light from each slit was thrown on to the same patch on the screen. The illuminating value of each slice was then determined, as well as that of the slices taken in pairs, and of the three together. In these measurements it was necessary to move the comparison-light,* as the slits must necessarily be stationary, and, as we have stated before, this method is not quite so accurate as the other.

We append some examples. The slide was so placed that one slit occupied a position in the violet, another in the green, and the third in the red. Measures were taken of these colours singly, in pairs, and altogether, and equations obtained by the method of least squares from these six measurements, with the following results:—

	Observed.	Calculated.
R	203	204·25
(R+G)	242	241·75
G	38·5	37·50
(G+V)	45	46·00
V	8·5	8·50
(R+V)	214·0	212·50
(R+G+V)	250·0	250·25

The accuracy of the results may perhaps be best shown by adding the values of the single colours together, and the pairs and single colours, and comparing these values with that obtained by the three colours combined.

* It will be seen in the addendum that this is now unnecessary if a rotating disc with moving sectors be employed in front of the comparison-light.

$$R+G+V=250$$
$$(R+G)+V=250{\cdot}5$$
$$(R+V)+G=252{\cdot}5$$
$$(G+V)+R=248{\cdot}0$$
$$(R+G+V)=250{\cdot}0$$
$$\text{,,}\quad =250{\cdot}25 \text{ by method of least squares.}$$

A measurement of the red and green was taken on the following day, the position slits in the spectrum not having been altered.

	Observed.	Least squares.
R+G	238	239
G	38	37
R	203	202

The slit of the spectroscope was then opened, and measurements of the same rays made as follows :—

	Observed.	Least squares.
R+G	904	903·3
R	760	760·7
G	142	142·6

It will be observed that in all these cases the relative proportions of red and green are the same within very close limits.

The slit card was then reversed, and the slits placed in other parts of the spectrum. The following results were obtained :—

	Observed.	Least squares.
R	19	19·9
G	130	131·19
V	7·5	7·32
R+G	151·0	150·38
G+V	140·0	138·51
R+V	27·0	26·51
R+G+V	156·0	157·70

Combining the observed results as before we get :—

$$R+G+V=156{\cdot}7$$
$$(R+G)+V=158{\cdot}5$$
$$(R+V)+G=157{\cdot}0$$
$$R+(G+V)=159{\cdot}0$$
$$(R+G+V)=156$$

The mean of these is 157·44, and by least squares $(R+G+V)=157{\cdot}70$.

Many other equally concordant measurements were made; we therefore feel satisfied that it is true that the impression caused by mixed light is equal to the sum of the impressions of its components, also that the method of measurement adopted is trustworthy. If this be true when the ordinary light of a candle or lamp is used for comparison with different parts of the spectrum, it must be true when using light of any other colour.

§ XIV. Hering's *Theory*.

According to Hering, accepting as correct an abstract by Dr. Pole of the various memoirs on the subject ('Nature,' vol. 20, pp. 611, 637, vol. 21, p. 14 (1879)), there are six fundamental sensations of colour arranged in pairs:—black and white, blue and yellow, green and red. The substance which causes the sensations of black and white receives impressions from the whole of the spectrum, being most receptive to the yellow and diminishing in receptivity towards both ends of the spectrum. The blue-yellow substance receives impressions from two parts, one blue and one yellow, of the spectrum, separated by a spot which receives no impression, and where pure green is to be seen. The red-green receives impressions from three parts, one green in the middle and one red at each end, the three parts being separated by two spots which do not affect this substance. Further, Hering supposes that the members of each pair of colours cause what might be called opposite chemical actions, which Dr. Pole has translated as "assimilation" and "dissimilation." When the amount of assimilation in a pair is equal to that of dissimilation no light belonging to the pair is present. As the white-black substance is affected by all the spectrum, he finds that "mixed light appears colourless when it acts on the blue-yellow or red-green substance with equal D (dissimilative) and A (assimilative) power, for then the effects neutralise one another, and the action of the black-white substance alone appears. For this reason two kinds of light which, when mixed, give white are not *complementary* but *antagonistic*; they do not produce white by their combination, but merely destroy each other, and leave visible the white which was already there." As our experiments show that the sum of the intensities of different colours equals that of the mixed light, the statement in the above paragraph would appear to require modification, as it is evident that if one stimulus destroyed the other, the white light resulting from a mixture would be feebler than if the colours were truly complementary. Further, in colour-blind sight there should be perception of light in those parts where it is proved absent.

§ XV. *Testing the Perceptions of Colour in Other People.*

The slit slide was placed in the spectrum focus, and we placed the comparison light ourselves, requesting the observer to oscillate the slide until he obtained a balance. The position of the slit in the spectrum was then noted by one of us by reading the scale at the back of the slide, the observer himself not being able to see the scale. The observation was repeated three times for each position of the candle. We have often found the first two or three measurements to be rather wild, but the observations soon get more accurate. We find it well, as a rule, to place the comparison light so that the first observations shall be a little on the D side of E and then to work up towards the maximum on both sides. By this plan the accuracy of the judgment increases as the maximum is reached. The observer then works back into the violet. This plan ensures a double comparison in the most difficult part of the spectrum, and in case of much difference between the first and second sets of readings a third set is taken; this we have invariably found to agree better with the second than with the first set. It should be recollected that few persons can use an ordinary RUMFORD photometer with much accuracy until they have had some practice with it; it is not therefore surprising that we have found that those who have had some practice in ordinary photometry are, as a rule, much better observers than those who have had no such practice. In some cases we have caused observations to be repeated on two or three separate occasions, and have always found that with untrained eyes the ease and accuracy of observation are much increased in the later trials.

On comparing the "curves of luminosity" plotted from the observations of different persons, we have found some to be identical, or nearly so, with our own; but in some cases, even of persons not actually colour-blind, this is not the case. The variable part may be in the green near the yellow, or in a want of normal perception of red, the appreciation of these colours not being so strong as with us.

Observer K.—We append a curve, fig. 4, Plate 24, obtained from a very careful observer whom we will call K.

K's CURVE.

This curve is the same as the normal curve, except at the following scale numbers:—

Scale reading.	Intensity.
47·0	100
47·8	87
48·3	80
49·0	68
49·4	60
49·6	54
49·8	47
50·0	42
50·1	40

It will be noticed that whereas in our curve there is a marked convexity about scale number 48·5, with K that part of the curve is much less convex. In his paper (Phil. Trans., vol. 150 (1860) p. 57), Prof. CLERK MAXWELL notices the same thing with regard to one of his observers at another part of the spectrum. It will, however, be seen that in comparing K's curve with ours, the total areas are nearly the same. It would appear that if the trichromatism of the normal eye be an established fact, the slight deficiency in K's curve would be accounted for by a slight want of sensitiveness in the curves which give the sensation of green.

§ XVI.—*Colour-Blindness.*

We are not aware that hitherto any direct and accurate determination has been attempted of the relative luminosities to a colour-blind person of different parts of the spectrum. We have been favoured by several gentlemen who have submitted to an examination of their perceptiveness of different colours. The luminosity curves of the electric arc light as perceived by four of these gentlemen, who are distinctly "colour-blind," were determined in the way above described. While at certain parts their curves fall below ours, their maxima are considerably higher than ours. The reason of this evidently is that their eyes not being equally sensitive to the whole of the spectrum, the light of the comparison-candle is to them depreciated by the proportion of the rays which do not affect their eyes, and their measurements are therefore proportionally exaggerated, this exaggeration being, of course, most evident in the parts of the spectrum to which their eyes are most sensitive. We have plotted each of their curves with the ordinates reduced in such a proportion in each case that no part of the curve lies outside the normal curve. Assuming for the moment that their eyes are not more sensitive than ours for any part of the spectrum, the difference between the curve of any one of them and the normal will show the amount and the position in the spectrum of the deficiency in his colour perception.

Observer G.—The first to whom we call attention is G, who kindly put his eyes at our disposal on two occasions for measuring the luminosity of the spectrum. Reducing the ordinates of his original curve by $\frac{1}{3}$ we obtained a curve which touched our curve, taken under the same circumstances, in the green, but dropped below it considerably in the red, and very slightly in the blue-green. Fig. 5, Plate 24, gives the two curves, the maximum of the normal curve being taken as 100. G's principal defect is evidently in the red; in fact, when he matched the luminosity in the part of the spectrum near C with the candlelight, to our eyes, the shadow illumined by the spectrum was a brilliant red, whilst that illumined by the candle was nearly black. There may be a very slight falling off in the blue-green, which on the trichromatic theory can be accounted for by a slight want of sensitiveness in the "blue-perceiving" nerves. Apparently, also, his perception of violet is not normal.

He describes it as blue, as if the red element which to us is present in violet were unperceived by him.

Having measured and plotted our luminosity curve of the candle, the area of which should represent the strength of the comparison-light as seen by us, we proceeded to derive from it a curve whose area should represent that as seen by him. This we did by reducing the ordinates in the same proportion that those of G's curve mentioned above were less than those of the normal curve. The areas of these two last curves were determined to be in the proportion of 1 : ·687. It would thus appear that the measurements made by G would be exaggerated as compared to ours in this proportion; and that, therefore, to compare his original curve with ours his ordinates should be multiplied by ·687. The factor which we actually used, as we have said above, was ·667, differing only ·02 from that obtained by the method described. This may, we think, be looked upon as a fair confirmation of our method of proceeding. Similarly in reducing the curves of R and H we used the factors ·465 and ·425, and the factor found as above was ·455 and ·408. For Dr. POLE's curve the factors were respectively ·952 and ·93.

The bearing of this on practical photometry of unanalysed light is worthy of attention. Colour-blind people can compare two lights of nearly the same tint without apparent error. But when there is any appreciable difference in colour their comparisons will differ from those of people with normal sight.

As an illustration—the ratio of the area of G's reduced electric-light curve to that of the normal curve is ·838 : 1, while the ratio of the areas of the candle curve is ·687 : 1; in comparing the electric light with that of a candle we should evidently estimate the former light as being $\frac{838}{687}$, or 1·22 times stronger than a normal-eyed person would; and this is the proportion of the area of his original curve to that of the normal curve.

G's observations.

Original readings.		Reduced readings.	
Scale reading.	Intensity.	Scale reading.	Intensity.
43·70	9·0	43·70	1·5
43·80	14·0	43·80	2·3
43·90	18·2	43·90	2·9
44·00	25·0	44·00	4·1
44·48	35·5	44·48	5·8
44·58	61·0	44·58	10·0
44·78	81·0	44·78	13·2
44·88	92·0	44·88	15·0
44·88	111·0	44·88	18·1
45·08	137·0	45·08	22·3
45·18	160·0	45·18	26·2
45·33	190·0	45·33	31·0
45·37	226·0	45·37	36·7
45·57	277·0	45·57	45·2
45·65	309·0	45·67	50·4
45·80	346·0	45·80	57·1
45·98	390·0	45·98	64·0
46·20	444·0	46·20	72·7
46·55	510·0	46·55	83·2
47·73	568·0	47·73	93
48·61	510·0	48·61	83·2
49·20	444·0	49·20	72·7
49·50	390·0	49·50	64·0
49·70	346·0	49·70	57·1
49·89	309·0	49·89	50·4
50·00	277·0	50·00	45·2
50·18	226·0	50·18	36·7
50·47	190·0	50·47	31·0
50·61	160·0	50·61	26·2
50·77	137·0	50·77	22·3
51·07	111·0	51·07	18·1
51·39	92·0	51·39	15·0
51·55	81·0	51·55	13·2
51·85	61·0	51·85	10·0
52·20	35·5	53·20	5·8
54·54	25·0	54·49	4·1
54·54	18·2	54·40	2·9
55·74	14·0	55·57	2·3
56·95	9·0	56·75	1·5

Dr. Pole.—In some cases of colour-blindness it does not seem as if there were any diminution in the range of vision along the spectrum as compared to that of normal sight. Such a case we had in Dr. Pole, F.R.S., who has given us permission to mention his name, as he has described in a very interesting paper in the Phil. Trans. (vol. 149 (1859) p. 323), an examination of his own eyes he himself made by another method.

From this paper it appears that with him the sensation of red is altogether absent, but that red appears to him as yellow. He described to us certain shades of orange

as yellow ochre diluted or mixed with white or black. The red about C, and below, appeared to him as yellow mixed with increasing quantities of black. A very interesting feature in his case is that one part of the spectrum which we definitely determined to be at λ 5020, scale number 51, is to him absolutely neutral in colour. He describes all tints on one side of this point as being composed of blue, and on the other of yellow, mixed in each case with varying proportions of white or black.

Dr. POLE is a most accurate observer; in very nearly every case did his three observations of intensity come to precisely the same point on the scale. Fig. 6, Plate 24, shows the comparison between his curve and ours. From this it would appear that there is a deficiency both in the red and green. To us his neutral point appeared a cerulean-blue, which to make white requires the addition of a large proportion of yellow.

The colours of the illuminated shadows for different parts of the spectrum he described as follows :—

	Candle.	Spectrum.
At 52 spectrum scale	Yellow.	Blue.
51	Yellowish.	Blue.
50	Certainly yellow, but grey.	Grey, slightly blue.
49	Grey.	Grey.
47	Grey, with a trifle of blue.	Yellowish.
45	PAINE's grey.	Decided yellow.

It thus appears that at 49, which is λ 5386 in the spectrum, the shadows seem of the same tint to Dr. POLE, while to us at that point the candle light appeared burnt sienna, and the spectrum light emerald-green, whilst the colours of the shadows appear to us the same at 46½ or about D.

DR. POLE'S observations.

Original readings.		Reduced readings.	
Scale reading.	Intensity.	Scale reading.	Intensity.
55·3	7·0	55·3	2·5
53·5	9·5	53 5	3·4
51·85	17·0	51·85	6·2
51·2	26·0	51·2	9·5
50·7	37·0	50·7	13·5
50·35	55·0	50·35	20·2
50·1	71·0	50·1	26·0
49·8	95·0	49·8	35·0
49·35	122·0	49·35	45·0
49·1	142·0	49·1	52·0
48·8	167·0	48·8	61·0
48·35	199·0	48·35	73·0
47·95	238·0	47·95	87·0
47·5	260·0	47·5	93·0
46·8	277·0	46·8	98·0
46·2	260·0	46·2	93·0
45·85	238·0	45·85	87·0
45·3	199·0	45·3	73·0
45·1	167·0	45·1	61·0
45·0	142·0	45·0	52·0
44·8	122 0	44·8	45·0
44·6	95·0	44·6	35·0
44·45	71·0	44·45	26·0
44·35	55·0	44·35	20·2
44·05	37·0	44·05	13·5
43·9	26·0	43·9	9·5
43·6	17·0	43·6	6·2
43·5	9·5	43 5	3·4
43·35	7·0	43·35	2·5
42·8	0	42·8	0

Observer R.—The next colour-blind person whom we examined we will call R. We were very anxious to obtain observations from him, as one of us had seen him some years ago tested by the wool test—an operation, we may remark, which appears to be so unpleasant to the victim, that some persons who had undergone it were deterred by the recollection of it from submitting themselves to our test. We had some slight difficulty in persuading R to do so. Subsequently, however, he remarked that our method was "not unpleasant."

R's curve and that of H (which we shall presently discuss) seem to us to be typical. It will be seen that with R there is scarcely any perception in the red. At C the luminosity of the spectrum to him is only $\frac{1}{4}$th and at D only about $\frac{2}{3}$rds of the normal perception. Again he is slightly deficient in perception of the b'ue rays; but, as will be seen, not so much as is H. Fig. 6, Plate 24, shows his curve compared with the normal. Like G's it had to be reduced in height by using a factor which represents the ratio of the illuminating value of the candle as seen by him to

that as seen by us. When thus reduced his curve touches the normal about E, and after following it a short way leaves it on both sides. On plotting the deficiency curves shown in fig. 6, Plate 24, we found that they both started from E. It was not until after we had done this that we referred to CLERK MAXWELL'S paper, and found that according to him the red curve and the blue curve should meet at E. We may therefore conclude that the observations made by R are trustworthy. It will be noticed that the maximum of the red curve lies about scale-number 45·5, and that his curve ends at 43·6.

R's observations.

Original readings.		Reduced readings.	
Scale reading.	Intensity.	Scale reading.	Intensity.
56·8	7	56·8	1·5
56·0	7·4	56	1·75
55·4	9·3	55	2·0
54·5	11·5	54	3·0
53·4	19·0	53·5	3·5
52·9	26	53	5·0
52·6	30	52·5	6·75
52·5	38	52·2	8·2
52·2	48·5	51·8	10·0
51·8	55	51·6	11·0
51·6	71	51·4	13·5
50·9	95	51	19
50·6	122	50·8	22
50·38	142	50·6	25
50·25	167	50·4	30
50·1	199	50·25	36
49·8	238	50	44
49·4	290	49·8	52
48·9	325	49·6	59
48·5	346	49·4	63
47·8	367	49	69·5
47·2	346	48·9	71
46·9	325	48·65	74·5
46·7	290	48	79
46·5	238	47·5	77·5
46·3	199	47·2	74
46·15	167	46·9	68
46·1	142	46·6	58
45·9	122	46·5	50
45·8	95	46·3	43
45·6	71	45·8	22
45·4	55	45·5	15
45·3	48·5	45·2	8·5
45·0	30	45	5·5
44·7	26	44	1
44·5	19	43·7	0
44·3	15		
44·1	7		

Observer H.—H had no idea that he was deficient in colour perception. He is a gentleman, aged 74, of an acute and educated mind. He attributed his miscalling colours to an ignorance of their names rather than to any deficiency of perception. At first we were not sanguine as to what the results of our tests might be, as we thought he might be right, having sometimes met people whose sole defect is a want of what may be called colour education. Our doubts were however very soon dispelled, and the observations proved to be among the most interesting which we have to record. He had never in his life made a photometrical observation so far as we could learn, so that his test thoroughly tried the practicability of our method as to ease of observation. One of us however moved the slit for him, and asked him to say "too light" or "too dark" as to the shadows, and by gently oscillating the slit we had no difficulty in arriving at his value of the luminosity. Considering his want of practice the readings were very fairly concordant, and can, we believe, be relied on, in which belief we were confirmed by comparing his curve with that of R. It will be seen, fig. 5, Plate 24, that, like R, his red deficiency curve starts from E, and that his luminosity curve ends at 43·6. His blue deficiency is most pronounced. Between C and 43·6 his curve is fuller than R's, as if his perception to red at that part of the spectrum were greater. But it may be that with the feeble illumination of that part his readings were not so correct, and they were certainly less concordant here than in other parts. We have, however, plotted a curve from the means of the readings, and this makes the maximum of the red deficiency curve rather nearer 46 than 45 on our scale.

We append notes of the colours of different parts of the spectrum as he described them.

At 46·6 (D)	Yellow or green, more green than yellow.
46	Orange.
44	(1st time) scarlet like sunset.
	(2nd time after several intermediate colours had been shown) dark green.
54	Blue.
49·6	White, a *slight* tinge of green.
49	Very light green.

When the two shadows were placed alongside one another as in the photometry he described.

	Candle.	Spectrum.
49	Brown.	Light violet.
49·6	Brown.	Green.
46·2	Dark blue.	Green.

H's observations.

Original readings.		Reduced readings.	
Scale reading.	Intensity.	Scale reading.	Intensity.
54·32	11·0	43·6	0
53·63	19·0	44·0	1·5
51·47	40·0	44·5	6·0
51·12	43·5	45·0	12·0
50·77	69·0	45·5	20·0
50·60	92·0	45·9	27·0
50·42	119·0	46·1	33·0
50·24	137·0	46·3	45·0
50·05	160·0	46·4	60·0
49·88	190·0	46·5	64·0
49·70	226·0	46·8	70·0
49·70	277·0	47·0	73·5
49·52	309·0	47·5	81·0
49·16	346·0	48·0	84·0
48·64	390·0	48·5	80·0
47·75	417·0	49·0	72·5
47·38	390·0	49·2	69·0
46·85	346·0	49·6	54·0
46·50	309·0	50·0	35·0
46 50	277·0	50·5	19·0
46·33	226·0	51·0	11·0
46·33	190·0	51·5	7·8
46·15	160·0	52 0	5·5
45·61	137·0	53·0	3·5
45·15	119·0	54·0	2·25
45·15	92·0	55·0	1·5
45·07	69·0		
44·72	43·5		
44·53	40·0		
44·35	19·0		
44 0	11·0		

§ XVII. *Summary of Observations on Colour-blindness.*

We have now described in some detail the results obtained from the observations of four persons who are more or less colour-blind. It may be assumed that H and R are totally blind to the red, and the curve of their red deficiency is therefore of more than passing interest, more especially as they so nearly agree in characteristics. An examination of their curves leads us to suspect that the colour entirely wanting in their colour perception is a red which lies about 43·6 on our scale, and would appear to us of a crimson hue. The fact that the red and blue deficiency curves meet at E is also instructive. In G we have an example of partial colour-blindness to red, his curve showing that he is not entirely insensitive to red radiation, but that the perception of it is damped in some peculiar manner. We may mention that some near relatives of his have the same peculiarity of vision.

Dr. Pole's curve is a most puzzling one, and, as we have already said, it might

appear at first sight that he was blind to red and green, which would correspond to a tetrachromatic theory. On the trichromatic theory also he is not only defective in perception of red, but also of green. Whether this last view is in accordance with his perceptions of colour as he describes them it is difficult to say; but be that as it may, we leave the deficiencies noted for theorists to deal with as they like, merely suggesting that in dealing with Dr. POLE'S curve the others should also be taken into account, more particularly G's, which indicates a deficiency for red intermediate between that of Dr. POLE and those of H and R.

§ XVIII. *Actual Intensity of Illumination seen by Colour-Blind People.*

The question as regards the intensity of illumination actually seen by colour-blind people is one which is very difficult to solve. There appear to us to be two plans by which this might be done, but both are open to doubt. The first is by the method of extinction by means, say, of a graduated wedge. The difficulties in this plan are evident. Our own experience is that the state of health at the time has a marked influence on the precision with which the observations can be made, as has also the time which the observer has spent in complete darkness before the observation. The other method, to which we have already alluded, is that of observing closely ruled lines, and estimating the intensity of the light by the distance at which it had to be placed from the lines in order that they might just not be "resolved."

This latter plan one of us (Captain ABNEY) tried with the observer G. Fine parallel lines, fairly closely ruled on white paper, were placed about 6 feet from an observing telescope, and the two observers, having focussed the lines sharply when they were illuminated by a bright light, alternately observed the disappearance of the lines as a candle was moved from and toward the screen on which the lines were ruled. The following results are the means of several (never less than four) observations in each case, none of which differed more than one inch from the mean.

			Inches.
With the lines vertical. . . .	A	distance of candle	108
	G	„ „	72
„ „ horizontal. . . .	A	„ „	64
	G	„ „	71
„ „ at 45° on each side of the vertical	A	„ „	81
	G	„ „	72

A had previously found his eyes to be astigmatic, and these observations would have proved it if proof were necessary. It appears from the above that except when the lines were horizontal less light was necessary to enable him to resolve them than for G. Apparently the lens of G's eye is spherical, and it is probable that although for one direction of the lines his definition may be inferior to A's, it may be superior for

another direction. Taking the mean of the observations made in the four different directions of the lines, we arrive at the result that the candle value to A : candle value to G : : 75 : 52, that is that G has less perception of light than A.

We have shown that to reduce G's curve so as to lie on ours his ordinates had to be multiplied by ·6, and that this corresponded to his candle value as derived from $\frac{\text{area of G's candle curve}}{\text{area of normal curve}}$. If we multiply 72 by ·6 we get 48 as the relative candle value of G, which differs from that obtained by the above method by 4, *i.e.*, about 8 per cent.

It seems probable, therefore, that the want of appreciation of red in G's case is not made up for by an increased sensitiveness to the other colours. We do not wish to lay too much stress on the above quantitative determinations, but we think that the observations made undeniably show that the actual intensity of the total light of a candle as seen by G is inferior to that as seen by us.

§ XIX. *Curve of Luminosity of the Solar Spectrum.*

As a rule estimations of luminosity have usually been made by comparison with the light of the sun or sky. We, therefore, think it may be of interest to give a general idea of the illuminating value of the different parts of the solar spectrum as determined by us, and we annex two diagrams made by independent methods of observation. The first was our early plan of noting the amount of light required to enable us to distinguish closely-ruled parallel lines. The required amount of light was given by opening or closing the slit of the collimator, the width of opening being measured by a micrometer screw. The lines and observing telescope were fixed, the only variables being the width of opening of the slit and the parts of the spectrum dealt with. This spectrum was taken about 10 A.M. on the 6th July, 1885. The other was obtained on the 17th November of the same year by the method we have described in this paper. We do not lay any particular stress on these curves, as the first was by a method which, as we have said, is particularly difficult to manage, and the latter was taken at a time of year when sunlight is particularly apt to vary. As curves taken on subsequent days agree with this latter, we believe it to be fairly representative of the sunlight as it came to us on that day. The sky was blue, but there was a slight haze which would necessarily alter slightly the proportions of the different parts of the spectrum, particularly the violet and blue, but would not alter the characteristics of the curve in the red, orange, green, and green-blue. This we have been able to determine on several occasions, though from the date given to the present time on no day has the sun been visible long enough to enable us to fully complete a curve. (See Addendum.)

We have put forward these curves to compare with that obtained by VIERORDT (Annal. Phys. Chem., vol. 137 (1869), p. 200), which has, we find, been generally accepted. His method was to take a certain portion of the spectrum, and having extinguished the colour by white light, to calculate the amount of it employed at the

moment of extinction, and thence to deduce the luminosity of the coloured light. This method was used subsequently by Dr. DRAPER.

OBSERVATIONS on Solar Spectrum.

6th July, 1885.		17th November, 1885.	
Scale reading.	Intensity.	Scale reading.	Intensity.
42·2	0	42·54	0
43	6	43·3	4·7
43·6	16	44·3	27
44·6	48	44·55	47
45·5	80	45·17	64
46·8	100	45·80	80
47·7	90	46·55	97·6
48·7	78	47	98·8
49·7	48	47·65	92
50·5	32	48·3	80
51·5	22	48·92	64
53·5	11	49·55	44·5
55·4	6	49·67	42·5
57	4	50·16	30
		50·8	19
		52·05	12·5
		52·2	12
		53·3	8
		54·5	5·5
		55·8	4

Fig. 7, Plate 25, also gives VIERORDT'S results on a bright summer's day; for the sake of comparison we have reduced his maximum to agree with ours.

His table is as follows:—

Position in Spectrum.	Light Intensity.
a — $a\frac{1}{3}B$	80
$B\frac{1}{3}C$ — C	493
C — $C\frac{8}{11}D$	1100
$C\frac{8}{11}$ — $C\frac{1}{11}D$	2773
$C\frac{1}{11}D$ — D	6985
D — $D\frac{1}{13}E$	7891
$D\frac{1}{13}E$ — E	3033
b — F	1100
F — $F\frac{1}{4}G$	493
$F\frac{1}{4}G$ — $F\frac{3}{4}G$	90
G — $G\frac{13}{20}H$	35·9
$G\frac{13}{20}H$ — $G\frac{19}{20}H$	13·1
$G\frac{19}{20}H$ — H	5·8
Beyond H	·9

Now in VIERORDT'S estimation of the intensity of another source of light he has taken the mean of the two positions in the spectrum, and indicated by straight lines joining these points the light intensity of the different parts. We show by a chain dotted line the curve formed in this way. The lines joining the less luminous ends of

those portions of the spectrum on which he operated will, however, be seen by the figure to lie nearer to our curve than the former. But on consideration it would appear as if the curve should be drawn through the more luminous ends of the portions, as these would be the parts of the spectrum where the colour was actually last extinguished by the measured amount of white light. The curve drawn thus differs, as will be seen by reference to the figure, very considerably from ours.

We believe that there are several disadvantages in VIERORDT'S method, and much uncertainty attending its use, at least we have found more uncertainty than in the method of separating lines. We may call attention to the luminosity curve given by Sir W. HERSCHEL in the Phil. Trans., 1801, p. 265, and even allowing for the impurity of the spectrum he worked with, it appears to us that Sir W. HERSCHEL considerably over-estimated the intensity of the blue and violet.

§ XX. CLERK MAXWELL'S *Diagrams of Colour Intensity.*

If reference be made to CLERK MAXWELL'S paper in the Phil. Trans. of 1860, to which we have already alluded, it will be seen that his intensity curves of sunlight are very different in character to ours. There is a dip in the curve at about the position in the spectrum where our curve is most concave, but his curve rises again in the blue, indicating far more intensity than is the case in ours. It is not cur purpose now to discuss the cause of this difference, but it should be remarked that his intensity curve was arrived at by mixing light of different colours to compare in whiteness and intensity with the light, the source of which was a white screen illuminated by sunshine.*

§ XXI. *Intensity of Light passing through Turbid Media.*

We have thought that in connexion with this curve of luminosity of the light of the sun, as it reaches us after having passed through our atmosphere, it would be interesting to give a comparison of the curve derived from the light of the crater of the electric arc, after it had passed through a turbid medium, with that of the light passing through the same medium but unclouded.

It also occurred to us that the theoretical results obtained by Lord RAYLEIGH regarding the colour of light passed through turbid media, as given in the Phil. Mag. (ser. 4, vol. xli. (1871), p. 447), required experimental verification. It is true that in the paper referred to he explains how he attempted to verify his results by comparing sunlight with skylight, and he remarks that "considering the difficulties and uncertainties of the case the two curves" (deduced from theory and experiment respectively)

* Lord RAYLEIGH has kindly pointed out to us that the scale of ordinates chosen by CLERK MAXWELL for the three colours is purely arbitrary. When his ordinates, at the three points where he supposes pure colour to be exhibited in the spectrum, are reduced to those of one normal curve, the agreement is better.—[July 23.]

"agreed very well." If a confirmation of the theoretical loss which light undergoes by passing through a turbid medium be obtained by experiment, it is evident that the law of the effect of small particles in scattering light is also confirmed, since the former is deduced from the latter.

According to Lord RAYLEIGH $I = I_0 \epsilon^{-kx\lambda^{-4}}$, where I_0 is the intensity of the incident light, x the thickness of the turbid medium, and k a constant independent of the wave-length which is denoted by λ. I then represents the intensity of the light which passes through.

To test the matter experimentally, a glass trough was filled with water which was apparently quite clear, and in which the beam from the electric lamp discovered but some few floating motes. This trough was placed between the slit of the spectroscope and the lens used to focus the image of the crater on it, the length of water traversed by the beam being 6 inches (exclusive of the glass walls). The curve of luminosity of the spectrum was then obtained. A solution of mastic varnish in alcohol was prepared, various quantities of which were dropped into the water in the trough for different experiments, the extremely fine precipitate so formed remained in suspension without apparent change for days, the trough remaining in its place throughout. "Curves of luminosity" of the beam through the clear and the turbid water were obtained and plotted on a large scale, and the relative intensities of the beam through the clear and turbid water thus determined. In order to obtain the best results it was found advisable to slightly increase the light admitted to form the spectrum when the water was turbid, on account of the difficulty of measuring a feeble light with accuracy.

The following are the measurements in detail of one pair of curves:—

λ.	Scale-reading of spectrum.	Intensity of colour	
		Through the clear water.	Through the turbid water.
	24	·85	2·3
	25	1·4	2·7
	27	2·3	4·0
	30	4·4	6·5
6448	31	5·0	8·0
6374	35	10·0	15·0
	36	11·2	17·0
6210	39	17·0	24·0
	42	22·5	29·7
6061	44	25·5	31·0
	47	27·3	32·0
5900	48·5	28·5	31·5
	52	29·2	29·2
	55	28·5	27·0
5589	59	27	23·5
5459	64·5	24·3	17·5
4602	66·5	22·5	15·6

It will be remembered that more light was admitted to the prisms with the turbid than with the clear water; y was taken to represent the ratio, and the formula $I_y = I_0 \epsilon^{-\alpha\lambda^{-4}}$ was applied to each of the above results, which gave eight equations containing x and y. By the method of least squares two normal equations were formed, from which the values $x = \cdot 146$, $y = 3 \cdot 72$ were obtained. From independent measurements of the apertures of the slit it was believed that the ratio was 3·8.

The following table gives the ratios of incident and transmitted light as obtained by observation and calculation from Lord RAYLEIGH's formula.

λ.	Observed ratio.	Calculated ratio.
6448	16·0	16·05
6374	15·3	15·4
6218	14·1	14·05
6061	12·2	12·6
5900	11·1	11·2
5589	8·7	8·37
5459	7·2	7·22
5179	5·0	4·90
4602	3·87	3·92

Fig. 8, Plate 25, gives the theoretical curve and the positions of the observed ratios.

On another occasion, with a different degree of turbidity, the values obtained were $x = \cdot 101$, $y = 2 \cdot 28$; from observation $y = 2 \cdot 31$.

λ.	Observed ratio.	Calculated ratio.
6448	13·1	12·7
6374	12·1	12·35
6210	11·85	11·60
5900	10·0	9·9
5589	8·25	8·1
5459	7·4	7·3
5180	5·6	5·6
4602	4·8	4·8

Other results of the same character were obtained, but it is unnecessary to give them. The values observed and calculated seem in every case to confirm the theoretical deductions made by Lord RAYLEIGH.

A comparison of the curves, fig. 9, Plate 25, of the absorbed and unabsorbed lights will show the modifications which are effected in the spectrum by passing the beam of light through a turbid medium. That the waves of shorter length are more absorbed than the longer is clear, from the fact that the position of maximum intensity is shifted towards the red end of the spectrum. It will be observed that the clear water and glass ends of the trough even have somewhat affected the spectrum, for there is less convexity in the curve on both sides of the maximum than in the normal

curve of the arc crater, in fact, this former curve more nearly resembles that of sunlight. Now we have already shown in previous papers that the solar spectrum gives evidence of the light having passed through a considerable quantity of water "stuff," and if the sunlight before it reaches our atmosphere has anything in common with the light emitted by the carbon, the modification of the latter in passing through the trough of water would be a further confirmation of our view. We hope to enter into this subject more fully in a subsequent communication, but we may say, in passing, that the observations which we have made of the diminution of the light in different parts of the spectrum, after passing through still greater thicknesses of water, are of great interest, and accord well with the results of corresponding observations made with the thermopile.

§ XXII. *Curves of Luminosity of Carbon at Different Temperatures.*

It was not only a matter of curiosity, but during our investigations it became a matter of necessity, to examine into the illuminating values of incandescence lamps at different temperatures. Although the full results of our experiments in this direction are scarcely germane to this paper, we think it well briefly to notice some of them.

Fig. 10, Plate 25, gives the curve of luminosity of a lamp with a carbon filament in a vacuum of $\frac{1}{4}$-millionth of an atmosphere after all gases had been exhausted, both from the filament and from the glass globe enclosing it. A secondary battery supplied the current, and it may be assumed that the temperature remained constant during the whole of each experiment.

Some of the measurements were made by the method of using a patch of monochromatic light, others by that of throwing on the screen the whole spectrum, which measured from A to H 18 inches.

The latter, as already pointed out, is not one which lends itself to accuracy unless precautions are taken; on the other hand, the greater brightness of the spectrum on the screen renders it preferable to the patch method for feeble lights. From a careful comparison of the observations made with the same source of light by the two methods, a constant was determined for converting the results of one method into the terms of the other.

During these experiments no part of the apparatus was even touched; everything remained unchanged, except the amount of current, and therefore the temperature of the filament. With the current of approximately $2\frac{1}{4}$ ampères from 26 boxes of a secondary battery, and an E.M.F. of about $49\frac{1}{2}$ volts, the filament was of a fairly bright orange colour, and its curve of luminosity is given as C, fig. 10, Plate 25. Curve B represents the result of passing the current from 28 boxes through the lamp. They gave a current of $2\frac{3}{4}$ ampères and an E.M.F. of 53 volts; curve A, that from 34 boxes, which approximately gave $3\frac{1}{4}$ ampères and $64\frac{1}{2}$ volts.

The light represented by curve B appeared to the naked eye a very bright orange, and would certainly be estimated to be as white as gaslight; the lamp would be

intended by the makers to be worked at about this intensity. The light represented by curve A was brilliant, and would certainly be supposed to be whiter than that of gas burnt in a good fishtail burner. On plotting curve D, however, which is that of gas burnt in a fishtail burner, an image of the flame having been thrown on the slit, we were somewhat astonished to find that it indicated a whiter light than our incandescence lamp would give with 34 boxes. On comparing the shadows of the two lights in an ordinary RUMFORD photometer, we found a confirmation of the truth of our curves, for the shadow illumined by the gas was distinctly blue-grey as compared to the yellow shadow illumined by the lamp. Our unaided eyes, therefore, had failed to appreciate the colour of the light. We found that with 34 boxes of the secondary battery the lamp gave a light of almost precisely the same colour as that of the candles which we used in our measurements.

We next determined the luminosity-curve of a candle, and found that it lay very closely on that of the lamp with 34 boxes. There was, however, a slightly larger amount of blue-green in the former than in the latter, which is a point of some interest.

Reverting to the curves of the incandescence lamps, it will be remarked that at comparatively low temperatures there is marked deficiency in the yellow-green of the spectrum about λ 5630, as shown by a decided bend in the curve, which, however, diminishes with increase of temperature, and in curve C is scarcely discernible, showing that the emissivity of carbon for that part of the spectrum increases proportionally more than for the adjacent parts. It would appear, then, that cold carbon should probably be less absorbent of these rays than of others, a point which we have not determined as yet. The scale of A must be multiplied 6·67 times, and that of B 1·8 times to arrive at the actually observed values compared with C.

LUMINOSITY of Incandescence Light and Spectra.

Intensity.	26 boxes. Scale reading. C	28 boxes. Scale reading. B	34 boxes. Scale reading. A	Gas. Scale reading. D	Intensity.	26 boxes. Scale reading. C	28 boxes. Scale reading. B	34 boxes. Scale reading. A	Gas. Scale reading. D
100	46·2	46·25	46·3	46·35	100	46·2	46·25	46·3	46·35
96	46·45	46·45	46·5	46·75	96	45·65	45·65	45·8	46·00
90	46·72	46·75	46·83	47·28	90	45·35	45·35	45·55	45·65
80	46·92	47·08	47·3	47·9	80	45·05	45·05	45·10	45·25
70	47·10	47·35	47·70	48·4	70	44·88	44·85	44·9	45·00
60	47·28	47·50	48·10	48·75	60	44·7	44·7	44·7	44·85
52	47·50	47·72	48·50	49·00					
50	47·60	47·85	48·60	49·05	50	44·60	44·6	44·6	44·78
46	47·80	48·10	48·75	49·15					
40	48·20	48·50	49·00	49·4	40	44·45	44·45	44·45	44·60
30	48·90	49·20	49·50	49·90	30	44·25	44·25	44·28	44·35
20	49·70	50·00	50·20	50·5	20	44·00	44·00	44·00	44·10
15	50·10	50·45	50 65	50·95					
10	50·80	51·00	51·30	51·60	10	43·6	43·6	43·6	43·6
6	51·5	51·8	52·2	52·6	6	43·4	43·4	43·4	43·4
4	52·00	52·5	53·2	54·00	4	43·2	43·2	43·2	43·2

§ XXIII. *Unit Surface Values of Illumination.*

As a matter of curiosity we have made a comparison between the illuminating values per unit of surface of the filament of the incandescence lamp with the current from 28 boxes and of the crater of the arc light.

This was done in the following manner: by means of a lens of about 9 inches focal length, the image of the crater was accurately focussed on a slit of moderate dimensions, which it well filled. The diverging beam of light from the slit was made to fall on a white screen, and its illuminating power compared with that of a candle by the RUMFORD method. The relative distances of all parts of the apparatus was carefully measured, that of the screen from the slit being in this case 498 inches, and of the lens to the slit 89·5. The mean distance of the candle was 38 inches from the screen.

The positions of the lens and slit being unchanged, the incandescence lamp was substituted for the arc, and the image of the filament was made to fall on the slit, which it more than filled. The screen was then placed at 93·5 inches from the slit, and the light compared as before with that of the same candle, which had to be placed at 157 inches from the screen. Combining these two measurements it was found that, area for area, the light of the crater was 484 times that of the filament in the incandescence lamp. We have found this to be a convenient plan of measuring the illuminating values of areas, and one of us extensively practised this method in a long series of experiments undertaken for the War Department at Chatham six years ago.

POSTSCRIPT.

(Added March 4, 1886.)

§ XXIV. *Practical Application of the Mode of Testing.*

As an example of the practical method of testing luminosity by rapid alternations of "too light and too dark," we would call attention to the method that can be adopted in evaluating the light from a candle and a glow-lamp. Suppose we wish to know the current, &c., which in a given glow-lamp will produce say 16 candles, a candle is placed say 3 feet and the glow-lamp 12 feet from the screen, before which a rod is placed. A current is passed through a glow-lamp and some carbon pellet resistances,* which can be screwed together more or less tightly. The resistance is increased or reduced as may be found necessary till it gives an equal illumination to the screen that the candle does. The pellet resistances are then tightened and slackened alter-

* Since the above experiments were made we have substituted for the pellet resistances carbonized cloth resistance, which was kindly supplied by Messrs. VARLEY of Mildmay Park. The change of resistance in the circuit which this can effect by screwing up is sufficient to allow an ordinary incandescence lamp of 100 ohms resistance to vary between a black heat and a full white heat.—[July 23.]

nately so as to make the illuminated shadows on one side too light and too dark. With rapid alternations the point of what we may call equilibrium is found, and the current read off by suitable means. The same method of procedure may be carried out to compare any light with an incandescence light, and the results show great concordance one with the other.

Addendum.

(Added June 9, 1886.)

§ XXV. *Elimination of the Variations in the Comparison-Candle.*

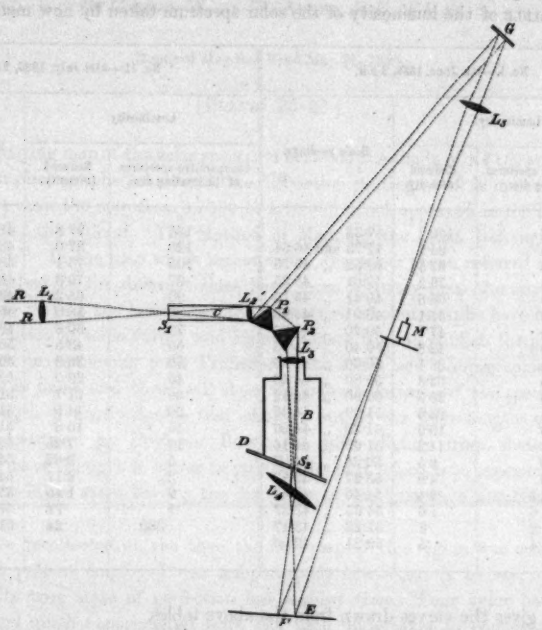
Since our last communication was made on this subject, we have introduced some improvements into our mode of using the instrument described therein, which we think will meet any objections which may be made to our use of a candle or other similar source of light in the comparison-light. Mr. Vernon Harcourt, F.R.S., whose colour perceptions we tried, found that the brand of candle we employed burnt more uniformly than the standard candle, and he gave us the results of some careful measurements which he made. Except for the first five minutes, when the candle he tested evidently was not in its normal condition, and readings with which we should have rejected, the light varied but slightly; and we have no doubt that the results we have already given are very nearly correct, particularly as we find our curves agree almost absolutely with those taken by our improved methods which we describe below. It may be remarked that fluctuations in the candle light are readily distinguished in plotting the curves, since observations are taken on each side of the maximum; and it is readily seen that if the pair of observations are too high or too low the candle has varied. Their variations disappear when the readings for a curve are taken in three sets. To meet, however, the possible objection that may be taken to the candle method, we have employed an incandescence lamp giving about 10 candles, and instead of shifting this comparison-light have kept it stationary, and altered the intensity of its light falling on the screen, by means of rotating discs in which are movable sectors to cut off more or less light. These discs are set to give various apertures, and the reading taken as before. We drive the discs by means of a small electromotor, which causes them to rotate about 40 times a second. We are glad to state that the results for the electric light do not vary from those originally obtained.

§ XXVI. *Measurement of Sunlight.*

Owing to the variability of sunlight during a set of experiments to determine the luminosity of its spectrum, we considered that it would be advisable to make the comparison-light vary in the same degree as the light which formed the spectrum.

To effect this we made the light which had passed through the slit and the collimator to be our comparison-light by using the reflection from the first surface of the first prism.

The general arrangement of the apparatus is seen in the accompanying diagram.



RR are rays coming from a heliostat, and a solar image is formed by a lens L_1, on the slit S_1 of the collimator C. The parallel rays produced by the lens L_2 are partially refracted and partially reflected. The former pass through the prisms P_1P_2, and are focussed to form a spectrum by a lens L_3 on D, a movable screen in which is a slit, S_2. The rays coming through S_2 are collected by a lens L_4 to form a monochromatic image of the near surface of the second prism on F. When D is removed the image of the surface of the prism is white, obtained by tilting the lens L_4 at an angle as shown.

The reflected rays from P_1 fall on G a silver-on-glass mirror. They are collected by L_5, and form a white image of the prism also at F.

At M is a small electromotor carrying moving sectors as already described.

It will be seen that both the comparison-light and the spectrum itself are formed by the light coming through the slit. Any small haze passing over the sun thus

affect both alike, and comparisons are easily made. Of course for the electric light the same procedure may be had recourse to, and gives the same curve as when the candle is used as the comparison-light.

TABLE of the luminosity of the solar spectrum taken by new method.

No. I.—4th June, 1886, 2 P.M.			* No. II.—21st July, 1886, 2 P.M.		
Luminosity.		Scale readings.	Luminosity.		Scale readings.
Comparative apertures of rotating disc.	Reduced Intensity.		Comparative apertures of the rotating disc.	Reduced Intensity.	
°			°		
105	100	47·33	135	97·8	47·24 and 46·77
100	95·2	47·73 and 46·54	120	87·0	48·16 „ 45·97
90	87·5	48·52 „ 45·75	105	76·1	48·61 „ 45·63
80	76·2	49·01 „ 45·35	97	70·3	48·84 „ 45·52
70	66·6	49·41 „ 45·15	90	65·2	49·18 „ 45·40
60	57·1	49·91 „ 44·96	80	58·0	49·64 „ 45·17
50	47·6	50·20 „ 44·70	70	50·8	49·87 „ 45·06
45	42·8	50·40 „ 44·66	60	43·5	50·09 „ 44·83
40	38·1	50·60 „ 44·56	50	36·3	50·53 „ 44·60
35	33·3	50·80 „ 44·46	40	29·0	50·76 „ 44·49
30	28·5	50·90 „ 44·42	30	21·7	51·10 „ 44·38
25	23·8	51·09 „ 44·36	20	14·5	51·56 „ 44·26
20	19·0	51·39 „ 44·26	15	10·8	51·90 „ 44·03
15	14·2	51·69 „ 44·16	10	7·25	52·92 „ 43·80
10	9·5	52·28 „ 44·07	5	3·62	53·84 „ 43·58
5	4·8	53·27 „ 43·87	3	2·17	54·75 „ 43·35
2·5	2·4	54·46 „ 43·47	2	1·45	55·89 „ 43·01
1·6	1·6	55·65 „ 43·37	1	·72	57·94 „ 42·67
·8	·8	57·23 „ 43·17	·330	·24	61·59 „ 42·21
·4	·4	59·21 „ 43·00			

Fig. 11 gives the curves drawn from the above tables.

It should be remarked that from the measurements we have made in London, the solar curve varies considerably according to the direction of the wind. A wind from the east diminishes the intensity of the blue end of the spectrum—a fact no doubt due to the smoke haze which is then prevalent.

We omitted in our paper to point out the method by which the prismatic spectrum curve can be converted into the normal spectrum curve. It will be seen that a very approximate curve can be obtained by using $\frac{dl}{\lambda^3}$ or $\frac{1}{\lambda^3}$ as the factor by which the ordinates of the curves in the former can be reduced to the latter.

* This table gives a fair idea of the luminosity of the solar spectrum in July of this year, the wind being in the S.W., and the sky blue. It agrees with other curves taken on different days, and is a representative one.—[July 23.]

XIII. *The Solar Spectrum, from* λ 7150 *to* λ 10,000.

By Captain W. de W. Abney, *R.E., F.R.S.*

Received May 6,—Read May 21, 1885.

[Plates 26–28.]

The accompanying map of the solar spectrum between the limits of λ 7150 and λ 10,000 is made from photographs taken with the diffraction gratings, and is more complete in every respect than the map from λ 7600 to λ 10,000 which appeared in the Phil. Trans. for 1880, under the title of "The Method of Mapping the Least Refrangible End of the Spectrum." In the map which accompanied the paper above referred to the scale numbers attached to the different lines have more accuracy than the wave numbers, and it was to correct the latter that the new series of photographs have been taken. It is my intention at some future and indeterminate time to publish the photographs of this region in connexion with Professor Rowland's new photographic spectrum which he has in hand, and these will show the minute features of the spectrum down to a wave-length of nearly double that shown; but as the wave-lengths adopted for the visible spectrum by Professor Rowland differ slightly from those given by Ångström, I have thought it better to publish the part which is to supersede the map of 1880 on the latter scale, leaving the discussion of the true wave-numbers to a later period.

It must be recollected at the time the first map of this region was made that the photographic process employed was comparatively new—that is to say, it had been brought to its true state of perfection but a short time. Four years have elapsed since then, and much experimental work has been undertaken in connexion with it; and moreover the instrumental defects which were then present have been remedied to a large extent, new apparatus having been procured and finer gratings having been employed.

Process employed.

A reference to the paper of 1880 will give the idea of the sensitive salt of silver which I then employed, and as regards its nature I have not found any marked improvement to note; but in the development I have succeeded in effecting an advance. My developer now, as then, is the ferrous oxalate developer, but I have found that by employing a glass plate which is previously coated with gelatine emulsion instead of a bare glass plate to receive the sensitive salt, a more powerful solution of ferrous oxalate, without any large addition of restrainer, is capable of being used. The mode of preparation of the collodion emulsion is nearly the same as

that given in the Phil. Trans. for 1880, except that I find the addition of nitric acid is not necessary to be present whilst the emulsion is formed, though in the subsequent washings it is convenient to use it. This may be avoided, however, by washing first with water, and using a dilute solution of iodine to eliminate the veil which is nearly always present after the emulsion is boiled. To the emulsion when prepared I add about ½ per cent. of good soluble cotton; that known as ANTHONY'S No. 1 I find the most satisfactory. A very sensitive gelatine emulsion plate is coated with the collodion emulsion, washed, and then allowed to dry in a warm chamber. The washing causes a minute portion of the underlying gelatine to mingle with the collodion film surrounding the sensitive salt, and to protect it from premature reduction by the ferrous oxalate solution. As I have pointed out in other communications, gelatine acts as a physical restrainer, in contradistinction to a soluble alkaline bromide, which acts as a chemical restrainer. When the former is used there is no partial obliteration of the effect of radiation on the salt, as is the case when the latter is employed; and as a consequence a shorter exposure is necessary.

Apparatus employed.

At first I used the grating with which I had worked for some time previously in my preliminary work. It had a surface of 1½ inches square and about 17,200 wires to the inch, and it is excellent in its performance though with spectra of high orders, ghosts can be traced. Wishing to use less dispersion to obtain a higher wavelength, I secured from Mr. CHAPMAN, of New York, a grating ruled on parallel glass which Mr. A. HILGER had prepared. For the visible spectrum this glass grating gives excellent results, but it is otherwise for the infra-red. When I obtained the grating I backed it with a mercury amalgam, intending to use it as a reflection grating through the glass, but on trying it I found that the spectrum in the first order ceased about λ 8300. This want of success I laid to the metallic backing, and removed it to replace it with one of silver. The glass was carefully cleaned with nitric acid and very dilute potash and alcohol, and silvered by MARTIN'S process. On taking it from the bath I was surprised to find that the layer of silver was so delicate that it took in the grooves formed by the rulings, and that reflection spectra could be taken from both surfaces, *i.e.*, from the silvered surface direct or through the glass. A grating such as this would have proved of the very greatest service, since all that is necessary to have a perfect reflecting surface is to remove the silver film and to re-silver again. Unfortunately the grating thus silvered proved to be useless for my purpose, for there was no infra-red spectrum, probably owing to some peculiarity in the shape of the grooves formed in ruling. That such phenomena are by no means uncommon would now appear to be recognised. Professor ROWLAND most generously gave me several gratings, and amongst them is one ruled on a spherical concave surface, of a radius of 7 feet 9 inches. This grating is so ruled that there is one really bright spectrum, and that one of the first order; all the other spectra are faint. In another grating the

spectrum of the third order is brighter than the second order on one side, and the second order brighter than the first order on the other side. This shows that the very greatest care must always be taken in deducing any value for radiation in any particular region unless the constants of the particular grating employed are known. Both of these last-mentioned gratings have been employed for the producing the photographs to which the map is referred. Besides these, Professor ROWLAND sent me for this work two beautiful concave gratings of about 3600 lines to the inch, of 5 feet 7 inches and 4 feet 8 inches radii respectively. These have been employed for the still lower regions of the spectrum.

Taking a hint from Professor ROWLAND, I used the concave gratings with their axes perpendicular to the photographic plate, but I adopted a plan which is somewhat different to that proposed and utilised by Professor ROWLAND. The annexed figure will give an idea of a mounting for the 7 feet 6 inches concave grating.

Fig. 1.

A A is a plank some 8 feet 6 inches long; at one end of it, E, is placed the grating, with its axis lying along the centre of the plank; G G is a camera, as shown, the plate being inserted at H at exactly the distance of the radius of the grating from its centre. At a point bisecting the radius of the grating a pivot is fixed in the plank A, round which the plank B can turn, and another pin or pivot is placed at D, also exactly at a distance of half the radius from the first pivot. Round D the plank C C can turn, which carries a tube, F, with the slit S attached. S is placed exactly over D. By this arrangement the slit S and the tube F can always be made to point towards E, no matter to what angle the plank B is turned away from A; and as D travels on the circumference of a circle in which the plane of the plate and the grating lie, the diffraction spectra are always in focus on H, and the scale of the spectrum is invariable. This plan I adopted for portability's sake, and also because my laboratory is somewhat confined in extent. At K is a slow motion screw, of which Fig. 2 is a representation on a larger scale, which turns the tube bearing the slit, and I found this arrangement better than turning the slit itself: a is a band encircling the brass tube, b is a screw turned by K, which screws into a swivel at c; when K is turned the tube F turns in

its bearings with a slow motion. In reference to this I may mention that nearly all friction between the brass tube and the wood is avoided by placing between them pieces of baize or cloth. With a concave grating the necessity of the slit being accurately parallel to the lines of the ruling cannot be too much insisted upon, since the definition very largely depends upon it.

Fig. 2.

The majority of the photographs which have been measured were taken with the flat grating, and as the apparatus differs slightly in its details from that previously described, I annex a drawing showing how it was employed.

S is the slit; a beam of light falls on the right-angle prism p, which reflects it to the mirror M′, whence it by successive reflections from G and M″ falls on P, the photographic plate. The mirrors M′ and M″ have radii of curvature of about 7 feet each; M′ is so placed that the rays from S are made parallel and fall on the grating G. The spectra thus formed are very bright and the definition excellent, and I can strongly recommend this arrangement for spectrum work, whether gratings or prisms are employed. The object in using this arrangement was to secure the focus of all overlapping rays being in the same plane, thus allowing coincidences between the first and second and the second and third orders to be discussed.

For filling the gratings with light a quartz or glass condensing lens was employed, though a silvered mirror was also utilised. The sunlight was reflected from a heliostat, the mirror being silvered on the front surface. In reference to this I may say that Colonel Festing and myself have examined the reflective power of silver and a

variety of other metals for radiation, extending from λ 6600 to 27,000, which we hope to submit shortly to the Royal Society.

Measurement of the negatives.

The negatives, some 30 in number, were all measured by a micrometer by A. Hilger, which was placed at my disposal by the Department of Science and Art. Of the performance of this instrument I cannot speak too highly. It is capable of measuring to the $\frac{1}{100000}$ of an inch, but for any photographs which I have obtained, or which I believe can be obtained, this is too delicate; the $\frac{1}{10000}$ of an inch is the unit which alone is necessary, any readings for a well defined line rarely differing more than 1 or 2 of these units from the mean of a series. In most of the photographs measured, a wave-length on Ångström's scale is nearly $\frac{1}{100}$ of an inch; any error in wave-length through bad measurement would therefore only be effected in the second place of decimals: a degree of accuracy which is, for the present, sufficient, since the wave-lengths are only carried to the first place.

In obtaining fiducial wave-lengths the numbers given by Ångström for the visible spectra of the higher order of spectrum with which the infra red had to be compared have been generally adopted, though in some cases the corrections found in Vogel's map have been utilised when it was found that the interpolation formula better agreed with the latter than with the former. Cornu's map was used for the ultra violet comparisons, together with photographs on the normal scale made by Professor Rowland. In many instances the wave-lengths have also been checked by photographs taken on the normal scale, the measures obtained being proportional to the differences in wave-lengths. The labour of mapping the spectrum has been much augmented by this process, as every difference in the first decimal place between Ångström's and my own determination had to be discussed.

In the tables the first column shows the number assigned to each line, the second column the measures of the photographs, the third the deduced wave-lengths, and the fourth check measures taken from other photographs.

Atmospheric Lines.

In previous papers by Colonel Festing and myself we have discussed the effect of water in the atmosphere, and shown that it has a marked effect in shortening the solar spectrum when present in any quantity. We have also shown that the A line is not due to aqueous vapour. It has been asserted by Eggeroff that this line is due to oxygen. This may be so; but if it be, the absorption by this gas is very intense, since I have found that it does not vary in intensity when observed and photographed at a height of about 9000 feet, at Zermatt at the end of June, from that observed in England at sea level. That many lines are due to aqueous vapour there is no doubt, but there are an equal number of lines which cannot be so disposed of. Part of the group, for instance, extending from A to 8200, alter in appearance at sunset and midday, whilst

other lines below it do not appear to vary. Again, $X_{,}$, $X_{,,}$, $X_{,,,}$ are unmistakably not due to ordinary atmospheric absorptions, nor are the pair which I have called Y. There are some metallic lines present which appear to be coincident with certain lines in the spectrum, and as I have before pointed out in the Proceedings of the Royal Society, these are all metals of a low melting-point. My observations have been confirmed recently by HENRI BECQUEREL in an unmistakable manner, since he was, at the time he made his research, I believe, unaware of the paper alluded to.

I have finally to record my thanks to Sergeant H. JACKSON, R.E., and Mr. WALTER BRADFIELD, my two assistants, for the care they have taken in the work, which has been long and arduous, extending over the best of five years to complete.

Line.	Measures.	Wave-lengths.	Check measures.	Line.	Measures.	Wave-lengths.	Check measures.
1	20542	7146·0	7146·0	45	25016	7234·5	7234·7
2	20691	7148·9		46	25168	7237·5	
3	20829	7151·6		47	25203	7238·1	
4	20843	7153·8		48	25251	7239·1	7239·1
5	21184	7157·9	7157·8	49	25397	7241·8	
6	21277	7160·3		50	25458	7243·0	
7	21375	7162·5		51	25503	7243·8	
8	21430	7163·6		52	25578	7245·4	
9	21527	7165·5	7165·5	53	25664	7247·1	
10	21555	7166·1		54	25735	7248·9	7248·8
11	21772	7170·3		55	25841	7250·6	
12	21802	7170·9		56	25874	7250·9	7251·0
13	21837	7171·6		57	25906	7251·6	7251·8
14	21867	7172·2		58	26091	7255·8	
15	21969	7174·3		59	26123	7256·4	
16	22026	7175·4	7175·3	60	26238	7259·1	7259·1
17	22249	7179·7	7179·8	61	26320	7260·3	
18	22397	7182·7		62	26412	7262·2	
19	22490	7184·6		63	26457	7263·1	
20	22537	7185·4	7185·4	64	26509	7264·1	7264·1
21	22751	7189·6	7189·6	65	26719	7268·3	
22	22856	7191·6		66	26877	7271·4	7271·6
23	22914	7192·8		67	27000	7274·1	
24	22973	7194·0		68	27060	7275·3	7275·4
25	23033	7195·2		69	27108	7276·2	7276·0
26	23070	7195·9		70	27202	7279·2	7279·2
27	23110	7196·7	7196·5	71	27357	7281·0	7281·0
28	23168	7197·9		72	27492	7283·7	7283·7
29	23207	7198·6	7198·7	73	27610	7286·0	
30	23247	7199·4		74	27655	7287·9	
31	23301	7200·2		75	27767	7289·1	7289·3
32	23344	7201·3		76	27883	7291·4	7291·2
33	23377	7201·9		77	27980	7293·3	
34	23409	7202·6	7202·4	78	28061	7294·9	
35	23510	7204·6	7204·7	79	28143	7296·5	
36	23552	7205·5		80	28238	7298·4	7298·6
37	23668	7207·7	7207·9	81	28366	7301·0	
38	24019	7214·7	7214·7	82	28415	7302·0	
39	24879	7221·9		83	28668	7306·7	
40	24576	7225·8	7225·8	84	28695	7307·2	
41	24818	7230·6	7230·7	85	28727	7307·8	7307·2
42	24853	7231·3	7231·3	86	28759	7308·4	7308·4
43	24906	7232·3		87	28790	7309·8	
44	24935	7233·9		88	28877	7311·3	7311·4

Line.	Measures.	Wave-lengths.	Check measures.
89	28974	7313·3	
90	29042	7314·4	7314·3
91	29093	7315·4	
92	29133	7316·5	7316·5
93	29188	7317·6	7317·8
94	29305	7319·9	7319·8
95	29455	7322·9	7323·1
96	29573	7325·2	
97	29627	7326·2	7326·4
98	29810	7327·9	
99	29920	7330·1	
100	29948	7330·6	
101	30052	7332·6	7332·4
102	30297	7338·1	
103	30490	7343·4	7343·1 7343·9
103	3584	7343·4	7343·1 7343·9
104	3802	7348·4	7348·6
105	3876	7350·4	
106	3947	7352·15	
107	4065	7354·9	7354·7
108	4117	7356·2	
109	4251	7359·4	
110	4319	7361·1	
111	4392	7362·9	7563·1

Line.	Measures.	Wave-lengths.	Check measures.
112	4465	7365·0	
113	4604	7368·0	7368·1
114	4707	7370·5	
115	4791	7372·6	
116	4873	7374·6	
117	4921	7375·7	
118	5001	7377·7	7377·6
119	5063	7379·2	
120	5140	7381·0	
121	5175	7381·9	
122	5228	7383·2	
123	5282	7384·5	
124	5334	7385·8	
125	5388	7387·1	
126	5465	7389·0	
127	5497	7389·7	
128	5546	7390·8	7390·8
129	5642	7393·2	
130	5745	7395·5	
131	5910	7399·4	
132	6150	7405·0	
133	6305	7408·7	7408·8
134	6384	7410·6	
135	6530	7413·9	
136	6593	7415·4	7415·4
137	6666	7417·0	
138	6718	7418·2	7418·2

Line.	Measures.	Wave-lengths.	Check measures.
134	5674	7410·7	7410·6
135	5825	7414·1	7413·9
136	5879	7415·4	7415·4
138	6006	7418·4	7418·2
139	6151	7421·8	
140	6212	7423·2	
141	6262	7424·4	
142	6954	7440·6	7440·7
143	7058	7443·0	7443·0
144	7163	7445·4	7445·4
145	7898	7462·2	
146	9179	7491·3	
147	9339	7494·7	7494·9
148	9891	7507·3	
149	10054	7511·0	7511·0
150	10588	7522·5	
151	10701	7524·9	7524·9
152	10816	7528·1	
153	10969	7530·7	
154	11073	7533·0	
155	11658	7545·5	
156	12089	7554·8	7554·8
157	12125	7555·6	
158	12710	7568·3	
159	12932	7573·1	
160	13410	7583·5	
161	13514	7585·8	7585·8
Edge of A . . 162	13875	7593·6	7593·6

Line.		Measures.	Wave-lengths.	Check measures.
Edge of A . .	162	31939	7593·6	
Centre of 1st line	163	31952	7593·9	
	164	32010	7595·1	
	165	32053	7596·0	
Edge of 3rd line	166	32079	7596·5	
	167	32123	7597·4	
	168	32176	7598·5	
	169	32255	7600·1	
Several of the	170	32328	7601·6	
lines between	171	32403	7603·1	
7596·5 and	172	32448	7604·0	
7616·2 are pro-	173	32508	7605·2	
bably double	174	32558	7606·2	
lines, but too	175	32627	7607·6	
difficult to	176	32692	7609·1	
measure in the	177	32740	7610·1	
micrometer.	178	32827	7611·9	
	179	32872	7612·8	
	180	32968	7614·8	
	181	33009	7615·7	
Edge of band . .		33033	7616·2	
Single line . .	182	33257	7620·6	
1st pair . . .	183	33368	7622·8	Centre 7623·3
	184	33418	7623·8	
2nd pair . . .	185	33553	7626·5	Centre 7626·9
	186	33601	7627·4	
3rd pair . . .	187	33740	7630·2	Centre 7630·7
	188	33798	7631·3	
4th pair . . .	189	33947	7634·3	Centre 7634·7
	190	33998	7635·3	
5th pair . . .	191	34165	7638·6	Centre 7639·0
	192	34211	7639·5	
6th pair . . .	193	34391	7643·1	Centre 7643·5
	194	34432	7643·9	
7th pair . . .	195	34627	7647·8	Centre 7648·3
	196	34674	7648·7	
8th pair . . .	197	34881	7652·8	Centre 7653·3
	198	34926	7653·7	
	199	35063	7656·4	
9th pair . . .	200	35143	7658·0	Centre 7658·5
	201	35191	7658·9	
10th pair . . .	202	35420	7663·4	Centre 7663·9
	203	35466	7664·4	
11th pair . . .	204	35705	7669·1	Centre 7669·6
	205	35752	7670·0	
12th pair . . .	206	36018	7675·2	Centre 7675·5
	207	36057	7676·1	
	208	36183	7678·6	
13th pair . . .	209	36318	7681·2	Centre 7681·7
	210	36366	7682·2	
	211	36640	7685·6	
	212	36760	7689·7	

Line.	Measures.	Wave-lengths.	Check measures.
212	39077	7689·8	7689·7
213	39290	7694·0	
214	39452	7697·0	7696·9
215	40037	7708·2	
216	40239	7712·2	7612·0
217	40931	7724·8	
218	41739	7740·6	7740·6
219	42045	7745·6	7745·6
220	42180	7748·1	
221	42385	7752·0	
222	43195	7767·4	
223	43400	7771·1	
224	43773	7778·2	
225	44246	7787·0	7787·0
226	44514	7792·1	
227	44614	7794·0	
228	45266	7806·4	
229	46695	7831·0	7831·0
230	46865	7833·6	

Line.	Measures.	Wave-lengths.	Check measures.	Line.	Measures.	Wave-lengths.	Check measures.
231	23175	7868·6		262	17755	7975·8	
232	22855	7875·1		263	17702	7976·9	
233	22368	7884·9		264	17637	7978·2	
234	21874	7894·4		265	16711	7996·7	
235	21568	7900·5		266	16679	7997·4	
236	21219	7907·3		267	16516	8000·6	
237	21170	7908·2		268	16476	8001·4	
238	21062	7910·3		269	16266	8005·6	
239	20994	7911·7		270	15991	8011·1	
240	20861	7914·3		271	15206	8026·8	8026·8
241	20790	7915·7		272	15028	8030·4	
242	20640	7918·7		273	15041	8032·1	8032·1
243	20455	7922·4		274	14980	8033·3	8033·3
244	20395	7923·5		275	14319	8044·5	8044·3
245	20205	7927·3		276	14112	8048·6	
246	20010	7931·1		277	13643	8058·0	
247	19802	7935·2		278	13125	8068·4	8068·3
248	19587	7939·4		279	12348	8083·8	
249	19361	7944·1		280	11656	8097·6	
250	19243	7946·4		281	11212	8106·5	
251	19065	7949·9		282	10894	8112·4	
252	18950	7952·1		283	10660	8117·6	8117·6
253	18742	7956·3		284	10451	8121·7	8121·3
254	18626	7958·6		285	10268	8125·4	
255	18488	7961·1		286	10052	8129·7	8129·5
256	18325	7966·2		287	9892	8132·6	8132·8
257	18136	7968·2		288	9576	8139·5	
258	18056	7969·8		289	9519	8140·6	
259	17980	7971·3		290	9436	8142·2	
260	17885	7973·2		291	9197	8146·8	
261	17810	7974·7		292	9126	8148·2	

Line.	Measures.	Wave-lengths.	Check measures.	Line.	Measures.	Wave-lengths.	Check measures.
289	27042	8140·7	8140·6	329	33075	8260·5	
290	27115	8142·2		330	33138	8261·7	
291	27350	8146·8		331	33561	8270·1	8269·8
292	27423	8148·2		332	33684	8273·2	
293	27548	8150·8		333	33787	8275·2	
294	27666	8153·2		334	33948	8278·4	
295	27830	8156·3		335	34062	8280·4	
296	27980	8159·3		336	34355	8286·2	
297	28025	8160·2		337	34374	8286·8	
298	28120	8162·1		338	34693	8293·0	
299	28412	8167·9		339	34990	8299·0	
300	28808	8175·9		340	35202	8303·2	
301	28874	8177·1		341	35575	8310·7	
302	29038	8180·3		342	35784	8314·5	
303	29088	8181·4		343	35894	8316·4	8316·3
304	29269	8184·9	8184·9	344	36050	8319·6	
305	20425	8188·0		345	36254	8323·6	
306	29610	8191·7		346	36463	8327·7	
307	29694	8193·4		347	36665	8331·7	
308	29830	8196·1		348	36741	8333·2	
309	29982	8199·1	8199·1	349	36948	8337·3	
310	29603	8210·4		350	37015	8338·6	
311	30864	8216·6	8216·5	351	37101	8340·3	8340·3
312	30955	8218·4		352	37313	8344·6	8344·6
313	30996	8219·2		353	37455	8347·4	8347·4
314	31041	8220·1	8220·3	354	37668	8351·7	
315	31160	8222·5	8222·4	355	37865	8355·6	
316	31313	8225·6		356	38085	8360·0	
Z 317	31357	8226·4		357	38320	8364·7	
318	31534	8229·9		358	38693	8372·2	
319	31663	8232·5		359	38828	8374·9	
320	31818	8235·6		360	39207	8382·4	
321	31959	8238·4	8238·4	361	39385	8386·0	
322	32131	8241·8		362	39795	8394·2	8394·3
323	32308	8245·2		363	40453	8407·4	
324	32363	8246·4		364	40799	8414·2	
325	32536	8249·8		365	41811	8434·2	
326	32584	8250·7		366	42388	8445·6	
327	32784	8254·7		367	43450	8466·6	8466·6
328	32938	8257·8		X_1 368	44988	8497·0	

Line.	Measures.	Wave-lengths.	Check measures.	Line.	Measures.	Wave-lengths.	Check measures.
369	11185	8497·0	8496·8	420	22020	9041·2	
370	11273	8501·4		421	22126	9046·5	
371	11516	8513·7		422	22233	9051·9	
372	11750	8525·6		423	22384	9059·4	
$X_{II.}$ 373	12071	8541·8	8541·7	424	22420	9061·2	
374	12355	8556·2		425	22555	9068·0	
375	12901	8583·5		426	22609	9070·7	
376	13427	8609·8		427	22651	9072·8	
377	13611	8619·0		428	22772	9078·8	9078·6
378	14053	8642·0		429	22913	9085·8	
379	14169	8647·8		430	22967	9088·5	
$X_{III.}$ 380	14441	8661·4	8661·4	431	23058	9093·0	
381	14969	8688·0		432	23168	9098·4	
382	15433	8711·2		433	23315	9105·7	
383	15766	8728·1		434	23419	9110·9	
384	15910	8735·4		435	23540	9116·9	
385	15985	8739·2		436	23668	9123·3	
386	16040	8742·0		437	23782	9128·9	
387	16211	8751·5		438	23907	9135·1	
388	16461	8763·0		439	24240	9151·7	
389	16647	8772·4		440	24292	9153·9	
$X_{IV.}$ 390	17320	8806·1		441	24377	9158·5	
391	17655	8822·9		442	24464	9162·8	
392	17837	8832·0		443	24543	9166·7	9166·6
393	17990	8839·7		444	24661	9172·8	
394	18075	8843·9		445	24722	9175·6	
395	18417	8861·0		446	24876	9183·2	
396	18518	8866·1		447	25092	9193·9	
397	19021	8891·2		448	25297	9204·1	
398	19417	8911·0		449	25452	9211·8	9211·7
399	19743	8927·4		450	25535	9215·9	
400	19975	8939·0		451	25576	9217·9	
401	20111	8945·9		452	25634	9220·8	
402	20163	8948·4		453	25684	9223·3	
403	20221	8951·3		454	25867	9232·4	
404	20265	8953·5		455	25966	9237·3	
405	20434	8961·9		456	26106	9244·2	
406	20496	8964·9		457	26207	9249·2	
407	20603	8970·3	8970·2	458	26302	9253·9	
408	20672	8973·7		459	26359	9256·8	9256·7
409	20787	8979·5		460	26400	9258·8	
Y { 410	20926	8986·5		461	26514	9264·4	
{ 411	21005	8990·4		462	26591	9268·3	9268·5
412	21181	8999·2		463	26653	9271·3	
413	21245	9002·4	9002·2	464	26777	9277·7	
414	21317	9006·0		465	26900	9283·6	
415	21411	9010·7	9010·6	466	26928	9285·0	
416	21506	9015·5	9015·5	467	26974	9287·2	
417	21629	9021·8	9021·5	468	27021	9289·6	
418	21689	9024·7		469	27095	9293·2	
419	21804	9030·4		470	27153	9296·1	

Line.	Measures.	Wave-lengths.	Check measures.	Line.	Measures.	Wave-lengths.	Check measures.
469	27215	9299·1		524	32034	9537·2	
470	27280	9302·3	9302·5	525	32086	9539·4	
471	27380	9307·3		526	32140	9542·4	
472	27532	9314·9		527	32162	9543·4	
473	27586	9317·6		528	32185	9544·5	9544·4
474	27663	9321·4		529	32230	9546·8	
475	27706	9323·5		530	32275	9549·0	
476	27765	9326·5		531	32333	9551·9	
477	27827	9329·5		532	32401	9555·2	
478	27894	9332·8	9332·8	533	32431	9556·7	
479	27995	9337·8		534	32512	9560·7	9560·5
480	28071	9341·5	9341·5	535	32524	9561·8	
481	28112	9343·6		536	32554	9563·3	
482	28286	9352·2		537	32605	9565·2	
483	28366	9356·1		538	32647	9567·3	
484	28529	9364·2		539	32698	9569·8	
485	28636	9369·4		540	32877	9578·6	
486	28770	9376·1		541	33040	9586·6	
487	28840	9379·5		542	33100	9589·6	
488	28957	9385·3		543	33166	9592·8	
489	29055	9390·1		544	33140	9591·5	
490	29126	9393·7		545	33247	9597·5	
491	29195	9397·1		546	33384	9603·0	9602·8
492	29239	9399·3		547	33488	9608·6	
493	29297	9402 2		548	33576	9612·9	
494	29343	9404·5		549	33631	9615·6	
495	29393	9407·0		550	33680	9618·0	
496	29431	9408·8		551	33792	9623·5	
497	29506	9412·5		552	34043	9635·8	
498	29573	9415·8		553	34202	9643·7	
499	29660	9420·1		554	34559	9661·2	
500	29772	9425·6		555	34711	9668·7	
501	29835	9428·7		556	34916	9678·7	
502	29986	9436·2	9436·3	557	35140	9689·9	
503	30051	9439·4		558	35347	9699·9	
504	31006	9442·1		559	35491	9707·0	
505	30169	9445·2		560	35606	9712·6	9712·6
506	30237	9448·6		561	35696	9717·1	
507	30330	9453·2		562	35782	9721·3	9721·3
508	30350	9454·2		563	35935	9728·3	
509	30443	9458·7		564	36038	9732·8	
510	30616	9467·3		565	36116	9737·7	
511	30742	9473·5		566	36183	9741·0	
512	30859	9479·3		567	36261	9744·8	
513	30980	9485·2		568	36328	9748·1	
514	31120	9492·2		569	36416	9752·4	
515	31189	9495·5		570	36498	9756·5	
516	31269	9499·5		571	36601	9761·5	
517	31517	9511·9		572	36650	9763·9	
518	31584	9515·0		573	36714	9767·1	
519	31694	9520·4		574	36764	9769·8	
520	31746	9523·0		575	36818	9772·6	
521	31826	9526·9		576	36847	9773·5	
522	31850	9529·8		577	36903	9776·3	
523	31978	9534·4					

Line.	Measures.	Wave-lengths.	Check measures.	Line.	Measures.	Wave-lengths.	Check measures.
562	11764	9721·3	9721·3	583	13850	9823·8	
570	12456	9756·4	9756·5	584	14130	9837·3	
575	12785	9772·6		585	14220	9841·6	
578	12958	9780·6.		586	14304	9846·6	
579	13163	9790·5	9790·4	587	14441	9853·4	
580	13396	9801·8	9801·7	588	14517	9856·9	
581	13598	9810·7		589	14582	9861·1	9861·2
582	13646	9813·0		590	14725	9867·0	

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XIV. *Description of Fossil Remains of Two Species of a Megalanian Genus* (Meiolania) *from "Lord Howe's Island."*

By Sir Richard Owen, *K.C.B., F.R.S., &c.*

Received March 15,—Read April 1, 1886.

[Plates 29–32.]

In 1884 I was favoured by Dr. Woodward, F.R.S., F.G.S., with the inspection of a series of fossil remains from "Lord Howe's Island," which had been transmitted by the Government of New South Wales (Department of Mines) to the Department of Geology in the British Museum of Natural History.

These fossils indicated a Saurian Reptile allied to the genus, characters of which are described and figured in the 'Philosophical Transactions of the Royal Society' for the years 1858, 1880, and 1881.

The geographical conditions of the island, its extent not exceeding six miles in length by one mile in average width, and its distance from the nearest continental tract of land—about 450 miles due east of Port Macquarie, Australia—lent unusual interest to acquisitions of knowledge of the kinds of vertebrate air-breathing animals which had there left evidence of their nature and affinities.

The most singular of the Saurians of Australia, now extinct, was the many-horned toothless kind, which has received the name of *Megalania*, an addition to the osteology of which, throwing light upon its medium and mode of locomotion, was the subject of a late communication to the Society.*

A reptile of this genus, or nearly allied thereto, but of a species inferior in size, I conclude, after comparison of the fossil remains, to claim a generic or sub-generic distinction, and propose to designate it *Meiolania*.

These fossils arrived imbedded in a coral-sand rock,† and are indicative of two species slightly differing in size and in characters not explicable on difference of individual age. They have been carefully worked out of their matrix by Mr. Barlow, of the British Museum, with his usual skill and care, to which the "Department of Geology" owes the display of instructive characters of many of its rare fossil specimens.

* "Description of Fossil Remains, including Foot-bones, of *Megalania prisca*." Read January 28, 1886.

† See Appendix to the present Paper.

The subject of figs. 1–2 (Plate 29) is the occipital and basi-sphenoidal part of the skull, showing the reptilian character of a single occipital condyle. Of this specimen the marginal portions, 2, 2, fig. 1, supplied by the ex-occipitals are slightly convex; the larger proportion (*ib.*, 1), due to the basi-occipital, is more feebly concave; but the sutures, still discernible in the corresponding part of *Megalania prisca*,* are obliterated, and the shape of the condyle is semicircular, not elliptical. Its breadth is 33 mm., its height is 22 mm.

The outlet of the brain case (foramen magnum, *ib.*, *n*) is of equal size to that in the larger genus, and is elliptical, not circular. Its vertical diameter is 25 mm., its transverse diameter is 23 mm.

The confluent side extensions of the occiput, formed by the basi-, ex-, and par-occipitals, constitute a massive extent of bone (*ib.*, 4, *p*) 2 inches in vertical diameter and 1½ inches transversely. The part directed downward (*p*) and extending below the condyle is divided by a large irregular cavity (*x*) from the transverse cuboidal mass (4) above. The super-occipital spine or ridge (*n s*) is sharper than in *Megalania prisca* (fig. *cit.*); its summit, showing a broader base, is broken off. The breadth of the occiput is 4 inches 8 lines (100 mm.), its height 2½ inches (65 mm.).

The cranial cavity at the fore part of this segment (Plate 29, fig. 2, *n*) of the skull gives 30 mm. in the transverse, 35 mm. in the vertical diameter. The portions of cranial wall continued forward from the sides of the brain case are thin, not exceeding 2 mm. The roof of the brain case here preserved is impressed by a transverse semicircular canal or sinus (*s*) 3 mm. in diameter. The lower border of the anterior fracture shows the canal leading down to the recess (*r*, fig. 2) lodging the pituitary remnant of the primitive pharyngeal tract. Half-way between the fore and hind apertures of the brain case a small tuberosity (*t*) rises from the middle of the floor. Behind this, where the side wall rises from the floor, are two small orifices, leading each to a canal which has its outlet (fig. 1, *u*) at the base of the ex-occipital.

In advance of these intracranial foramina is the entry of a larger canal leading to an irregular cavity (fig. 1, *x*), widely opening, in the present mutilated fossil, at *y* (fig. 2) upon the under part of the broad and deep par-occipital process, *p*. From this cavity or chamber a canal, 12 mm. by 10 mm. diameter, leads obliquely inwards and forwards (fig. 2, *y*) towards the wall of the pituitary chamber. The under surface of the basi-occipital is concave, expanding as it descends and advances. Its junction, and here coalescence, with the basi-sphenoid is indicated by a pair of ridges which diverge from near the mid-line outward and backward. The par-ex-occipitals (4, 4) are sub-quadrate masses, terminated each by a hemispheroid cavity with a thin sharp border. Each ex-occipital is excavated behind and below by the chamber, into which opens the canal from the larger of the three foramina on each side of the occipital division of the cranial chamber. A large foramen (fig. 1, *m*) on the upper and fore part of the ex-occipital leads also to the sinus (*x*), from which cavity is continued a wider

* 'Phil. Trans.,' 1880, vol. 171, Plate 36, figs. 1 and 2.

channel (*y*) opening on the outer side of the "sella turcica"; with this opening communicates the "trigeminal canal" (fig. 2, *e*).

Returning to the base of this part of the skull, will be seen, half an inch from the mesial ends of the basi-sphenoidal ridges, a second basal floor formed by a flattened horizontal plate of bone with a smooth under-surface 2 inches broad at the hind border and narrowing as it advances to an inch and a quarter's breadth, where it terminates with a forward production of its angles, between and above which protrudes a continuation of the proper "basis cranii" constituting the floor of the pituitary cavity. The interspace between the true and false basis cranii is 2 mm. vertically at the fore part, 3 mm. at the hind part of the supplementary plate. The breadth of the hinder interspace is 40 mm., of the anterior interspace 25 mm. The extreme breadth of this singularly complex part of the skull is 4 inches, the vertical diameter of the occiput is 3 inches; but this would be exceeded if the superoccipital spine were entire.

What I regard as the cranial outlet of the trigeminal nerve opens into a wider passage leading to the large irregular smooth-surfaced chamber (fig. 1, *x*), excavated at the under part of the par-occipital production (*p*). The flatness of the basi-sphenoidal tract contrasts with the concavity of that region shown at 5, fig. 2, Plate 36 ('Phil. Trans.,' 1880, vol. 171) in *Megalania prisca*. The amount and degree of modification of the hinder part of the skull of *Meiolania* add to other grounds for generic or subgeneric distinction of the present horned Saurian, manifested by the fossils next to be described.

In the same block with the occipital part of the skull was sent the fore end of, apparently, the same skull (Plate 29, figs. 3–6), judging by relative size, colour, and state of fossilisation. This specimen included the floor of the outer nostril (figs. 3 and 6, *n*, *n*) with a partial bony septum (*ol*) commencing half an inch behind the nasal aperture. The breadth of this nostril is 1½ inches; the smooth surface of the cavity to which it leads contrasts with the coarse reticulate sculpturing of the surface at the fore part of the upper jaw, which describes a transverse convex curve to the fractures (figs. 4, 6, *m*, *m*), severing the fossil from the rest of the skull. The alveolar border (fig. 4) is a trenchant plate (*a*) with a semi-circular notch at the mid-line. The plate is thickened by a slight buttress of bone on each side of the notch; there is not a trace of dental alveolus. The smooth channel (*b*), of which the trenchant border (*a*) forms the outer wall, is half an inch in breadth, and is bounded behind by a shorter and lower pair of ridges (*ib.*, *c*, *c*) almost parallel with the foremost. Each of the hind ridges (*c*, *c*) commences opposite the thicker mesial part of the outer ridge (*a*). The intervening channel (*d*) sinks deeper than the lateral ones into the jaw's substance. The bony palate extends backward from the mesial ends of the hind ridges, where it has a breadth of half an inch; it narrows as it extends outwards to the terminations of those ridges.

The floor of the nasal cavity (fig. 6, *n*) extends backward for half an inch behind the anterior opening, then abruptly sinks to a plane (fig. 5, *g*) half an inch below the hind

borders of the partially divided passages. This seemingly second floor also extends beneath the floor of that fore part of the outer nostril, as it were, undermining it, and contracting to a blind end two-thirds of an inch in advance of the overlying base of the partial bony "septum narium" (fig. 6, *ol*). The smooth surface of the naso-palatal cavity extends outward and downward on each side of the above part of the floor and behind the outer ends of the hinder trenchant walls of the alveolar channels.

The thickness of the jaw-bone forming the side walls of the palatal tracts at a transverse line drawn across the lower floor (fig. 6, *g*) of the nasal chamber is two-thirds of an inch. The fracture of this part of the upper jaw on the right side exposes a small smooth-surfaced sinus (fig. 5, *i*). The under-surface of the mid-portion (fig. 4, *g*) of the bony palate shows a pair of depressions (*ib.*, *l*, *l*), from each of which a small canal is continued into the substance of the upper jaw.

If the fossil above described (Plate 29, figs. 3–6) be compared with the corresponding part of the skull of *Megalania prisca*,* the bony narial septum will be seen to be similarly incomplete, dividing the lower half only of the nostril. The trenchant border of the beak (fig. 3, Plate 38, *ib.*) is similarly incomplete; the trenchant border of the upper mandible of *Megalania* shows also a median notch, which I deemed might be "due to accident" (*tom. cit.*, p. 1043), but it is only partially so.

If the under-surface of the fore end of the upper jaw of *Meiolania* (fig. 4) be compared with that part of *Megalania* (fig. 3, Plate 38), the difference, added to those already and subsequently noticed in other parts of the skull, may be admitted to exceed mere specific distinction. The second trenchant ridge in *Megalania* is co-extensive and parallel with the outer one, and is followed by a third nearly parallel ridge, but lower, and of less extent. In both Reptiles these ridges were, most probably, sheathed with horn, as in the Chelonian Reptiles. The bony roof of the mouth is continued in *Megalania* a short way behind the third trenchant ridge. The narrower medial backward production of the bony palate has not, in *Megalania*, the mid-longitudinal rising with the lateral and perforated depressions as in *Meiolania*. The anterior transverse curve of the upper mandible (premaxillaries) is greater in the smaller edentulous Saurian.

The partial bony septum (fig. 3, *ol*) terminates obtusely in *Meiolania* without trace of fracture. The vertical extent of the fore and outer plate of the upper jaw in *Meiolania* is relatively greater than in *Megalania*, and no trace of the outer channel impressing the interspace between the nostril and orbit is present in the portion of Meiolanian skull above described.†

Plate 30 represents the upper surface, natural size, of the portion of the skull of *Meiolania platyceps* which could be exposed in the block of matrix in which it was buried. The skull had suffered fracture, with some dislocation of parts, but retained

* 'Phil. Trans.,' 1880, vol. 171, Plate 37, fig. 1, and Plate 38, fig. 3.

† See Plate 38 *tom. cit.*, fig. 1, *o n*.

the positions in which they had become imbedded. The partially detached right horn-core, fig. 2, is figured below, in the position to show its curvature.

Three eminences of the outer table of the cranium, Plate 30, *a*, *b*, *c*, answer to the three protruding from each side of the corresponding parts of the skull in *Megalania prisca*.* The homologues of the shorter horn-cores are represented in *Meiolania* by low obtuse eminences, *a*, *c*. The main horn-core, *b*, rises from a broadly elliptical base to a height of 4 inches, following the convex curvature, which is backward and slightly outward; its basal circumference is 6½ inches. The right side of the cranial wall had suffered a greater degree of fracture, and the main horn-core had become detached; its relative position in the mass of matrix is nearly parallel transversely with the left undisturbed horn-core. The side of the cranial wall, from which the right horn is detached, shows the hinder rudimental core, *c*; it differs from the anterior one, *a*, in its lower development, and narrower, more elongate, basal proportions.

The tract of the upper wall of the skull, continued forward from the main horn-core and anterior protuberance, *a*, includes the upper border of the left orbit, *o*, and that preserved portion of the frontal region extends to the right orbit *o*. The skull-wall is continued forward to the outer nostril, *ol*, of which the right border is preserved, with a portion of that side of the upper jaw, 21, 22.

On the under-side of this maxillo-premaxillary part of the skull, Plate 29, fig. 4, is shown the absence of tooth-sockets, and their replacement, in the present species of *Meiolania*, by three curved, nearly parallel, trenchant ridges, with the two intervening channels. The osseous parts, which had been ensheathed by the perishable corneous beak, differ from that of *Megalania prisca* (Plate 38, fig. 3, 'Phil. Trans.,' 1880) inasmuch as the intermediate ridge subsides sooner at its fore end, leaving the mesial portion of this surface of the bony mouth as a broader and shallower depression undivided by the forward continuation of the second ridge. In *Meiolania minor* a third trenchant ridge is not developed.

The outer surface of the skull-wall of *Meiolania platyceps*, Plate 30, is sub-reticulate, the boundaries of the depressions being barely prominent. The only traceable suture on the exposed tract of skull is a medial one between the parial frontals and nasals. The basal mutilation of this part of the skeleton of *Meiolania* (Plate 31, fig. 1) is such as to preclude reliable description. The right main horn-core (Plate 30, fig. 2) is so displaced, by probably disturbing movements of the matrix, as to show the degree of backward curvature.

The characters above described indicate a larger species than *Meiolania minor*, but add grounds for the generic distinction from the still larger Australian horned Saurian. The fracture at the upper surface of the skull, marked *x* in Plate 30, may relate to a seventh horn-core; if so, it has a hinder position in relation to the orbit than in the subject of fig. 1, Plate 38, of the 'Phil. Trans.' for 1880.

* Plate 37, fig. 1, *b*, *e*, *f*, 'Phil. Trans.,' 1880.

Subsequent discoveries in Lord Howe's Island may indicate sexual modifications in *Meiolania*, but the occipital and mandibular characters above defined, the subjects of Plates 1 and 2, are of higher taxonomic value.

The main horn-core, *b*, in *Meiolania*, is shorter in proportion to its basal breadth; the hinder protuberance is lower and has a longer base; the reticular sculpturing of the superficies of the larger horns and contiguous parts of the cranial wall is more strongly marked than in *Megalania*. Here, also, I may observe that, if the part marked *x* in Plate 30 be merely fracture, the nasal horn in the Moloch Saurian ('Phil. Trans.,' 1880, Plate 37, fig. 8) has not a bony support, and such might be the case in the horned Saurians of Lord Howe's Island.

The length of the imbedded portion of the skull of *Meiolania platyceps*, above described, is 9½ inches; the breadth across the bases of the biggest horn-cores is 6½ inches.

The subject of fig. 2, Plate 31, is the symphysial extremity of the mandible of a larger *Meiolania*, including similar modifications for support of a lower horny beak, and the commencement of the mandibular rami thence diverging. The anterior of the trenchant ridges, *a*, *a*, is notched in the middle; the next is divided by a median channel. The partially preserved oral surface extends backward on a tract of bone, *e*, half an inch in vertical breadth or thickness, which overlies the smooth inner surface, *f*, of the attached parts of the mandibular rami. The anterior trenchant border terminates at each outer end in a tuberosity, *t*, from which the corresponding border of the ramus descends, and expands to form a smooth, broad, slightly concave surface, *f*, defining the upper part of so much of the ramus as is here preserved. The rough granulo-reticulate outer surface of the lower jaw, continued at right angles from the smooth upper surface, makes an abrupt bend inward to a broad, flattened, similarly sculptured, under-surface of the beginning of the ramus, where it extends freely backwards from the masticatory and symphysial part of the lower jaw. The breadth of the lower surface, at the fracture, is 1⅓ of an inch; the depth of the thick ramus here, between the upper smooth and lower sculptured surfaces, is half an inch. The concave inner surface of the ramus, where it is continued from and undermines the hinder masticatory part of the symphysial end, is smooth. The breadth of the mandible at the fractured ends of the rami is 5 inches. This portion of jaw indicates a massive powerful bone.

The above-described fossil reached me in a detached state.

A second block of matrix includes an anterior vertebra, part of a scapula, and fragment of a humerus.

Of the vertebra, the hinder part is exposed, the fore part of the centrum is imbedded in the matrix. A deep and well-defined concavity occupies the hind part of the centrum, showing an epicœlian character. This articular surface is continued upon a pair of processes extending backward, and slightly diverging from the hinder and under part of the centrum. Each process is short and thick, and terminates obtusely with an articular surface slightly convex, apparently for the attachment of a rib. The

proportion of the margin of the articular concavity of the centrum, not continued upon the parapophysis, is sharply defined; this articular surface is 1⅓ inch by 1 inch 2 lines.

In Part III., 'Phil. Trans.,' 1881, vol. 172, portions of the dermal skeleton, with productions meriting the name of "caudal horns," were described and figured. *Meiolania* repeats, with modifications, these characters of *Megalania*.

An osseous sub-cylindrical sheath, with similar defensive processes, was brought to light in one of the blocks of matrix forming part of the series of fossils from Lord Howe's Island (figs. 3 and 4, Plate 31). To get clear demonstration that this sheath included vertebræ, a portion was removed, and careful detachment of inclosed matrix brought to light caudal centrums, one sufficiently well preserved to show the elongate sub-cylindrical shape of the vertebral body, and the reduced area of the neural canal overarched by a simple thin bridge of bone. The three segments, subjects of fig. 3, consist each of an exo- and endo-skeleton. The former, *x*, *x*, is a cylinder, 2⅓ inches long, with an upper and a lower lateral pair of obtuse processes, *d*, *d'*. The thickness of the sheath varies from 2 mm. to 10 mm., the interspace between it and the included vertebra varies from 13 mm. to 3 mm. The osseous processes may have supported horny sheaths. The tail was relatively as long, probably, as in the Moloch Lizard (Plate 37, fig. 9, 'Phil. Trans.,' 1880); and, as in that surviving representative, the terminal segments ceased to develope caudal horns. A portion of this part of the tail is the subject of fig. 4, Plate 31. The endo-skeleton, reduced to the vertebral centrums, is partially exposed.

A singular appendage this long and seemingly inflexible tail must have been in the living *Meiolania*, contrasting with that part of the frame in the Chelonian order, whilst the edentulous modification at the opposite end of the body resembled that of the Tortoises and Turtles. The greater and closer affinity to the Saurian section of Reptilia is manifested by the vertebræ and parts of the limbs which have been brought to light.

The portion of scapula includes the glenoid cavity, a little mutilated at the margin, and part of the expanded body or blade; the "neck" supporting the cavity is well marked. The breadth of the preserved portion of this bone is 4 inches, the opposite diameter is 2 inches. The absence of any trace of suture between the blade and neck, or in the articular cavity, precludes a reference of the fossil to a coracoid.

The portion of humerus includes the expanded proximal end, the shaft supporting which has undergone fracture; this discloses a large medullary cavity, the wall varying from 5 to 3 mm. in thickness. Though fragmentary, this fossil is acceptable as testifying to a terrestrial stage of active life.

In a former memoir * was noted the coalescence of the centrum and neural arch of a trunk vertebra;† and in the series of comparisons with the spinal column in other

* "Description of Remains of *Megalania prisca*," Part II., 'Phil. Trans.,' 1880, vol. 171, p. 1037.

† P. 1038.

Reptilia, both extinct and existing, a like coalescence of the vertebral elements was found in the small existing Australian lizard (*Moloch*).

As this osteological character is the rule in the "warm-blooded" Vertebrates, the neural arch retaining, for the most part, its earlier stage of development in the cold-blooded air-breathers, I was, in some measure, prepared by structures above described to recognise a corresponding advance of osteogeny in the piers of the pelvic arch of *Meiolania*.

In Anthropotomy the pelvic elements, recognised as becoming ossified from three chief centres, received the names of "ilium," "ischium," and "pubis"; but, coalescing into a single bone in the adult, this was described as the "nameless bone"—*os innominatum*. Such coalescence is rare in the class *Reptilia*, and it adds to the singular characters of the Megalanian Family of the Antipodes. I subjoin three figures of this "innominate" fossil from Lord Howe's Island, and on the grounds of locality and contiguity refer it to the genus *Meiolania*. The coalesced pelvic elements are denoted in Plate 32, figs. 1, 2, and 3, by the numerals indicative of them in the descriptions of the modifications of that part of the reptilian skeleton in my 'Anatomy of Vertebrates.'* Reference to those figures will show the contrast between the expanded (*Mammalian*) character of the bone, 62, in Crocodiles, and the almost columnar form in the Lacertian pelvis. This latter form is attained or retained in *Meiolania*. The ilium, 62, is a triedral column, moderately expanded at the proximal end to join by a rough syndesmosis with a short sacrum; and, at the distal end, to contribute its share to the acetabulum, *a*. One angle of the shaft or body of the bone looks outward, one plane inward. The anchylosis of the osseous elements, which has dominated to an unusual extent in the skull of the Megalanian family, prevails in the pelvis. There is no trace of a suture which would indicate the share taken by the ilium in the formation of the acetabular cavity.

The upper or "proximal" end of the triedral ilium expands in a moderate degree to unite by a rough surface (fig. 2, *s*) with a sacrum. The opposite, outer, surface of this expansion (fig. 1) is also roughened, but in a minor degree, for attachment of a "glutæus" muscle. Acetabular sutures indicative of the shares respectively taken by the ischium, 63, and pubis, 64, are also obliterated by the firm anchylosis of the pelvic bones of this powerful extinct Saurian. The contour of the acetabulum deviates slightly from the circular toward the triangular form: the ischial and pubic angles are rounded off (fig. 1, *a*). The diameter of the acetabular outlet gives two inches: the depth of the cavity is rather less than one inch. These dimensions indicate the femur of *Meiolania* to have equalled that of the largest known Crocodilian in size.

Of the pubic element, 64, little more than two inches are preserved: the bone contracts to a triedral form as it recedes from the articular cavity for the thigh-bone. A smaller proportion of the ischium, 63, is preserved, and it indicates an assumption of a more lamellate figure as it recedes from the acetabulum. The length of the

* Vol. 1, 1866, p. 188, fig. 119, *Crocodilus*; p. 191, fig. 121, *Monitor*.

above-described portion of pelvis is 6⅞ inches; the fractured ischio-pubic end is 3½ inches. The size of this part of the fossil skeleton agrees with that indicated by the Meiolanian trunk-vertebræ, which hardly differ, save in size, from those previously described and figured in the type-species of the Megalanian family.* The subjects of a former paper indicated a pentadactyle and unguiculate character of one, probably the fore, pair of limbs in *Megalania*.

Geological Appendix.

"Lord Howe's Island is formed of three elevated basaltic masses, connected by low-lying grounds of blown coral-sand formation. The latter consists almost entirely of fine rounded grains and comminuted fragments of corals and shells." "The coral-sand rock is, in some places, very hard, but in others soft and easily disintegrated; and was found to consist, when analysed, of carbonate of lime, with traces of phosphoric acid."†

The authorities of the Geological Department of the Museum of Natural History report that: "The matrix in which the remains" (Meiolanian) "forwarded by Dr. Fitzgerald were embedded corresponds exactly with the description given by Mr. H. Wilkinson." "It consists of the minutely comminuted and semi-rounded remains of coral fragments, shells, and spines and plates of Echini, with occasional grains of volcanic substances, such as augite, magnetite, and altered lava-fragments. It appears to be in a great measure simply an agglomerate, perhaps of an Æolian‡ nature; for a cementing medium is only partially present. So little cohesion is there between the particles, which are of a tolerably uniform size, that it was with the greatest difficulty slices sufficiently thin for the microscope could be prepared."

Mr. T. Davies, F.G.S., of the "Department of Mineralogy, Natural History Museum," who has kindly examined slides, writes:—"The whole is cemented by a clear crystalline calcite, which does not, however, entirely fill the interstitial spaces, but is sufficient to cement together adjoining grains. The fragments appear to owe their rounded aspect not alone to attrition, but probably partly to the solvent action which yields the crystalline calcite for the partially cementing medium."

"The block of coral-rock in which portions of the remains were entombed contained a number of irregular cavities ranging from less than an inch in size up to as much as six and nine inches in diameter. These cavities were partially, or sometimes wholly, filled with a deep chocolate-brown soil, having all the appearance of the usual rich volcanic soils of the Australian continent. By dissolving the soluble parts in water, a residue of volcanic products was left."

* 'Phil. Trans.,' 1880, vol. 171, Plates 34 and 35.

† "Report," by Mr. Henry Wilkinson, of the Department of Mines, Sydney.

‡ This term signifies "wind" as the moving agent in contradistinction to "aqueous" and "igneous" forces.

Description of the Plates.*

PLATE 29.

Meiolania minor.

Fig. 1. Occipital segment of the skull; back view.
Fig. 2. Occipital segment of the skull; front view.
Fig. 3. Fore end of upper jaw; front view.
Fig. 4. Fore end of upper jaw; under view.
Fig. 5. Fore end of upper jaw; upper view.
Fig. 6. Fore end of upper jaw; back view.

PLATE 30.

Fig. 1. Skull of *Meiolania platyceps*; upper surface, natural size.
Fig. 2. Right horn-core, showing curvature.

PLATE 31.

Meiolania platyceps.

Fig. 1. Under surface of the skull; one-third natural size.
Fig. 2. Upper view of the fore end of the mandible of a larger individual.
Fig. 3. Upper view of three caudal vertebræ.
Fig. 4. Upper view of caudal vertebræ, nearer the tail-end. In both views a portion of the exo-skeleton is removed, to show its thickness and the inclosed endo-skeletal vertebræ.

PLATE 32.

Pelvic Bones of *Meiolania*.

Fig. 1. Outer view, showing the acetabular cavity.
Fig. 2. Inner view.
Fig. 3. Side view.

* All the figures are of the natural size, save where otherwise expressed. The "letters" and "numerals" are explained in the text.

XV. *On Systems of Circles and Spheres.*

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INTRODUCTION.

THIS Memoir is divided into three Parts: Part I. treats of systems of circles in one plane; Part II. treats of systems of circles on the surface of a sphere; and Part III. of systems of spheres; the method of treatment being that indicated in two papers among CLIFFORD'S 'Mathematical Papers,' viz., "On Power-Coordinates" (pp. 546–555) and "On the Powers of Spheres" (pp. 332–336). These two papers probably contain the notes of a paper which was read by CLIFFORD to the London Mathematical Society, Feb. 27, 1868, "On Circles and Spheres," which was not published ('Lond. Math. Soc. Proc.,' vol. 2, p. 61). The method of treatment indicated in these papers of CLIFFORD'S was successfully applied by the author to prove some theorems given by him in a paper "On the Properties of a Triangle formed by Coplanar Circles" (1885) ('Quarterly Journal of Mathematics,' vol. 21), and then to the extension of those theorems to the case of spheres. But as CLIFFORD'S papers contained some suggestions as to the application of the same method to the treatment of Bi-circular Quartics, he was induced to develop these ideas and extend the results to the case of the analogous curves on spheres—called by Professor CAYLEY Spheri-quadrics—and also of cyclides. It is impossible to say whether, if at all, CLIFFORD was indebted to DARBOUX for any of the ideas contained in the two papers cited above; but it is noticeable that they coincide in a great measure with those expressed by DARBOUX in several papers published during the years 1869–1872.

In Part I. (§§ 1–124) of this Memoir a general relation is first shown to subsist between the powers of any two groups of five circles; the definition of the power of two circles, as the extension of STEINER'S "power of a point and a circle," being due to DARBOUX, but the definition is here slightly modified so as to include the case when the radius of either (or each) circle is infinite. In Chapter II. an extension of the definition so as to apply to a certain system of conics is given; this is practically adapted from Chapter II. in Professor CASEY'S Memoir "On Bicircular Quartics"

(1867) ('Irish Acad. Trans.,' vol. 24). In Chapter III. the general theorem is applied to several interesting cases of circles; some of the results of this chapter are believed to be new. In Chapter IV. the problem of drawing a circle to cut three given circles at given angles is considered, and the circles connected with a triangle formed by three circles, which are analogous to the circumcircle, the inscribed and escribed, and the nine-points circle of an ordinary triangle are discussed. The results are the same, with one or two exceptions which may be new, as arrived at, but in a different manner, in the paper by the author in the 'Quarterly Journal' (vol. 21). In Chapter V. the power-coordinates of a point (or circle) are defined, and the equations of circles, &c., discussed; and it is shown that there are two simple coordinate systems of reference; one consisting of four orthogonal circles, mentioned by CLIFFORD (CASEY and DARBOUX consider five orthogonal spheres), the other consisting of two orthogonal circles and their two points of intersection, which seems to have been indicated for the first time by Mr. HOMERSHAM COX in a paper "On Systems of Circles and Bicircular Quartics" ('Quarterly Journal,' vol. 19, 1883). In Chapter VI. the general equation of the second degree in power-coordinates is discussed, and in Chapter VII. Bi-circular Quartics are classified according to the number of principal circles which they possess. In Chapter VIII. the connexion between Bi-circular Quartics and their focal conics is briefly indicated, the circle of curvature is found, and an expression for the radius of curvature at any point of a bi-circular quartic is investigated. In these last three chapters the results are probably all old, but as the method employed is different from any previously used to discuss these curves in detail, it may not be without interest.

In Part II. (§§ 125–198) almost all the results given in Part I. are extended, with occasionally some slight modifications, to the case of small circles on a sphere and spheri-quadrics.

In Part III. (§§ 199–287) the same order is followed as in Part I.; most of the results in Chapter III., Part I., are extended to the analogous systems of spheres. In Chapter III., however, it is shown that though there is a group of spheres corresponding to the circum-sphere of a tetrahedron, and though several analogous theorems are true for what correspond to the inscribed and escribed spheres, yet there is no analogy to FEUERBACH'S theorem. Chapter IV. corresponds exactly to Chapter V. in Part I., and in Chapter V. the general equation of the second degree in power-coordinates is shown to represent a cyclide, and the equation is discussed in the same manner as in Part I., Chapter VI. The reduction, however, of the general equation to its simplest form presents some difficulty. In Chapter VI. cyclides are briefly classified in the order in which they present themselves in reducing the general equation, and in Chapter VII. a few miscellaneous propositions are discussed, as, for instance, the determination of the locus of the centres of the bitangent spheres, *i.e.*, the Focal Quadrics.

It may be convenient to state here the Memoirs consulted :—

CASEY.

"On Bicircular Quartics" (1867), 'Irish Acad. Trans.,' vol. 24.

"On Cyclides and Sphero-Quartics" (1871), 'Phil. Trans.,' vol. 161, pp. 585–721.

CLIFFORD.*

"On the Powers of Spheres" (1868), 'Mathematical Papers.' 1882. Pp. 332–336.

"Of Power-Coordinates in general" (1866), 'Mathematical Papers.' 1882. (Appendix.) Pp. 546–555.

COX, H. "On Systems of Circles and Bicircular Quartics," 'Quart. Journ. Math.,' vol. 19, 1883, pp. 74–124.

DARBOUX.

"Sur les Relations entre les groupes de Points, de Cercles et de Sphères dans le plan et dans l'espace," 'Annales de l'École Normale Supérieure,' vol. 1, 1872, pp. 323–392.

'Sur une Classe remarquable de Courbes et de Surfaces Algébriques.' Paris, 1873.

SALMON.

'Higher Plane Curves.' 3rd edition, 1879, pp. 240–253.

'Geometry of Three Dimensions.' 4th edition, 1882, pp. 527–536.

PART I.—SYSTEMS OF CIRCLES IN ONE PLANE.

CHAPTER I.—GENERAL SYSTEMS OF CIRCLES.

Definitions. §§ 1–5.

1. The *power* of two circles (or of one circle with respect to the other) is the square of the distance between the centres of the circles, less the sum of the squares of their radii.

Thus denoting the power of two circles whose radii are r_1, r_2 by $\pi_{1,2}$: if $d_{1,2}$ be the distance between their centres, we have

$$\pi_{1,2}=d^2_{1,2}-r_1^2-r_2^2,$$

or if $\omega_{1,2}$ be the angle at which the circles intersect we have

$$\pi_{1,2}=2r_1r_2\cos\omega_{1,2}.$$

2. If one of the circles reduces to a point, then the power becomes equal to the square of the tangent from the point to the circle. In this case the definition agrees with STEINER'S definition of the power of a point with respect to a circle ('CRELLE, Journ. Math.,' vol. 1, 1826, p. 164). The use of the word power is of great antiquity—the area

[* The probable date of these papers is given as 1866 and 1868 respectively. *Cf.* Preface to 'Math. Papers,' pp. xxi, xxii; and also note on page 332.—October, 1886.]

of the parallelogram formed by joining the points, in which two parallel tangents to a hyperbola meet the asymptotes, was called the "Power of the Hyperbola"—and the name was borrowed by STEINER, in the paper quoted above, which was written in 1826, to express the constant rectangle of the segments of any chord of a circle through a point, and this rectangle he called the power of the point with respect to the circle. STEINER also extended his definition thus: if O be one of the centres of similitude of two circles whose centres are A, B: and if a chord through O cut the circles in P and Q respectively, but so that AP, BQ are not parallel, then he proposed to call the rectangle OP, OQ the power of the two circles with respect to O.

DARBOUX seems to have been the first to give the definition of the power of two circles, as used in this memoir, in a paper written in 1872 and published in the 'Annales de l'École Normale Supérieure,' vol. 1. CLIFFORD also gives the same definition in a paper, the probable date of which is said to be 1866, given in the Appendix to his 'Mathematical Papers' (1882); but the paper itself does not seem to have ever been published.

Mr. HOMERSHAM COX, in a paper published in the 'Quarterly Journal of Mathematics' (vol. 19, 1883), has shown that the power of two circles may also be defined as the product of two circles.

3. If the equations of two circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

we have

$$\pi_{1,2}=c_1+c_2-2g_1g_2-2f_1f_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

4. It will be convenient to define the power of a straight line and a circle as twice the perpendicular from the centre of the circle on the line; and the power of two straight lines as twice the cosine of the angle between them.*

Thus the power of (1) and the straight line

$$-2x\cos\alpha-2y\sin\alpha+2p=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

is given by

$$\pi=2p+2g_1\cos\alpha+2f_1\sin\alpha. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus regarding (4) as a degenerate form of (2), (5) may be considered as a particular case of (3).

[* We can easily show that these definitions are included in that given in § 1. Thus, considering a straight line as a circle of infinite radius, R say, the power of a circle, radius r, with respect to it $=(p+R)^2-r^2-R^2=2pR$, in the limit, p being the perpendicular from the centre of the circle on the straight line. Similarly the power of two straight lines, inclined at an angle ω, $=2R^2\cos\omega$. Consequently, as we are going to deduce our results from a certain symmetrical determinant, we may ignore these factors R, R^2, and define these powers as in § 4.—22nd October, 1886.]

Again, the power of (4) and

$$-2x\cos\beta-2y\sin\beta+2q=0,$$

is given by

$$\pi=-2\cos\beta(\alpha-\beta)=-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta;$$

which may also be considered as a particular case of (3).

The equation to the line at infinity may be written

$$0x+0y+1=0.$$

Hence, denoting this line by the symbol θ, we shall have

$\pi_{\theta,x}=1$, if x denote any circle, or point,

and

$\pi_{\theta,x}=0$, if x denote any straight line,

and

$\pi_{\theta,\theta}=0$, of course.

5. It will be convenient to observe here, that if π' denote the power of the two circles which are respectively the inverse curves of (1) and (2) with respect to any circle, whose centre is the origin O and radius R; then

$$\pi'=\frac{R^4}{c_1c_2}\pi;$$

i.e., denoting the circles by S_1, S_2, and the circles inverse to them by S'_1, S'_2, since $c_1=\pi_{0,s_1}$, $c_2=\pi_{0,s_2}$,

$$\frac{\pi_{s'_1,s'_2}}{\sqrt{\pi_{0,s'_1}\pi_{0,s'_2}}}=\frac{\pi_{s_1,s_2}}{\sqrt{\pi_{0,s_1}\pi_{0,s_2}}},$$

and the formula is still true if either or both circles degenerate into straight lines.

Thus if x, y denote any circles, straight lines, or points, the expresssion

$$\frac{\pi_{x,y}}{\sqrt{\pi_{0,x}\pi_{0,y}}}$$

is unaltered if the circles be inverted with respect to any circle whose centre is O.

General Theorems. §§ 6–8.

6. If we have given a system of five circles (1, 2, 3, 4, 5), their powers with respect to any five other circles (6, 7, 8, 9, 10) are connected by identical relation, which may be expressed in the umbral notation by

$$\Pi\begin{pmatrix}1,2,3,4,5\\6,7,8,9,10\end{pmatrix}=0.$$

The word "circle" being intended to include a point, a straight line, or the line at infinity.

This is easily proved by multiplying together the two matrices

$$\begin{vmatrix} 1, & 2g_1, & 2f_1, & c_1 \\ 1, & 2g_2, & 2f_2, & c_2 \\ 1, & 2g_3, & 2f_3, & c_3 \\ 1, & 2g_4, & 2f_4, & c_4 \\ 1, & 2g_5, & 2f_5, & c_5 \end{vmatrix} \quad \begin{vmatrix} c_6, & -g_6, & -f_6, & 1 \\ c_7, & -g_7, & -f_7, & 1 \\ c_8, & -g_8, & -f_8, & 1 \\ c_9, & -g_9, & -f_9, & 1 \\ c_{10}, & -g_{10}, & -f_{10}, & 1 \end{vmatrix}$$

and we obtain at once the equation

$$\begin{vmatrix} \pi_{1,6}, & \pi_{1,7}, & \pi_{1,8}, & \pi_{1,9}, & \pi_{1,10} \\ \pi_{2,6}, & \pi_{2,7}, & \pi_{2,8}, & \pi_{2,9}, & \pi_{2,10} \\ \pi_{3,6}, & \pi_{3,7}, & \pi_{3,8}, & \pi_{3,9}, & \pi_{3,10} \\ \pi_{4,6}, & \pi_{4,7}, & \pi_{4,8}, & \pi_{4,9}, & \pi_{4,10} \\ \pi_{5,6}, & \pi_{5,7}, & \pi_{5,8}, & \pi_{5,9}, & \pi_{5,10} \end{vmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

which may be conveniently written

$$\Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 6, 7, 8, 9, 10 \end{pmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

7. An important particular case is when θ is a member of both systems: then we have

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4 \\ \theta, 5, 6, 7, 8 \end{pmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

or

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & \pi_{1,5}, & \pi_{1,6}, & \pi_{1,7}, & \pi_{1,8} \\ 1, & \pi_{2,5}, & \pi_{2,6}, & \pi_{2,7}, & \pi_{2,8} \\ 1, & \pi_{3,5}, & \pi_{3,6}, & \pi_{3,7}, & \pi_{3,8} \\ 1, & \pi_{4,5}, & \pi_{4,6}, & \pi_{4,7}, & \pi_{4,8} \end{vmatrix} = 0.$$

Whence, denoting the angle of intersection of the two circles (x, y) by $\omega_{x,y}$, we have, provided none of the circles reduce to points,

$$\begin{vmatrix} 0, & \frac{1}{r_5}, & \frac{1}{r_6}, & \frac{1}{r_7}, & \frac{1}{r_8}, \\ \frac{1}{r_1}, & \cos\omega_{1,5}, & \cos\omega_{1,6}, & \cos\omega_{1,7}, & \cos\omega_{1,8} \\ \frac{1}{r_2}, & \cos\omega_{2,5}, & \cos\omega_{2,6}, & \cos\omega_{2,7}, & \cos\omega_{2,8} \\ \frac{1}{r_3}, & \cos\omega_{3,5}, & \cos\omega_{3,6}, & \cos\omega_{3,7}, & \cos\omega_{3,8} \\ \frac{1}{r_4}, & \cos\omega_{4,5}, & \cos\omega_{4,6}, & \cos\omega_{4,7}, & \cos\omega_{4,8} \end{vmatrix} = 0. \quad . \quad . \quad (9)$$

This is true if any of the circles are replaced by straight lines.

If we take the two systems (1, 2, 3, 4), (5, 6, 7, 8) as coincident, we have an equation which gives us the radii of the two circles which cut three given circles at given angles.

Thus, if three given circles, radii r_1, r_2, r_3 cut at angles α, β, γ, the radii of the circles which cut them at angles ϕ_1, ϕ_2, ϕ_3 respectively, are the roots of the equation

$$\begin{vmatrix} 0, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{\rho}, \\ \frac{1}{r_1}, & -1, & \cos\gamma, & \cos\beta, & \cos\phi_1 \\ \frac{1}{r_2}, & \cos\gamma, & -1, & \cos\alpha, & \cos\phi_2 \\ \frac{1}{r_3}, & \cos\beta, & \cos\alpha, & -1, & \cos\phi_3 \\ \frac{1}{\rho}, & \cos\phi_1, & \cos\phi_2, & \cos\phi_3, & -1 \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad (10)$$

8. Another important theorem is easily deduced by the method employed in § 6, thus

$$\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} = \begin{vmatrix} 1, & 2g_1, & 2f_1, & c_1 \\ 1, & 2g_2, & 2f_2, & c_2 \\ 1, & 2g_3, & 2f_3, & c_3 \\ 1, & 2g_4, & 2f_4, & c_4 \end{vmatrix} \times \begin{vmatrix} c_5, & -g_5, & -f_5, & 1 \\ c_6, & -g_6, & -f_6, & 1 \\ c_7, & -g_7, & -f_7, & 1 \\ c_8, & -g_8, & -f_8, & 1 \end{vmatrix};$$

hence we have at once

$$\left\{\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix}\right\}^2 = \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} \times \Pi\begin{pmatrix} 5, 6, 7, 8 \\ 5, 6, 7, 8 \end{pmatrix} \quad . \quad . \quad . \quad . \quad . \quad (11)$$

CHAPTER II.—EXTENSION OF RESULTS OF CHAPTER I.

Preliminary Remarks.—§§ 9–12.

9. Dr. CASEY has shown in his memoir "On Bicircular Quartics,' that any two conics whose equations can be put in the form $S-L^2=0$, and $S-M^2=0$, possess a pair of angles which he calls their anharmonic angles, and which he shows to be analogous to the angle of intersection of two circles. Thus, if S′, S″ denote the results of substituting the coordinates of the poles of L, M respectively in S, and R

the result of substituting the coordinates of the pole of L in M, then the anharmonic angles θ, ϕ of the two conics are defined by :—

$$1-R=\sqrt{(1-S')(1-S'')}.\cos\theta,$$
$$1+R=\sqrt{(1-S')(1-S'')}.\cos\phi.$$

A proof of a theorem similar to that given in § 6 of the present memoir is to be found in SALMON's 'Conics,' p. 366.

10. Let us take as the equations of the three conics

$$S\equiv x^2+y^2+z^2=0,$$
$$a^2(x^2+y^2+z^2)-(xf'+yg+zh)^2=0,$$
$$a'^2(x^2+y^2+z^2)-(xf''+yg'+zh')^2=0.$$

Then following the method used in CASEY's paper (§ 126), we form the discriminant of

$$aS^{\frac{1}{2}}\pm(xf+yg+zh)+\lambda\{a'S^{\frac{1}{2}}\pm(xf''+yg'+zh')\}=0;$$

and we obtain

$$(a'^2-S')\lambda^2+2(aa'-R)\lambda+a^2-S=0;$$

and so, if we take

$$aa'-R=\sqrt{(a'^2-S')(a^2-S)}.\cos\theta,$$

where $R=ff''+gg'+hh'$, $S=f^2+g^2+h^2$, $S'=f''^2+g'^2+h'^2$; we have for the tact-invariant of the conics a^2S-L^2, a'^2S-M^2, $(a^2-S)(a'^2-S')\sin^2\theta=0$; or $\theta=0$.

11. Again, forming the discriminant of

$$aS^{\frac{1}{2}}\pm(xf+yg+zh)+\lambda\{a'S\mp(xf''+yg'+zh'\}=0,$$

we obtain for the tact-invariant, $\phi=0$, where

$$aa'+R=\sqrt{(a'^2-S')(a^2-S)}\cos\phi.$$

12. Either of these expressions, $aa'\pm R$, might be defined as the power of the two conics a^2S-L^2, a'^2S-M^2. For if $\theta=\frac{\pi}{2}$, it is evident, as CASEY has shown, that the pencil formed by the lines L, M, and the chords of contact of the two line pairs which can be drawn to touch S from the points of intersection of a'^2L-a^2M with a^2S-L^2, is harmonic; and so $\theta=\frac{\pi}{2}$ is the condition corresponding to the case of two circles cutting orthogonally. In this case the power vanishes.

Let us define, then, the power of the two conics a^2S-L, and a'^2S-M^2, as the expression $aa'+R$, and let us denote this by ϖ. Thus

$$\varpi=aa'+ff'+gg'+hh';$$

the conic S being supposed reduced to its standard form, and f, g, h being the co-ordinates of the pole of L with respect to S.

General Theorem.—§§ 13–15.

13. If we have any two systems of conics, say (1, 2, 3, 4, 5), (6, 7, 8, 9, 10), inscribed in the same conic S, the powers of the conics of one system with respect to the conics of the other system are connected by the relation

$$\Pi\begin{pmatrix}1,2,3,4,5\\6,7,8,9,10\end{pmatrix}=0, \quad \ldots\ldots\ldots\ldots \quad (12)$$

i.e., the same equation as in § 6.

Thus taking the same equations as in § 10, by multiplying the matrices

$$\begin{vmatrix} a_1, & f_1, & g_1, & h_1 \\ a_2, & f_2, & g_2, & h_2 \\ a_3, & f_3, & g_3, & h_3 \\ a_4, & f_4, & g_4, & h_4 \\ a_5, & f_5, & g_5, & h_5 \end{vmatrix} \quad \begin{vmatrix} a_6, & f_6, & g_6, & h_6 \\ a_7, & f_7, & g_7, & h_7 \\ a_8, & f_8, & g_8, & h_8 \\ a_9, & f_9, & g_9, & h_9 \\ a_{10}, & f_{10}, & g_{10}, & h_{10} \end{vmatrix},$$

we have at once the relation

$$\begin{vmatrix} \varpi_{1,6}, & \varpi_{1,7}, & \varpi_{1,8}, & \varpi_{1,9}, & \varpi_{1,10} \\ \varpi_{2,6}, & \varpi_{2,7}, & \varpi_{2,8}, & \varpi_{2,9}, & \varpi_{2,10} \\ \varpi_{3,6}, & \varpi_{3,7}, & \varpi_{3,8}, & \varpi_{3,9}, & \varpi_{3,10} \\ \varpi_{4,6}, & \varpi_{4,7}, & \varpi_{4,8}, & \varpi_{4,9}, & \varpi_{4,10} \\ \varpi_{5,6}, & \varpi_{5,7}, & \varpi_{5,8}, & \varpi_{5,9}, & \varpi_{5,10} \end{vmatrix} =0. \quad \ldots\ldots \quad (13)$$

14. This equation has been proved for two systems of conics inscribed in the same conic. The result is also true if any of the conics be replaced by straight lines, provided that we define the power of a straight line and a conic of the system to be the power of the straight line and the chord of contact of the conic and the conic S; the power of two straight lines being defined as the perpendicular distance from the pole of one line with respect to S to the other line. Thus let any conic of the system be

$$u\equiv a^2S-(fx+gy+hz)^2=0,$$

and any straight line

$$\alpha \equiv lx+my+nz=0;$$

then the power of the line and conic is $lf+mg+nh$, and the power of this straight line and the line

$$\alpha' \equiv l'x+m'y+n'z=0,$$

is $ll'+mm'+nn'$.

Again, we must define the power of S and u to be a, and the power of S and α to be zero; the power of S with respect to S to be unity.

15. Exactly as in § 8 we can show that for any two systems of four conics inscribed in S,

$$\left\{\Pi\begin{pmatrix}1,&2,&3,&4\\5,&6,&7,&8\end{pmatrix}\right\}^2=\left\{\Pi\begin{pmatrix}1,&2,&3,&4\\1,&2,&3,&4\end{pmatrix}\right\}\times\left\{\Pi\begin{pmatrix}5,&6,&7,&8\\5,&6,&7,&8\end{pmatrix}\right\}.$$

Chapter III.—Special Systems of Circles.

Circles Touching three Straight Lines.—§§ 16, 17.

16. Denoting the four circles which touch the sides of a triangle by (1, 2, 3, 4), and the sides of the triangle by a, b, c, we have, if x denote any other circle,

$$\Pi\begin{pmatrix}\theta,&a,&b,&c,&x\\\theta,&1,&2,&3,&4\end{pmatrix}=0,$$

i.e.,

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 0, & \pi_{a,1}, & \pi_{a,2}, & \pi_{a,3}, & \pi_{a,4} \\ 0, & \pi_{b,1}, & \pi_{b,2}, & \pi_{b,3}, & \pi_{b,4} \\ 0, & \pi_{c,1}, & \pi_{c,2}, & \pi_{c,3}, & \pi_{c,4} \\ 1, & \pi_{x,1}, & \pi_{x,2}, & \pi_{x,3}, & \pi_{x,4} \end{vmatrix} = 0;$$

and since

$$\pi_{a,1}=r_1,\ \pi_{a,2}=-r_2,\ \&c.,$$

we have at once

$$\frac{1}{r_1}=\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}.$$

This theorem will be subsequently extended.

17. Again let x denote the nine-points circle of the triangle, z the inscribed circle, and let (1, 2, 3) denote the mid-points of the sides; then since

$$\pi_{x,1}=\pi_{x,2}=\pi_{x,3}=0;$$

$$\pi_{z,1}=(b-c)^2;\quad \pi_{z,2}=(c-a)^2;\quad \pi_{z,3}=(a-b)^2;$$

the equation

$$\Pi\begin{pmatrix} x, z, 1, 2, 3 \\ x, z, 1, 2, 3 \end{pmatrix}=0,$$

gives us

$$\begin{vmatrix} \pi_{x,x}, & \pi_{x,z}, & 0, & 0, & 0 \\ \pi_{z,x}, & \pi_{z,z}, & (b-c)^2, & (c-a)^2, & (a-b)^2 \\ 0, & (b-c)^2, & 0, & \frac{c^2}{4}, & \frac{b^2}{4} \\ 0, & (c-a)^2, & \frac{c^2}{4}, & 0, & \frac{a^2}{4} \\ 0, & (a-b)^2, & \frac{b^2}{4}, & \frac{a^2}{4}, & 0 \end{vmatrix} =0\,;$$

whence we have

$$\pi_{x,x}\pi_{z,z}=\pi^2_{x,z},$$

or the circle which passes through the mid-points of the sides of a triangle touches the inscribed circle. Similarly it can be proved to touch the escribed circles.

Circle Cutting three Circles Orthogonally.—§§ 18–20.

18. Let the circle cutting the system (1, 2, 3) orthogonally be denoted by (x): then since

$$\Pi\begin{pmatrix} \theta, x, 1, 2, 3 \\ \theta, x, 1, 2, 3 \end{pmatrix}=0\,;$$

we have

$$\pi_{x,x}.\Pi\begin{pmatrix} \theta, 1, 2, 3 \\ \theta, 1, 2, 3 \end{pmatrix}=\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} \quad \ldots\ldots\ldots \quad (14)$$

But

$$\Pi\begin{pmatrix} \theta, 1, 2, 3 \\ \theta, 1, 2, 3 \end{pmatrix}= \begin{vmatrix} 0, & 1, & 1, & 1 \\ 1, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3} \\ 1, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3} \\ 1, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3} \end{vmatrix}\,;$$

and if the equations of the circles be

$$x^2+y^2+2g_r x+2f_r y+c_r=0,$$

where $r=(1, 2, 3)$;

we have at once

$$\Pi\begin{pmatrix}\theta, 1, 2, 3\\ \theta, 1, 2, 3\end{pmatrix}=\begin{vmatrix} 0, & 0, & 0, & 1\\ 1, & 2g_1, & 2f_1, & c_1\\ 1, & 2g_2, & 2f_2, & c_2\\ 1, & 2g_3, & 2f_3, & c_3\end{vmatrix}\times\begin{vmatrix} 1, & 0, & 0, & 0\\ c_1, & -g_1, & -f_1, & 1\\ c_2, & -g_2, & -f_2, & 1\\ c_3, & -g_3, & -f_3, & 1\end{vmatrix}$$

$$=-16\{\Delta(1, 2, 3)\}^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where $\Delta(1, 2, 3)$ denotes the area of the triangle formed by the centres of the circles $(1, 2, 3)$.

Hence by (14) we see that, if r denote the radius of the circle which cuts $(1, 2, 3)$ orthogonally,

$$r^2=\frac{\Pi\begin{pmatrix}1, 2, 3\\ 1, 2, 3\end{pmatrix}}{32\{\Delta(1, 2, 3)\}^3}=\frac{r_1^2r_2^2r_3^2}{4\{\Delta(1, 2, 3)\}^2}\times\begin{vmatrix} -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}\\ \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}\\ \cos\omega_{1,3}, & \cos\omega_{2,3}, & -1\end{vmatrix};$$

or

$$r^2=\frac{r_1^2r_2^2r_3^2}{\{\Delta(1, 2, 3)\}^2}\cos s.\cos(s-\omega_{2,3}).\cos(s-\omega_{3,1}).\cos(s-\omega_{1,2}); \quad . \quad . \quad (16)$$

$\omega_{2,3}$, $\omega_{3,1}$, $\omega_{1,2}$ being the angles of intersection of the circles $(1, 2, 3)$, and $2s$ being equal to $\omega_{2,3}+\omega_{3,1}+\omega_{1,2}$.

19. Since any point on a circle may be considered as a circle of infinitely small radius cutting the circle orthogonally, it follows that if three circles meet in a point, the radius of their orthogonal circle must vanish. Hence the condition that three circles meet in a point is

$$\Pi\begin{pmatrix}1, 2, 3\\ 1, 2, 3\end{pmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

20. The radius of the orthogonal circle of the system $(1, 2, 3)$ is infinite when $\Delta(1, 2, 3)=0$, *i.e.*, when the centres of the circles lie on a straight line; in which case the orthogonal circle degenerates into the straight line through their centres.

Four circles having a Common Orthogonal Circle.—§§ 21–24.

21. Suppose that the system $(1, 2, 3, 4)$ has a common orthogonal circle, x say; then since

$$\Pi\begin{pmatrix}x, 1, 2, 3, 4\\ y, 1, 2, 3, 4\end{pmatrix}=0;$$

we have

$$\begin{vmatrix} \pi_{x,y}, & \pi_{y,1}, & \pi_{y,2}, & \pi_{y,3}, & \pi_{y,4} \\ 0, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4} \\ 0, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4} \\ 0, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4} \\ 0, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4} \end{vmatrix} = 0 ;$$

whence

$$\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which is the condition that the system has a common orthogonal circle.

22. It is evident also that (5, 6, 7, 8) being any other system of circles, we must have, since

$$\Pi\begin{pmatrix} x, 5, 6, 7, 8 \\ y, 1, 2, 3, 4 \end{pmatrix} = 0,$$

$$\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

It follows by symmetry that this result must be true if either of the two systems (1, 2, 3, 4) or (5, 6, 7, 8) have a common orthogonal circle.

Hence we must have

$$\left\{\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix}\right\}^2 = \Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} \times \Pi\begin{pmatrix} 5, 6, 7, 8 \\ 5, 6, 7, 8 \end{pmatrix}.$$

23. We may notice that any three circles, whose centres lie on a straight line, may be considered as having, with the line at infinity, a common orthogonal circle.

24. The system (1, 2, 3, 4) having a common orthogonal circle, and (5, 6, 7, 8) being any other system of circles, we have

$$\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} = 0 ;$$

and, as in § 8, we may prove that

$$\left\{\Pi\begin{pmatrix} 1, 2, 3 \\ 5, 6, 7 \end{pmatrix}\right\}^2 = \Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} \times \Pi\begin{pmatrix} 5, 6, 7 \\ 5, 6, 7 \end{pmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

As a particular case, we have

$$\Pi\begin{pmatrix} x, 1, 2, 3 \\ 1, 2, 3, 4 \end{pmatrix} = 0 ;$$

whence

$$\pi_{x,1}.\Pi\begin{pmatrix} 1, 2, 3 \\ 2, 3, 4 \end{pmatrix} + \pi_{x,2}.\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 4, 3 \end{pmatrix} + \pi_{x,3}.\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 4 \end{pmatrix} - \pi_{x,4}.\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} = 0.$$

Hence by (20)

$$\pi_{x,4}\left\{\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}\right\}^{\frac{1}{2}}=\pi_{x,1}\left\{\Pi\begin{pmatrix}2,3,4\\2,3,4\end{pmatrix}\right\}^{\frac{1}{2}}+\pi_{x,2}\left\{\Pi\begin{pmatrix}1,4,3\\1,4,3\end{pmatrix}\right\}^{\frac{1}{2}}+\pi_{x,3}\left\{\Pi\begin{pmatrix}1,2,4\\1,2,4\end{pmatrix}\right\}^{\frac{1}{2}}.$$

But if r be the radius of the common orthogonal circle, we have by (16),

$$r^2=\frac{\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}}{32\{\Delta(1,2,3)\}^2}=\frac{\Pi\begin{pmatrix}1,2,4\\1,2,4\end{pmatrix}}{32\{\Delta(1,2,4)\}^2}=\&c.$$

Therefore

$$\pi_{x,4}.\Delta(1,2,3)=\pi_{x,1}.\Delta(2,3,4)+\pi_{x,2}.\Delta(1,4,3)+\pi_{x,3}.\Delta(1,2,4). \quad . \quad . \quad (21)$$

Thus if any four circles have a common orthogonal circle, and the areal coordinates of the centre of one of them referred to the triangle formed by joining the centres of the other three be α, β, γ: then the powers of any other circle with respect to these four circles are connected by the relation

$$\pi_{x,4}=\alpha.\pi_{x,1}+\beta.\pi_{x,2}+\gamma.\pi_{x,3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

As a particular case we obtain the well-known theorem that if A, B, C be the centres of any three circles, P any point on the circle which cuts these orthogonally, and O any other point, then

$$OP^2=(OA^2-r_1^2)\alpha+(OB^2-r_2^2)\beta+(OC^2-r_3^2)\gamma;$$

where α, β, γ are the areal coordinates of P referred to the triangle ABC.

Orthogonal Systems.—§§ 25–29.

25. Four circles may be said to form an orthogonal system if they cut one another orthogonally: it is clear that the centre of any one of them is the orthocentre of the triangle formed by the other three.

If the system be denoted by (1, 2, 3, 4), then (x, y) being any other circles, the equation

$$\Pi\begin{pmatrix}x,1,2,3,4\\y,1,2,3,4\end{pmatrix}=0$$

becomes

$$\begin{vmatrix}\pi_{x,y}, & \pi_{x,1}, & \pi_{x,2}, & \pi_{x,3}, & \pi_{x,4}\\ \pi_{y,1}, & \pi_{1,1}, & 0, & 0, & 0\\ \pi_{y,2}, & 0, & \pi_{2,2}, & 0, & 0\\ \pi_{y,3}, & 0, & 0, & \pi_{3,3}, & 0\\ \pi_{y,4}, & 0, & 0, & 0, & \pi_{4,4}\end{vmatrix}=0;$$

i.e.,

$$\pi_{x,y}=\frac{\pi_{x,1}\pi_{y,1}}{\pi_{1,1}}+\frac{\pi_{x,2}\pi_{y,2}}{\pi_{2,2}}+\frac{\pi_{x,3}\pi_{y,3}}{\pi_{3,3}}+\frac{\pi_{x,4}\pi_{y,4}}{\pi_{4,4}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

As particular cases, we have, since $\pi_{1,1}=-2r_1^2$, &c.,

$$\frac{\pi_{x,1}}{r_1^2}+\frac{\pi_{x,2}}{r_2^2}+\frac{\pi_{x,3}}{r_3^2}+\frac{\pi_{x,4}}{r_4^2}=-2.\pi_{x,\theta}=-2 \text{ or } 0; \quad . \quad . \quad . \quad . \quad . \quad (24)$$

according as x denotes a circle or a straight line.

If x represents a circle, radius r_x, we have

$$4r_x^2=\frac{\pi_{x,1}^2}{r_1^2}+\frac{\pi_{x,2}^2}{r_2^2}+\frac{\pi_{x,3}^2}{r_3^2}+\frac{\pi_{x,4}^2}{r_4^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25)$$

And if x represent a straight line, so that $\pi_{x,x}=-2$, we have

$$4=\frac{\pi_{x,1}^2}{r_1^2}+\frac{\pi_{x,2}^2}{r_2^2}+\frac{\pi_{x,3}^2}{r_3^2}+\frac{\pi_{x,4}^2}{r_4^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Again, since $\pi_{\theta,\theta}=0$, we have

$$0=\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

whence it appears that one at least of the three circles must be imaginary, and one at least real.

26. By equation (16) we have

$$r_4^2=-\frac{r_1^2r_2^2r_3^2}{4\{\Delta(1,2,3)\}^2}$$

Hence

$$4\{\Delta(1,2,3)\}^2=r_2^2r_3^2+r_3^2r_1^2+r_1^2r_2^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and we can easily find that, if ρ be the radius of the nine-points circle of the triangle formed by the centres of three of the circles,

$$(r_1^2+r_2^2)(r_1^2+r_3^2)(r_1^2+r_4^2)(r_2^2+r_3^2)(r_2^2+r_4^2)(r_3^2+r_4^2)+256.\rho^4r_1^2r_2^2r_3^2r_4^2=0. \quad (29)$$

27. If the circles (1, 2, 3, 4) be any system not having a common orthogonal circle, we may find four other circles, (5, 6, 7, 8) say, each of which is orthogonal to three of the former. Two such systems are connected by several interesting formulæ, and one system may be called the "orthogonal system" of the other.

Thus, x and y denoting any two circles, we have, since

$$\Pi\begin{pmatrix}x, 1, 2, 3, 4\\ y, 5, 6, 7, 8\end{pmatrix}=0,$$

the relation

$$\begin{vmatrix} \pi_{x,y}, & \pi_{x,5}, & \pi_{x,6}, & \pi_{x,7}, & \pi_{x,8} \\ \pi_{y,1}, & \pi_{1,5}, & 0, & 0, & 0 \\ \pi_{y,2}, & 0, & \pi_{2,6}, & 0, & 0 \\ \pi_{y,3}, & 0, & 0, & \pi_{3,7}, & 0 \\ \pi_{y,4}, & 0, & 0, & 0, & \pi_{4,8} \end{vmatrix} = 0 ;$$

which may be written

$$\pi_{x,y} = \frac{\pi_{x,5}\pi_{y,1}}{\pi_{1,5}} + \frac{\pi_{x,6}\pi_{y,2}}{\pi_{2,6}} + \frac{\pi_{x,7}\pi_{y,3}}{\pi_{3,7}} + \frac{\pi_{x,8}\pi_{y,4}}{\pi_{4,8}} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

whence as particular cases, we have, x denoting any circle whose radius is r_x—

$$\frac{\pi_{x,5}}{\pi_{1,5}} + \frac{\pi_{x,6}}{\pi_{2,6}} + \frac{\pi_{x,7}}{\pi_{3,7}} + \frac{\pi_{x,8}}{\pi_{4,8}} = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

$$\frac{\pi_{x,5}\pi_{x,1}}{\pi_{1,5}} + \frac{\pi_{x,6}\pi_{x,2}}{\pi_{2,6}} + \frac{\pi_{x,7}\pi_{x,3}}{\pi_{3,7}} + \frac{\pi_{x,8}\pi_{x,4}}{\pi_{4,8}} = -2r_x^2. \quad . \quad . \quad . \quad . \quad . \quad (32)$$

If x denote any straight line, we have

$$\frac{\pi_{x,5}}{\pi_{1,5}} + \frac{\pi_{x,6}}{\pi_{2,6}} + \frac{\pi_{x,7}}{\pi_{3,7}} + \frac{\pi_{x,8}}{\pi_{4,8}} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

$$\frac{\pi_{x,5}\pi_{x,1}}{\pi_{1,5}} + \frac{\pi_{x,6}\pi_{x,2}}{\pi_{2,6}} + \frac{\pi_{x,7}\pi_{x,3}}{\pi_{3,7}} + \frac{\pi_{x,8}\pi_{x,4}}{\pi_{4,8}} = -2, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

and

$$\frac{1}{\pi_{1,5}} + \frac{1}{\pi_{2,6}} + \frac{1}{\pi_{3,7}} + \frac{1}{\pi_{4,8}} = \pi_{\theta,\theta} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

28. This last result gives us an interesting theorem—in the case when the given system (1, 2, 3, 4) consists of four points; then $\pi_{1,5}$ is equal to the square of the tangent from the point (1) to the circle passing through the points (2, 3, 4), and so on; thus the sum of the reciprocals of the powers of each of four given points with respect to the circle passing through the remaining three is zero. One of these quantities must be negative, so that one of the four points must lie within the corresponding circle.

Also by (31) the sum of the powers of these points with respect to any other circle, divided respectively by their powers with respect to the orthogonal system, is equal to unity. And by (33) the sum of the quotients of the perpendiculars from each point on any straight line, divided by the power of that point with respect to the circle which does not pass through it, is zero.

29. There is another special system of circles which is closely allied to the orthogonal

system discussed in § 25; which is of some importance. Thus, let (1, 2) be any two circles cutting orthogonally, and let (3, 4) be their two points of intersection; the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 1, 2, 3, 4 \end{pmatrix} = 0$$

becomes

$$\begin{vmatrix} \pi_{x,y}, & \pi_{x,1}, & \pi_{x,2}, & \pi_{x,3}, & \pi_{x,4} \\ \pi_{y,1}, & \pi_{1,1}, & 0, & 0, & 0 \\ \pi_{y,2}, & 0, & \pi_{2,2}, & 0, & 0 \\ \pi_{y,3}, & 0, & 0, & 0, & \pi_{3,4} \\ \pi_{y,4}, & 0. & 0, & \pi_{4,3}, & 0 \end{vmatrix} = 0;$$

which may be written, if we put $\pi_{3,4} = e^2$,

$$\pi_{x,y} = \frac{\pi_{x,1}.\pi_{y,1}}{\pi_{1,1}} + \frac{\pi_{x,2}.\pi_{y,2}}{\pi_{2,2}} + \frac{\pi_{x,3}.\pi_{y,4} + \pi_{x,4}.\pi_{y,3}}{e^2}, \quad \ldots \ldots \quad (36)$$

a particular case of which is

$$0 = \frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{4}{e^2}.$$

Circles touching one another.—§§ 30–34.

30. Two circles may be said to touch externally, or internally, according as their angle of intersection $= 0$ or π, *i.e.*, if we denote the circles by x, y, then they touch externally if $\pi_{xy} = +(\pi_{x,x}.\pi_{y,y})^{\frac{1}{2}}$, and internally if $\pi_{xy} = -(\pi_{x,x}.\pi_{y,y})^{\frac{1}{2}}$.

If the four circles (1, 2, 3, 4) touch externally, we have at once, from the equation

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix} = 0;$$

$$\begin{vmatrix} 0, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4} \\ \frac{1}{r_1}, & -1, & 1, & 1, & 1 \\ \frac{1}{r_2}, & 1, & -1, & 1, & 1 \\ \frac{1}{r_3}, & 1, & 1, & -1, & 1 \\ \frac{1}{r_4}, & 1, & 1, & 1, & -1 \end{vmatrix} = 0.$$

Whence

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}=2\left(\frac{1}{r_1r_2}+\frac{1}{r_1r_3}+\frac{1}{r_1r_4}+\frac{1}{r_2r_3}+\frac{1}{r_2r_4}+\frac{1}{r_3r_4}\right); \quad . \quad . \quad . \quad (37)$$

or if r_1, r_2, r_3 be known,

$$\frac{1}{r_4}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\pm 2\left(\frac{1}{r_2r_3}+\frac{1}{r_3r_1}+\frac{1}{r_1r_2}\right)^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

31. This formula is given by STEINER ('CRELLE, Journ. Math.,' vol. 1, 1826), in a paper, in which several interesting cases of series of circles touching one another are discussed: two cases may be noticed here.

Let two circles (1, 2), radii a, c, be described touching each other externally and touching another circle, radius r, internally, whose centre lies on the common diameter of the other two. Now let a circle S_1 be described touching the two former externally and the latter internally, and let a series of circles S_2, S_3, &c., be described, all touching (1, 2) externally, as well as the preceding one in the series. Let their radii be r_1, r_2, &c.

Clearly r_{n-1}, r_{n+1} are the roots of the equation

$$\frac{1}{r_n^2}+\frac{1}{\rho^2}+\frac{1}{a^2}+\frac{1}{c^2}=\frac{2}{\rho}\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{r_n}\right)+\frac{2}{ac}+\frac{2}{r_na}+\frac{2}{r_nc},$$

therefore

$$\frac{1}{r_{n-1}}-\frac{2}{r_n}+\frac{1}{r_{n+1}}=2\left(\frac{1}{a}+\frac{1}{c}\right)=\frac{2r}{ac}.$$

And we easily find that

$$r_n=\frac{acr}{n^2r^2-ac}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39)$$

If, however, S'_1 be drawn touching (1, 2) externally, and also the line joining their centres, we shall have, if S'_n be the n^{th} circle of this second series

$$r'_n=\frac{4acr}{(2n-1)^2r^2-4ac}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

32. If the system of circles (1, 2, 3, 4) have a common tangent circle x, say, the equation

$$\Pi\begin{pmatrix}x, 1, 2, 3, 4\\ x, 1, 2, 3, 4\end{pmatrix}=0$$

may be written

$$\begin{vmatrix} 1, & \sqrt{\pi_{1,1}}, & \sqrt{\pi_{2,2}}, & \sqrt{\pi_{3,3}}, & \sqrt{\pi_{4,4}} \\ \sqrt{\pi_{1,1}}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4} \\ \sqrt{\pi_{2,2}}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4} \\ \sqrt{\pi_{3,3}}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4} \\ \sqrt{\pi_{4,4}}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4} \end{vmatrix}=0; \quad . \quad . \quad . \quad (41)$$

where the positive sign is to be taken with the radicals for external contact, and the negative for internal contact.

Now if $t_{r,s}$ denote the direct common tangent of the circles (r, s) we have

$$t^2_{r,s}=\pi_{r,s}+\sqrt{\pi_{r,r}.\pi_{s,s}};$$

and if $t'_{r,s}$ denote the transverse common tangent, then

$$t'^2_{r,s}=\pi_{r,s}-\sqrt{\pi_{r,r}.\pi_{s,s}}.$$

We can then deduce at once from (41)

$$\begin{vmatrix} 0, & t_{1,2}^2, & t_{1,3}^2, & t_{1,4}^2 \\ t_{2,1}^2, & 0, & t_{2,3}^2, & t_{2,4}^2 \\ t_{3,1}^2, & t_{3,2}^2, & 0, & t_{3,4}^2 \\ t_{4,1}^2, & t_{4,2}^2, & t_{4,3}^2, & 0 \end{vmatrix}=0,$$

or

$$t_{1,2}.t_{3,4}\pm t_{1,3}.t_{4,2}\pm t_{1,4}.t_{2,3}=0;$$

which is Dr. CASEY'S well-known formula.*

33. When the condition in the last article is satisfied, we can find the radius of the tangent circle by equation (31).

Thus, let the system (5, 6, 7, 8) be the system orthogonal to the system (1, 2, 3, 4), then we have

$$\frac{\pi_{x,1}}{\pi_{1,5}}+\frac{\pi_{x,2}}{\pi_{2,6}}+\frac{\pi_{x,3}}{\pi_{3,7}}+\frac{\pi_{x,4}}{\pi_{4,8}}=1.$$

Thus if (x) touch each of the circles externally, we have

$$\frac{1}{r_x}=\frac{2r_1}{\pi_{1,5}}+\frac{2r_2}{\pi_{2,6}}+\frac{2r_3}{\pi_{3,7}}+\frac{2r_4}{\pi_{4,8}}. \quad \ldots\ldots\ldots \quad (42)$$

34. If the system (1, 2, 3, 4) be such that four other circles (5, 6, 7, 8) can be drawn to touch them all, symmetrically: say each of the latter touches three of the former externally and one internally: then the equation

$$\Pi\begin{pmatrix} x, & 1, & 2, & 3, & 4 \\ y, & 5, & 6, & 7, & 8 \end{pmatrix}=0$$

becomes

$$-8\pi_{x,y}=2\Sigma\frac{\pi_{y,1}.\pi_{x,5}}{r_1r_5}-\left(\Sigma\frac{\pi_{y,1}}{r_1}\right)\left(\Sigma\frac{\pi_{x,5}}{r_5}\right);$$

[* If the circle (x) touches the circles (1, 2) in opposite senses, then $t_{1,2}$ must be replaced by $t'_{1,2}$ in this formula.—October, 1886.]

whence as a particular case

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)\left(\frac{1}{r_5}+\frac{1}{r_6}+\frac{1}{r_7}+\frac{1}{r_8}\right)=2\left(\frac{1}{r_1r_5}+\frac{1}{r_2r_6}+\frac{1}{r_3r_7}+\frac{1}{r_4r_8}\right) \quad . \quad . \quad . \quad (43)$$

Chapter IV.—Circles Connected with a Triangle.

The properties of several of these circles have been discussed in detail by the author in a paper published in the 'Quarterly Journal of Mathematics' (vol. 21, 1885). It is only proposed to discuss here some of the more general cases, which can be deduced at once from the general equation in § 6, and which by this method admit of immediate extension. The first case considered, viz., the circles which cut three given circles at given angles, is discussed by Darboux ('Annales de l'École Normale,' vol. 1, 1872). By triangle is meant the general case of a triangle formed by three circles.

Circles cutting Three Given Circles at Given Angles.—§§ 35–38.

35. Let (1, 2, 3) denote any given system of three circles, which cut at angles α, β, γ. If then S be any circle which cuts them at angles θ, ϕ, ψ, we have at once by § 7—denoting the radius by ρ—

$$\begin{vmatrix} 0, & \frac{1}{\rho}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3} \\ \frac{1}{\rho}, & -1, & \cos\theta, & \cos\phi, & \cos\psi \\ \frac{1}{r_1}, & \cos\theta, & -1, & \cos\gamma, & \cos\beta \\ \frac{1}{r_2}, & \cos\phi, & \cos\gamma, & -1, & \cos\alpha \\ \frac{1}{r_3}, & \cos\psi, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad (44)$$

Thus we obtain a quadratic for ρ; and two circles can in general be drawn cutting the given circles at the given angles.

Now let either of these circles cut the orthogonal circle of the system (1, 2, 3), at the angle ω—denoting the orthogonal circle by the symbol 4, and its radius by r; the equation

$$\Pi\begin{pmatrix} S, 4, 1, 2, 3 \\ S, 4, 1, 2, 3 \end{pmatrix}=0$$

becomes

$$\begin{vmatrix} -1, & \cos\omega, & \cos\theta, & \cos\phi, & \cos\psi \\ \cos\omega, & -1, & 0, & 0, & 0 \\ \cos\theta, & 0, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & 0, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & 0, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0 ;$$

or

$$\sin^2\omega \begin{vmatrix} -1, & \cos\gamma, & \cos\beta \\ \cos\gamma, & -1, & \cos\alpha \\ \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = \begin{vmatrix} 0, & \cos\theta, & \cos\phi, & \cos\psi \\ \cos\theta, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} . \quad (45)$$

It thus appears that each of the circles, which can be drawn to cut the system (1, 2, 3) at angles θ, ϕ, ψ, cuts the orthogonal circle at one of the angles ω or $\pi-\omega$. It is otherwise evident that these two circles are such that one is the inverse of the other with respect to the circle which cuts the system (1, 2, 3) orthogonally.

Let us denote the two circles by S, S′, and their radii by ρ, ρ'. We have by

$$\Pi\begin{pmatrix} x, 4, 1, 2, 3 \\ \theta, 4, 1, 2, 3 \end{pmatrix} = 0,$$

the equation

$$\begin{vmatrix} \frac{1}{\rho}, & \frac{1}{r}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, \\ \cos\omega, & -1, & 0, & 0, & 0 \\ \cos\theta, & 0, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & 0, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & 0, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0 ; \quad . \quad . \quad . \quad (46)$$

and we also get a similar equation for ρ'.

It appears, then, that the circle S′, which is the inverse of S, cuts the orthogonal circle at the angle $\pi-\omega$. It may happen, however, that the roots of equation (44) are of opposite sign ; in this case, the circle S′ evidently cuts the given system (1, 2, 3) at angles $\pi-\theta, \pi-\phi, \pi-\psi$, and the circle orthogonal to these at the angle ω.

ρ and ρ' being the roots of (44), we have at once by (46)

$$\frac{1}{\rho}-\frac{1}{\rho'}+\frac{2\cos\omega}{r}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47)$$

i.e.,

$$\frac{1}{\pi_{S,4}}+\frac{1}{\pi_{S',4}}=-\frac{1}{r^2}=\frac{2}{\pi_{4,4}}.$$

Hence the two circles S, S′ are real, coincident, or imaginary, according as $\frac{\cos^2\omega}{r^2}$ is positive, zero, or negative.

But by § 18 the sign of r^2 is the same as the sign of

$$\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix} \quad \text{or} \quad \cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma-1.$$

Hence, by (45), S, S′ are real, coincident, or imaginary, according as

$$\begin{vmatrix} -1, & \cos\theta, & \cos\phi, & \cos\psi \\ \cos\theta, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix}$$

is negative, zero, or positive, *i.e.*, according as

$$\Pi\begin{pmatrix}S,1,2,3\\S,1,2,3\end{pmatrix}$$

is negative, zero, or positive.

36. It is clear that four pairs of circles can be drawn to cut the given system (1, 2, 3) at angles equal to θ, ϕ, ψ, or their supplements. The radii of these eight circles are connected by a remarkable relation. Let them be denoted by ρ, ρ'; ρ_1, ρ'_1; ρ_2, ρ'_2; ρ_3, ρ'_3. By equation (46) we have

$$\begin{vmatrix} \frac{1}{\rho}+\frac{1}{\rho'}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3} \\ \cos\theta, & -1, & \cos\gamma, & \cos\beta \\ \cos\phi, & \cos\gamma, & -1, & \cos\alpha \\ \cos\psi, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0,$$

or

$$\frac{1}{\rho}+\frac{1}{\rho'}=\mathrm{F}\cos\theta+\mathrm{G}\cos\phi+\mathrm{H}\cos\psi; \quad \ldots\ldots\ldots \quad (48)$$

similarly, we shall have,

$$\frac{1}{\rho_1}+\frac{1}{\rho'_1}=-\mathrm{F}\cos\theta+\mathrm{G}\cos\phi+\mathrm{H}\cos\psi,$$

$$\frac{1}{\rho_2}+\frac{1}{\rho'_2}=\ \ \mathrm{F}\cos\theta-\mathrm{G}\cos\phi+\mathrm{H}\cos\psi,$$

$$\frac{1}{\rho_3}+\frac{1}{\rho'_3}=\ \ \mathrm{F}\cos\theta+\mathrm{G}\cos\phi-\mathrm{H}\cos\psi.$$

Hence

$$\frac{1}{\rho}+\frac{1}{\rho'}=\frac{1}{\rho_1}+\frac{1}{\rho'_1}+\frac{1}{\rho_2}+\frac{1}{\rho'_2}+\frac{1}{\rho_3}+\frac{1}{\rho'_3}. \quad \ldots\ldots\ldots \quad (49)$$

37. We also obtain some interesting results by considering another group of circles allied to the pair considered in § 35. Thus let $2s=\theta+\phi+\psi$; let ρ_1, ρ'_1 be the radii of the pair of circles cutting the system (1, 2, 3) at angles $2s$, ψ, ϕ; let ρ_2, ρ'_2 be the radii of the pair cutting the system at angles ψ, $2s$, θ; and ρ_3, ρ'_3 the radii of the pair cutting the system at angles ϕ, θ, $2s$.

As in (48), we have

$$\frac{1}{\rho}+\frac{1}{\rho'}=\mathrm{F}\cos\theta+\mathrm{G}\cos\phi+\mathrm{H}\cos\psi,$$
$$\frac{1}{\rho_1}+\frac{1}{\rho'_1}=\mathrm{F}\cos 2s+\mathrm{G}\cos\psi+\mathrm{H}\cos\phi,$$
$$\frac{1}{\rho_2}+\frac{1}{\rho'_2}=\mathrm{F}\cos\psi+\mathrm{G}\cos 2s+\mathrm{H}\cos\theta,$$
$$\frac{1}{\rho_3}+\frac{1}{\rho'_3}=\mathrm{F}\cos\phi+\mathrm{G}\cos\theta+\mathrm{H}\cos 2s.$$

Hence, by addition,

$$\begin{aligned}&\frac{1}{\rho}+\frac{1}{\rho'}+\frac{1}{\rho_1}+\frac{1}{\rho'_1}+\frac{1}{\rho_2}+\frac{1}{\rho'_2}+\frac{1}{\rho_3}+\frac{1}{\rho'_3}\\&=(\mathrm{F}+\mathrm{G}+\mathrm{H})(\cos\theta+\cos\phi+\cos\psi+\cos 2s)\\&=\left(\frac{1}{\mathrm{R}}+\frac{1}{\mathrm{R}'}\right)4\cos\tfrac{1}{2}(\phi+\psi).\cos\tfrac{1}{2}(\psi+\theta).\cos\tfrac{1}{2}(\theta+\phi);\quad\ldots\quad(50)\end{aligned}$$

where R, R′ are the radii of the circles which touch the given circles externally.

Similarly

$$\begin{aligned}&\frac{1}{\rho}+\frac{1}{\rho'}+\frac{1}{\rho_1}+\frac{1}{\rho'_1}-\frac{1}{\rho_2}-\frac{1}{\rho'_2}-\frac{1}{\rho_3}-\frac{1}{\rho'_3}\\&=(\mathrm{F}-\mathrm{G}-\mathrm{H})(\cos\theta+\cos 2s-\cos\phi-\cos\psi)\\&=\left(\frac{1}{\mathrm{R}_1}+\frac{1}{\mathrm{R}'_1}\right)4\cos\tfrac{1}{2}(\phi+\psi).\sin\tfrac{1}{2}(\psi+\theta).\sin\tfrac{1}{2}(\theta+\phi);\end{aligned}$$

where R_1, R'_1 are the radii of the circles which touch the circle (1) internally and the circles (2, 3) externally.

38. The problem of drawing a circle to cut four given circles at equal angles has been discussed by DARBOUX, in his paper cited above, who makes the solution depend on that of drawing a circle to cut three given circles at given angles. Given four circles, say (1, 2, 3, 4), we can easily find the angle at which a circle can cut them, and the radius of the circle. For let this angle be ϕ, then, if we denote the circle by S, we have

$$\Pi\begin{pmatrix}\mathrm{S}, 1, 2, 3, 4\\ \mathrm{S}, 1, 2, 3, 4\end{pmatrix}=0,$$

whence

$$\begin{vmatrix} -1, & \cos\phi, & \cos\phi, & \cos\phi, & \cos\phi \\ \cos\phi, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ \cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ \cos\phi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ \cos\phi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix} = 0;$$

where $\omega_{1,2}$, $\omega_{1,3}$, &c., are the angles of intersection of the system (1, 2, 3, 4).

We have then to determine ϕ, the equation

$$\begin{vmatrix} -\sec^2\phi, & 1, & 1, & 1, & 1, \\ 1, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ 1, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ 1, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ 1, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix} = 0 \quad . \quad . \quad . \quad (51)$$

Also since

$$\Pi\begin{pmatrix} S, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix} = 0;$$

we obtain at once, if ρ be the radius of the circle,

$$\begin{vmatrix} \frac{\sec\phi}{\rho}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4} \\ 1, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ 1, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ 1, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ 1, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix} = 0 \quad . \quad . \quad . \quad (52)$$

We thus see that only one circle can be drawn; (51) determines the angle at which this circle cuts the given system.

For instance, if the four given circles cut orthogonally, we have

$$\sec\phi = 2,$$

$$\frac{2}{\rho} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4}.$$

This circle will be imaginary, since one of the four (1, 2, 3, 4) is so.

The Circles which pass through three of the points of intersection of three Circles.

§§ 39–41.

39. Let the given systems of circles be denoted by (1, 2, 3), and their orthogonal circle by the symbol (4). Let P, Q, R, P′, Q′, R′ be the six points of intersection of the three circles, the points P, Q, R being situated within the triangle formed by the centres of the circles. Let S be the circle which passes through the points P, Q, R.

We have, then,

$$\Pi\begin{pmatrix}S,\ 2,\ 3\\S,\ 2,\ 3\end{pmatrix}=\Pi\begin{pmatrix}S,\ 3,\ 1\\S,\ 3,\ 1\end{pmatrix}=\Pi\begin{pmatrix}S,\ 1,\ 2\\S,\ 1,\ 2\end{pmatrix}=0.$$

Hence we have by a theorem of determinants

$$\left.\begin{aligned}\left\{\Pi\begin{pmatrix}S,\ 1,\ 3\\S,\ 2,\ 3\end{pmatrix}\right\}^2&=-\Pi\begin{pmatrix}S,\ 3\\S,\ 3\end{pmatrix}\times\Pi\begin{pmatrix}S,\ 1,\ 2,\ 3\\S,\ 1,\ 2,\ 3\end{pmatrix}\\ \text{and}\quad\left\{\Pi\begin{pmatrix}S,\ 2,\ 3\\1,\ 2,\ 3\end{pmatrix}\right\}^2&=-\Pi\begin{pmatrix}2,\ 3\\2,\ 3\end{pmatrix}\times\Pi\begin{pmatrix}S,\ 1,\ 2,\ 3\\S,\ 1,\ 2,\ 3\end{pmatrix}\end{aligned}\right\}\quad\ldots\ldots\quad(53)$$

But since

$$\Pi\begin{pmatrix}S,\ 1,\ 2,\ 3,\ \theta\\S,\ 1,\ 2,\ 3,4\end{pmatrix}=0,$$

we have

$$\pi_{S,4}\Pi\begin{pmatrix}1,\ 2,\ 3,\ \theta\\1,\ 2,\ 3,\ S\end{pmatrix}-\Pi\begin{pmatrix}S,\ 1,\ 2,\ 3\\S,\ 1,\ 2,\ 3\end{pmatrix}=0\ ;$$

or

$$\Pi\begin{pmatrix}S,\ 1,\ 2,\ 3\\S,\ 1,\ 2,\ 3\end{pmatrix}=\pi_{S,4}\left\{\Pi\begin{pmatrix}1,\ 2,\ 3\\1,\ 2,\ 3\end{pmatrix}-\Pi\begin{pmatrix}S,\ 2,\ 3\\1,\ 2,\ 3\end{pmatrix}-\Pi\begin{pmatrix}S,\ 3,\ 1\\1,\ 2,\ 3\end{pmatrix}-\Pi\begin{pmatrix}S,\ 1,\ 2\\1,\ 2,\ 3\end{pmatrix}\right\},\quad\ldots\quad(54)$$

and since

$$\Pi\begin{pmatrix}S,\ 1,\ 2,\ 3,\ 4\\S,\ 1,\ 2,\ 3,\ 4\end{pmatrix}=0,$$

$$\pi_{4,4}\Pi\begin{pmatrix}S,\ 1,\ 2,\ 3\\S,\ 1,\ 2,\ 3\end{pmatrix}=\pi^2_{S,4}\Pi\begin{pmatrix}1,\ 2,\ 3\\1,\ 2,\ 3\end{pmatrix}\quad\ldots\ldots\quad(55)$$

But by § 18,

$$\Pi\begin{pmatrix}1,\ 2,\ 3\\1,\ 2,\ 3\end{pmatrix}=-16\pi_{4,4}\left\{\Delta(1,\ 2,\ 3)\right\}^2,$$

hence by means of (53), (54) becomes

$$\frac{\pi_{4,4}-\pi_{4,S}}{\pi_{4,S}}.4\Delta(1,\ 2,\ 3)=\left\{\Pi\begin{pmatrix}2,\ 3\\2,\ 3\end{pmatrix}\right\}^{\frac{1}{2}}+\left\{\Pi\begin{pmatrix}3,\ 1\\3,\ 1\end{pmatrix}\right\}^{\frac{1}{2}}+\left\{\Pi\begin{pmatrix}1,\ 2\\1,\ 2\end{pmatrix}\right\}^{\frac{1}{2}}.$$

But

$$\Pi\begin{pmatrix}2,\ 3\\2,\ 3\end{pmatrix}=16\{\Delta(\mathrm{P},\ 2,\ 3)\}^2.$$

Thus we may write this equation

$$\frac{\pi_{4,4}-\pi_{4,8}}{\pi_{4,8}}=\frac{\Delta(\mathrm{P},\,2,\,3)+\Delta(\mathrm{Q},\,3,\,1)+\Delta(\mathrm{R},\,1,\,2)}{\Delta(1,\,2,\,3)};$$

or

$$\frac{\rho\cos\omega}{r}=\frac{\Delta(1,\,2,\,3)}{\Delta(1,\,2,\,3)+\Delta(\mathrm{P},\,2,\,3)+\Delta(\mathrm{Q},\,3,\,1)+\Delta(\mathrm{R},\,1,\,2)}, \quad . \quad . \quad . \quad . \quad (56)$$

where ω is the angle of intersection of S and the orthogonal circle of the system (1, 2, 3), and ρ, r are the radii of the circles (S, 4).

This formula is easily adapted for the circles (PQR'), &c., by taking the area of the triangle (P', 2, 3) as of opposite sign to $\Delta(\mathrm{P},\,2,\,3)$.

Again, if $\kappa_{8,8}$, $\kappa_{8,1}$, &c. denote the minors of $\pi_{8,8}$, $\pi_{8,1}$, &c. in $\Pi\begin{pmatrix}\mathrm{S},\,1,\,2,\,3\\ \mathrm{S},\,1,\,2,\,3\end{pmatrix}$; and if $\mu_{1,1}$, $\mu_{1,2}$, &c. denote the minors of $\pi_{1,1}$, $\pi_{1,2}$, &c. in $\Pi\begin{pmatrix}1,\,2,\,3\\ 1,\,2,\,3\end{pmatrix}$, we shall have from (55)

$$\frac{\pi^2_{4,8}}{\pi_{4,4}}.\Pi\begin{pmatrix}1,\,2,\,3\\ 1,\,2,\,3\end{pmatrix}\times\mu\begin{pmatrix}1,\,2,\,3\\ 1,\,2,\,3\end{pmatrix}=\Pi\begin{pmatrix}\mathrm{S},\,1,\,2,\,3\\ \mathrm{S},\,1,\,2,\,3\end{pmatrix}\times\mu\begin{pmatrix}1,\,2,\,3\\ 1,\,2,\,3\end{pmatrix}$$

$$=\begin{vmatrix}\pi_{8,8}.\kappa_{8,8}, & \kappa_{8,1}, & \kappa_{8,2}, & \kappa_{8,3}\\ \kappa_{1,8}, & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3}\\ \kappa_{2,8}, & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3}\\ \kappa_{3,8}, & \mu_{3,1}, & \mu_{3,2}, & \mu_{3,3}\end{vmatrix}.$$

But by (53)

$$\kappa^2_{8,1}=\left\{\Pi\begin{pmatrix}\mathrm{S},\,2,\,3\\ 1,\,2,\,3\end{pmatrix}\right\}^2=-\Pi\begin{pmatrix}2,\,3\\ 2,\,3\end{pmatrix}.\Pi\begin{pmatrix}\mathrm{S},\,1,\,2,\,3\\ \mathrm{S},\,1,\,2,\,3\end{pmatrix}=-\mu_{1,1}.\Pi\begin{pmatrix}\mathrm{S},\,1,\,2,\,3\\ \mathrm{S},\,1,\,2,\,3\end{pmatrix}.$$

Therefore

$$\frac{\pi_{8,8}.\pi_{4,4}-\pi^2_{4,8}}{\pi^2_{4,8}}\times\left\{\Pi\begin{pmatrix}1,\,2,\,3\\ 1,\,2,\,3\end{pmatrix}\right\}^2=\begin{vmatrix}0, & \sqrt{\mu_{1,1}}, & \sqrt{\mu_{2,2}}, & \sqrt{\mu_{3,3}}\\ \sqrt{\mu_{1,1}}, & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3}\\ \sqrt{\mu_{2,2}}, & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3}\\ \sqrt{\mu_{3,3}}, & \mu_{3,1}, & \mu_{3,2}, & \mu_{3,3}\end{vmatrix}. \quad . \quad . \quad (57)$$

But

$$\frac{\pi_{8,8}\pi_{4,4}-\pi^2_{4,8}}{\pi^2_{4,8}}=\tan^2\omega;$$

and

$$\mu_{1,1}=16\{\Delta(\mathrm{P},\,2,\,3)\}^2.$$

So that we may write the above equation

$$64\,r^4\tan^2\omega\{\Delta(1,2,3)\}^4=\begin{vmatrix} 0, & \Delta(P,2,3), & \Delta(Q,3,1), & \Delta(R,1,2) \\ \Delta(P,2,3), & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3} \\ \Delta(Q,3,1), & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3} \\ \Delta(R,1,2), & \mu_{3,1}, & \mu_{3,2}, & \mu_{3,3} \end{vmatrix} . \quad (58)$$

Similarly we can find ω for the circles (P′, Q, R), &c.

40. If the given system (1, 2, 3) intersect at angles α, β, γ, equation (57) reduces easily to

$$\cos^2\omega=\sec s\,.\cos(s-\alpha)\,.\cos(s-\beta)\,.\cos(s-\gamma), \quad . \quad . \quad . \quad . \quad (59)$$

where

$$2s=\alpha+\beta+\gamma.$$

Also for circle (P′QR) we shall find

$$\cos^2\omega=\cos s\,.\sec(s-\alpha)\,.\cos(s-\beta)\,.\cos(s-\gamma).$$

41. It may be noticed that the system (P, Q, R, 4) is the orthogonal system to (1, 2, 3, S)—(see § 27).

We have at once from equation (35)

$$\frac{1}{\pi_{P,1}}+\frac{1}{\pi_{Q,2}}+\frac{1}{\pi_{R,3}}+\frac{1}{\pi_{S,4}}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

If S′ be the circle through (P′, Q′, R′), we have

$$\frac{1}{\pi_{P',1}}+\frac{1}{\pi_{Q',2}}+\frac{1}{\pi_{R',3}}+\frac{1}{\pi_{S',4}}=0.$$

But by (47), since S′ is the inverse of S with respect to the circle (4),

$$\frac{1}{\pi_{S,4}}+\frac{1}{\pi_{S',4}}=\frac{2}{\pi_{4,4}}.$$

Hence we have

$$\frac{1}{\pi_{P,1}}+\frac{1}{\pi_{P',1}}+\frac{1}{\pi_{Q,2}}+\frac{1}{\pi_{Q',2}}+\frac{1}{\pi_{R,3}}+\frac{1}{\pi_{R',3}}+\frac{2}{\pi_{4,4}}=0\,; \quad . \quad . \quad . \quad . \quad . \quad (61)$$

or the sum of the reciprocals of the squares of the tangents, from the points of intersection of three circles to the circles, is equal to the reciprocal of the square of the radius of the circle which cuts the circles orthogonally.

Again, from equation (31), we shall have

$$\frac{\pi_{S,1}}{\pi_{P,1}}+\frac{\pi_{S,2}}{\pi_{Q,2}}+\frac{\pi_{S,3}}{\pi_{R,3}}+\frac{\pi_{S,4}}{\pi_{S,4}}=1, \quad \ldots \quad (62)$$

or, if S cut the system (1, 2, 3) at angles ϕ_1, ϕ_2, ϕ_3, this may be written

$$\frac{1}{\rho}=\frac{2r_1\cos\phi_1}{\pi_{P,1}}+\frac{2r_2\cos\phi_2}{\pi_{Q,2}}+\frac{2r_3\cos\phi_3}{\pi_{R,3}}-\frac{1}{r\cos\omega}. \quad \ldots \quad (63)$$

This with equation (60) is sufficient to determine ω and ρ. Also by drawing a figure we shall see at once that

$$\phi_2+\phi_3+\alpha=\phi_3+\phi_1+\beta=\phi_1+\phi_2+\gamma=\pi.$$

So that, if $2s=\alpha+\beta+\gamma$,

$$\phi_1=\tfrac{1}{2}\pi-(s-\alpha); \quad \phi_2=\tfrac{1}{2}\pi-(s-\beta); \quad \phi_3=\tfrac{1}{2}\pi-(s-\gamma).$$

The Circles which touch three given Circles.—§§ 42–46.

42. Let (1, 2, 3) be the given system, and let (4) denote the orthogonal circle of the system: then if S be the circle which touches all the circles externally, and ω its angle of intersection with (4), we shall have, since,

$$\pi^2_{S,1}=\pi_{1,1}.\pi_{S,S}; \text{ \&c.,}$$

and

$$\pi_{4,4}.\Pi\begin{pmatrix}S,1,2,3\\S,1,2,3\end{pmatrix}=\pi^2_{4,S}.\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix};$$

$$(\cos^2\omega-1).\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}=\begin{vmatrix} 0, & \sqrt{\pi_{1,1}}, & \sqrt{\pi_{2,2}}, & \sqrt{\pi_{3,3}} \\ \sqrt{\pi_{1,1}}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3} \\ \sqrt{\pi_{2,2}}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3} \\ \sqrt{\pi_{3,3}}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3} \end{vmatrix} \quad \ldots \quad (64)$$

By giving the expressions $\sqrt{\pi_{1,1}}$, &c., different signs, we obtain the values of $\cos\omega$ for the other pairs of tangent circles; and it is clear that there are four pairs of such circles.

43. If α, β, γ be the angles of intersection of the system (1, 2, 3), and ω, ω_1, ω_2, ω_3 the angles of intersection of the four pairs of tangent circles with the orthogonal circle of the system, we can easily deduce from (64) the formulæ

$$\left.\begin{aligned} K\cos^2\omega &= 2(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma)\\ K\cos^2\omega_1 &= 2(1+\cos\alpha)(1-\cos\beta)(1-\cos\gamma)\\ K\cos^2\omega_2 &= 2(1-\cos\alpha)(1+\cos\beta)(1-\cos\gamma)\\ K\cos^2\omega_3 &= 2(1-\cos\alpha)(1-\cos\beta)(1+\cos\gamma) \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad (65)$$

where

$$K = \cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma - 1.$$

44. The radii of the circles are given at once by the formula

$$\Pi\begin{pmatrix} S, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix} = 0.$$

Thus, if ρ be the radius of circle touching the system (1, 2, 3) externally, we have

$$\begin{vmatrix} \frac{1}{\rho}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r} \\ 1, & -1, & \cos\gamma, & \cos\beta, & 0 \\ 1, & \cos\gamma, & -1, & \cos\alpha, & 0 \\ 1, & \cos\beta, & \cos\alpha, & -1, & 0 \\ \cos\omega, & 0, & 0, & 0, & -1 \end{vmatrix} = 0,$$

or,

$$\begin{vmatrix} \frac{1}{\rho}+\frac{\cos\omega}{r}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, \\ 1, & -1, & \cos\gamma, & \cos\beta \\ 1, & \cos\gamma, & -1, & \cos\alpha \\ 1, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad (66)$$

If we denote the radii of the pairs of tangent circles by (ρ, ρ') (ρ_1, ρ'_1) (ρ_2, ρ'_2) (ρ_3, ρ'_3), we have, by (49)

$$\frac{1}{\rho}+\frac{1}{\rho'} = \frac{1}{\rho_1}+\frac{1}{\rho'_1}+\frac{1}{\rho_2}+\frac{1}{\rho'_2}+\frac{1}{\rho_3}+\frac{1}{\rho'_3},$$

a theorem first given by Mr. Cox.—('Quart. Journ. Math.,' vol. 19, 1883, p. 99.)

45. Let (5, 6, 7, 8) denote a system of circles formed by taking one of each pair of tangent circles of the system (1, 2, 3). This can be done in sixteen ways:—We may show that eight of these sixteen groups are touched respectively by eight other circles.

Let z be the circle which touches the group (5, 6, 7, 8): let z touch 5 internally and (6, 7, 8) externally; then, since

$$\Pi\begin{pmatrix} z, 4, 1, 2, 3 \\ x, 5, 6, 7, 8 \end{pmatrix} = 0,$$

we have, giving x the successive values 4, 1, 2, 3:—

$$\left.\begin{aligned} A_1 \cos \omega_{x,4} &= 4 \\ A_1 \cos \omega_{x,1} &= A_2 - A_3 \cos\gamma - A_4 \cos\beta \\ A_1 \cos \omega_{x,2} &= -A_2 \cos\gamma + A_3 - A_4 \cos\alpha \\ A_1 \cos \omega_{x,3} &= -A_2 \cos\beta - A_3 \cos\alpha + A_4 \end{aligned}\right\}, \qquad (66^*)$$

where

$$\left.\begin{aligned} A_1 &= \cos\omega_{4,5} - \cos\omega_{4,6} - \cos\omega_{4,7} - \cos\omega_{4,8} \\ A_2 &= -\cos\omega_{4,5} + \cos\omega_{4,6} - \cos\omega_{4,7} - \cos\omega_{4,8} \\ A_3 &= -\cos\omega_{4,5} - \cos\omega_{4,6} + \cos\omega_{4,7} - \cos\omega_{4,8} \\ A_4 &= -\cos\omega_{4,5} - \cos\omega_{4,6} - \cos\omega_{4,7} + \cos\omega_{4,8} \end{aligned}\right\}.$$

But we also have

$$\Pi\begin{pmatrix} z, 4, 1, 2, 3 \\ z, 4, 1, 2, 3 \end{pmatrix} = 0.$$

Therefore

$$-A_1 + A_2 \cos\omega_{1,z} + A_3 \cos\omega_{2,z} + A_4 \cos\omega_{3,z} + 4\cos\omega_{4,z} = 0.$$

Hence we must have

$$16 - A_1^2 + A_2^2 + A_3^2 + A_4^2 - 2A_3A_4\cos\alpha - 2A_4A_2\cos\beta - 2A_2A_3\cos\gamma = 0,$$

which may be written

$$\begin{aligned} &16 + 2\cos^2\omega_{4,5}(1 - \cos\alpha - \cos\beta - \cos\gamma) \\ &\quad + 2\cos^2\omega_{4,6}(1 - \cos\alpha + \cos\beta + \cos\gamma) \\ &\quad + 2\cos^2\omega_{4,7}(1 + \cos\alpha - \cos\beta + \cos\gamma) \\ &\quad + 2\cos^2\omega_{4,8}(1 + \cos\alpha + \cos\beta - \cos\gamma) \\ &\quad + 2(1 - \cos\alpha)\cos\omega_{4,5}.\cos\omega_{4,6} - 2(1 + \cos\alpha)\cos\omega_{4,7}.\cos\omega_{4,8} \\ &\quad + 2(1 - \cos\beta)\cos\omega_{4,5}.\cos\omega_{4,7} - 2(1 + \cos\beta)\cos\omega_{4,6}.\cos\omega_{4,8} \\ &\quad + 2(1 - \cos\gamma)\cos\omega_{4,5}.\cos\omega_{4,8} - 2(1 + \cos\gamma)\cos\omega_{4,6}.\cos\omega_{4,7} = 0. \end{aligned}$$

Referring to (65) we see that this equation is satisfied, provided we choose the groups of circles so that

$$\cos\omega_{4,5}.\cos\omega_{4,6}.\cos\omega_{4,7}.\cos_{4,8} \text{ is positive.}$$

Thus if we denote the tangent circles of the system (1, 2, 3) by the symbols (τ, τ'), (τ_1, τ'_1), (τ_2, τ'_2), (τ_3, τ'_3), where τ, τ_1, τ_2, τ_3 correspond to the positive values of $\cos\omega$, $\cos\omega_1$, &c., as given by (65), then we see that the groups

$$\begin{array}{ll} \tau, \tau_1, \tau_2, \tau_3 & \tau', \tau'_1, \tau'_2, \tau'_3 \\ \tau, \tau_1, \tau'_2, \tau'_3 & \tau', \tau'_1, \tau_2, \tau_3 \\ \tau, \tau'_1, \tau_2, \tau'_3 & \tau', \tau_1, \tau'_2, \tau_3 \\ \tau, \tau'_1, \tau'_2, \tau_3 & \tau', \tau_1, \tau_2, \tau'_3 \end{array}$$

are touched by circles S, S_1, S_2, S_3; S′, S'_1, S'_2, S'_3. Also each of these circles, S say, touches τ internally and the others externally; and if S touch the group $\tau, \tau_1, \tau_2, \tau_3$, and S′ the group $\tau', \tau'_1, \tau'_2, \tau'_3$, then S′ is the inverse of S with respect to the orthogonal circle of the system (1, 2, 3). These circles are usually called Dr. HART'S circles.

46. We can easily deduce from formulæ (66) that

$$\left.\begin{array}{lll} \text{S, S}' & \text{cut (1, 2, 3) at angles} & \beta-\gamma, \gamma-\alpha, \alpha-\beta \\ \text{S}_1, \text{S}'_1 & \text{,, (1, 2, 3)} \quad \text{,,} & \beta-\gamma, \gamma+\alpha, \alpha+\beta \\ \text{S}_2, \text{S}'_2 & \text{,, (1, 2, 3)} \quad \text{,,} & \beta+\gamma, \gamma-\alpha, \alpha+\beta \\ \text{S}_3, \text{S}'_3 & \text{,, (1, 2, 3)} \quad \text{,,} & \beta+\gamma, \gamma+\alpha, \alpha-\beta \end{array}\right\} \quad . \quad . \quad . \quad . \quad (67)$$

Also, if the angles of intersection with the orthogonal circle of (1, 2, 3) of the pairs (S, S′), &c., be ϖ, ϖ_1, ϖ_2, ϖ_3, we shall have

$$\left.\begin{array}{l} \cos^2 \varpi = 4 \sec s \,.\, \cos(s-\alpha) \,.\, \cos(s-\beta) \,.\, \cos(s-\gamma) \\ \cos^2 \varpi_1 = 4 \cos s \,.\, \sec(s-\alpha) \,.\, \cos(s-\beta) \,.\, \cos(s-\gamma) \\ \cos^2 \varpi_2 = 4 \cos s \,.\, \cos(s-\alpha) \,.\, \sec(s-\beta) \,.\, \cos(s-\gamma) \\ \cos^2 \varpi_3 = 4 \cos s \,.\, \cos(s-\alpha) \,.\, \cos(s-\beta) \,.\, \sec(s-\gamma) \end{array}\right\} .$$

Referring to § 40 we see that if the given circles (1, 2, 3) intersect in the points P, Q, R, P′, Q′, R′, and the circle (P, Q, R) cut the orthogonal circle to (1, 2, 3) at the angle ω, then

$$\cos \varpi = 2 \cos \omega.$$

Hence, if ρ, ρ' be the radii of the circles S, S′, and R, R′ the radii of the circles (P, Q, R), (P′, Q′, R′), then

$$\frac{1}{\rho} - \frac{1}{\rho'} = 2\left(\frac{1}{R} - \frac{1}{R'}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Similarly each pair of Dr. HART'S circles is connected with a corresponding pair of the circles which can be drawn through the points P, Q, R, P′, Q′, R′ by a formula which is analogous to that which connects the radius of the nine-points circle of a plane triangle with the radius of the circum-circle.

Three circles produce, as may be seen by drawing a figure, four pairs of triangles, each pair consisting of a triangle, and its inverse with respect to the orthogonal circle. Thus, supposing α, β, γ are the angles of the triangle P, Q, R, the angles of the triangle P′, Q′, R′ are either α, β, γ or else $\pi-\alpha$, $\pi-\beta$, $\pi-\gamma$. Again, the angles of P′, Q, R are α, $\pi-\beta$, $\pi-\gamma$. Hence, having obtained the formulæ for the radius of any circle connected with a particular triangle, we can easily obtain the formulæ for the other circles. It is also evident that there must be eight circles corresponding to the circum-circle, and eight circles corresponding to the nine-points circle of a plane triangle.

Chapter V.—Power-Coordinates.

Definition.—§§ 47–50.

47. We have already seen that any circle (straight line or point) is completely determined when its powers are known with respect to any four circles which have not a common orthogonal circle. Hence, given four such circles, which may be called the system of reference, any multiples, the same or different, of the powers of a circle (straight line or point) with respect to them, may be defined as its power-coordinates.

We shall find it convenient to denote the coordinates of any circle by $\xi\eta\zeta\omega$; the coordinates of any point by $xyzw$; and the coordinates of any straight line by $\lambda\mu\nu\rho$.

If α, β be the Cartesian coordinates of the centre of any circle whose power-coordinate with respect to a circle be ξ; if a, b be the Cartesian coordinates of the centre of the latter; and R, r be the radii of the two circles: we shall have

$$\xi \propto (\alpha-a)^2+(\beta-b)^2-R^2-r^2;$$

so that the power-coordinates of any *circle* are quadric functions of a particular form of the Cartesian coordinates of the centre of the circle.

Similarly, if x be the power-coordinate of a point whose Cartesian coordinates are α, β:

$$x \propto (\alpha-a)^2+(\beta-b)^2-r^2;$$

or the power-coordinates of a *point* are quadric functions of a particular form of the Cartesian coordinates of the point.

In the case of a straight line, whose Cartesian equation is

$$x\cos\alpha+y\sin\alpha=p,$$

we shall have

$$\lambda \propto p-a\cos\alpha-b\sin\alpha.$$

Thus the power-coordinates of a *straight line* are linear functions of a particular form of what may be called the Cartesian coordinates of the straight line.

48. If θ denote as previously the line at infinity, and the system of reference be denoted by (1, 2, 3, 4), then P being any point; since $\pi_{P,\theta}=1$, $\pi_{P,P}=0$, we see that the coordinates of P must satisfy:—

i. a homogeneous quadric relation,

$$\Pi\begin{pmatrix}P,\,1,\,2,\,3,\,4\\P,\,1,\,2,\,3,\,4\end{pmatrix}=0\,;$$

ii. a non-homogeneous linear relation,

$$\Pi\begin{pmatrix}P,\,1,\,2,\,3,\,4\\\theta,\,1,\,2,\,3,\,4\end{pmatrix}=0.$$

Let us suppose that $xyzw$, the coordinates of P, are given by

$$x=k_1.\pi_{P,1}\,;\ y=k_2.\pi_{P,2}\,;\ z=k_3.\pi_{P,3}\,;\ w=k_4.\pi_{P,4}\,;$$

then $xyzw$ must satisfy the relation, which is called the *absolute*:—

$$\psi(x,y,z,w)\equiv\begin{vmatrix} 0, & \frac{x}{k_1}, & \frac{y}{k_2}, & \frac{z}{k_3}, & \frac{w}{k_4}\\ \frac{x}{k_1}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4}\\ \frac{y}{k_2}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4}\\ \frac{z}{k_3}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4}\\ \frac{w}{k_4}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4}\end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Then the relation (ii) may be written

$$\left.\begin{aligned} k_1\frac{\partial\psi}{\partial x}+k_2\frac{\partial\psi}{\partial y}+k_3\frac{\partial\psi}{\partial z}+k_4\frac{\partial\psi}{\partial w}&=K,\\ \text{or}\qquad x\frac{\partial\psi}{\partial k_1}+y\frac{\partial\psi}{\partial k_2}+z\frac{\partial\psi}{\partial k_3}+w\frac{\partial\psi}{\partial k_4}&=K;\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (70)$$

where K is some constant.*

49. Similarly if S denote any circle, since $\pi_{S,\theta}=1$, we see that its coordinates (ξ,η,ζ,ω) must satisfy the linear relation

$$\Pi\begin{pmatrix}S,\,1,\,2,\,3,\,4\\\theta,\,1,\,2,\,3,\,4\end{pmatrix}=0,$$

or

$$\xi\frac{\partial\psi}{\partial k_1}+\eta\frac{\partial\psi}{\partial k_2}+\zeta\frac{\partial\psi}{\partial k_3}+\eta\frac{\partial\psi}{\partial k_4}=K \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

* [If we write $\psi\,(x,y,z,w)\equiv(a_{1,1},a_{1,2},\ldots)\,(x,y,z,w)^2$, then we shall have,

$$\frac{K}{\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}}=\frac{2a_{1,1}.k_1^2}{\Pi\begin{pmatrix}2,3,4\\2,3,4\end{pmatrix}}=\&c.—\text{October, 1886.}]$$

50. Again if L denote any straight line, since $\pi_{L,\theta}=0$, and $\pi_{L,L}=-2$; we see that its coordinates $(\lambda, \mu, \nu, \rho)$ must satisfy, (i) the homogeneous linear relation,

$$\Pi\begin{pmatrix} L, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix}=0,$$

or

$$\lambda\frac{\partial\psi}{\partial k_1}+\mu\frac{\partial\psi}{\partial k_2}+\nu\frac{\partial\psi}{\partial k_3}+\rho\frac{\partial\psi}{\partial k_4}=0; \quad \ldots\ldots\ldots \quad (72)$$

and (ii) the non-homogeneous quadric relation,

$$\Pi\begin{pmatrix} L, 1, 2, 3, 4 \\ L, 1, 2, 3, 4 \end{pmatrix}=0,$$

or

$$\psi(\lambda, \mu, \nu, \rho)=-K. \quad \ldots\ldots\ldots\ldots \quad (73)$$

The Circle.—§§ 51–55.

51. Let P be any point on a circle S, then $\pi_{S,P}=0$; hence the equation

$$\Pi\begin{pmatrix} P, 1, 2, 3, 4 \\ S, 1, 2, 3, 4 \end{pmatrix}=0,$$

leads to

$$\frac{\partial\psi}{\partial\xi}x+\frac{\partial\psi}{\partial\eta}y+\frac{\partial\psi}{\partial\zeta}z+\frac{\partial\psi}{\partial\omega}w=0. \quad \ldots\ldots\ldots \quad (74)$$

Thus the equation of a circle is of the first degree.

It follows that the general equation of the first degree

$$ax+by+cz+dw=0$$

represents in general a circle, whose coordinates are given by,

$$\frac{\frac{\partial\psi}{\partial\xi}}{a}=\frac{\frac{\partial\psi}{\partial\eta}}{b}=\frac{\frac{\partial\psi}{\partial\zeta}}{c}=\frac{\frac{\partial\psi}{\partial\omega}}{d}=\frac{K}{ak_1+bk_2+ck_3+dk_4}, \quad \ldots\ldots\ldots \quad (75)$$

by equation (71).

52. Given any two circles whose coordinates are $(\xi, \eta, \zeta, \omega)$, $(\xi', \eta', \zeta', \omega')$; their power π is given by

$$\Pi\begin{pmatrix} S, 1, 2, 3, 4 \\ S', 1, 2, 3, 4 \end{pmatrix}=0,$$

or

$$\left.\begin{aligned} \pi.K&=\xi\frac{\partial\psi}{\partial\xi}+\eta\frac{\partial\psi}{\partial\eta}+\zeta\frac{\partial\psi}{\partial\zeta}+\omega\frac{\partial\psi}{\partial\omega} \\ &=\xi\frac{\partial\psi}{\partial\xi'}+\eta\frac{\partial\psi}{\partial\eta'}+\zeta\frac{\partial\psi}{\partial\zeta'}+\omega\frac{\partial\psi}{\partial\omega'} \end{aligned}\right\}. \quad \ldots\ldots\ldots \quad (76)$$

In particular the radius of the circle $\xi\eta\zeta\omega$ is given by

$$-2r^2.\mathrm{K}=\xi\frac{\partial\psi}{\partial\xi}+\eta\frac{\partial\psi}{\partial\eta}+\zeta\frac{\partial\psi}{\partial\zeta}+\omega\frac{\partial\psi}{\partial\omega},$$

or

$$-r^2.\mathrm{K}=\psi(\xi,\eta,\zeta,\omega). \quad \ldots\ldots\ldots\ldots \quad (77)$$

53. Hence the radius of the circle

$$ax+by+cz+dw=0,$$

is given by

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d \\ a, & b, & c, & d, & \mathrm{M}r^2 \end{vmatrix} =0; \quad \ldots\ldots \quad (78)$$

where

$$\mathrm{M}=-\frac{4}{\mathrm{K}}(ak_1+bk_2+ck_3+dk_4)^2,$$

and $a_{1,1}$, $a_{1,2}$, &c., are the coefficients in the equation to the absolute, so that

$$\psi(xyzw)\equiv a_{1,1}x^2+a_{2,2}y^2+\ldots+2a_{1,2}xy+\ldots=0.$$

54. Again the power of the circle $(\xi\eta\zeta\omega)$ with respect to the circle

$$ax+by+cz+dw=0,$$

is clearly given by

$$\pi=\frac{a\xi+b\eta+c\zeta+d\omega}{ak_1+bk_2+ck_3+dk_4} \quad \ldots\ldots\ldots \quad (79)$$

55. And again the power of the two circles

$$ax+by+cz+dw=0,$$
$$a'x+b'y+c'z+d'w=0,$$

is given by

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d \\ a', & b', & c', & d', & \mathrm{M}\pi \end{vmatrix} =0;$$

where

$$\mathrm{M}=\frac{2}{\mathrm{K}}(ak_1+bk_2+ck_3+dk_4)(a'k_1+b'k_2+c'k_3+d'k_4). \quad \ldots\ldots \quad (80)$$

Whence, if the two circles cut at an angle ϕ, we have

$$\cos\phi=-\frac{a'\frac{\partial\Psi}{\partial a}+b'\frac{\partial\Psi}{\partial b}+c'\frac{\partial\Psi}{\partial c}+d'\frac{\partial\Psi}{\partial d}}{2\sqrt{\Psi(a,b,c,d).\Psi(a',b',c',d')}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

where

$$\Psi(a,b,c,d)\equiv-\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d \\ a, & b, & c, & d, & 0 \end{vmatrix},$$

The straight Line.—§§ 56–58.

56. Proceeding as in § 51, we see that the equation to the straight line, whose coordinates are $(\lambda, \mu, \nu, \rho)$, is

$$\frac{\partial\psi}{\partial\lambda}x+\frac{\partial\psi}{\partial\mu}y+\frac{\partial\psi}{\partial\nu}z+\frac{\partial\psi}{\partial\rho}w=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82)$$

But by equation (72)

$$\frac{\partial\psi}{\partial\lambda}k_1+\frac{\partial\psi}{\partial\mu}k_2+\frac{\partial\psi}{\partial\nu}k_3+\frac{\partial\psi}{\partial\rho}k_4=0;$$

hence the equation

$$ax+by+cz+dw=0,$$

will represent a straight line, provided that

$$ak_1+bk_2+ck_3+dk_4=0; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (83)$$

and if this condition be satisfied the coordinates of the lines are given by,

$$\frac{\frac{\partial\psi}{\partial\lambda}}{a}=\frac{\frac{\partial\psi}{\partial\mu}}{b}=\frac{\frac{\partial\psi}{\partial\nu}}{c}=\frac{\frac{\partial\psi}{\partial\rho}}{d}=2\sqrt{\frac{-\mathrm{K}.\Delta}{\Psi}}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84)$$

where

$$\Delta\equiv\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4} \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4} \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4} \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4} \end{vmatrix},$$

and Ψ denotes the same expression as in § 55.

57. The power of the straight line

$$ax+by+cz+dw=0,$$

and the circle $(\xi, \eta, \zeta, \omega)$ is given by,

$$\pi\mathrm{K}=\xi\frac{\partial\Psi}{\partial\lambda}+\eta\frac{\partial\Psi}{\partial\mu}+\zeta\frac{\partial\Psi}{\partial\nu}+\omega\frac{\partial\Psi}{\partial\rho}$$

$$=2\sqrt{\frac{-\mathrm{K}\Delta}{\Psi}}\cdot(a\xi+b\eta+c\zeta+d\omega) \quad \ldots\ldots\ldots \quad (85)$$

And the loci represented by the equations

$$ax+by+cz+dw=0,$$
$$a'x+b'y+c'z+d'w=0,$$

intersect at the angle ϕ, given by

$$\cos\phi=-\frac{a'\frac{\partial\Psi}{\partial a}+b'\frac{\partial\Psi}{\partial b}+c'\frac{\partial\Psi}{\partial c}+d'\frac{\partial\Psi}{\partial d}}{\sqrt{\Psi(a,b,c,d).\Psi(a',b',c',d')}} \quad \ldots\ldots\ldots \quad (86)$$

58. The coordinates of the line at infinity are k_1, k_2, k_3, k_4; hence the equation to the line at infinity is

$$x\frac{\partial\Psi}{\partial k_1}+y\frac{\partial\Psi}{\partial k_2}+z\frac{\partial\Psi}{\partial k_3}+w\frac{\partial\Psi}{\partial k_4}=0.$$

The Point.—§§ 59–61.

59. The power of the point $(xyzw)$ with respect to the circle

$$ax+by+cz+dw=0,$$

is

$$\frac{ax+by+cz+dw}{ak_1+bk_2+ck_3+dk_4}.$$

But if the equation represents a straight line, then we see that the perpendicular on it from the point

$$=(ax+by+cz+dw)\sqrt{\frac{-\Delta}{\mathrm{K}\Psi}} \quad \ldots\ldots\ldots \quad (87)$$

60. The power of the two points $(xyzw)$, $(x'y'z'w')$ is given by equation (76)

$$\pi\mathrm{K}=x\frac{\partial\Psi}{\partial x'}+y\frac{\partial\Psi}{\partial y'}+z\frac{\partial\Psi}{\partial z'}+w\frac{\partial\Psi}{\partial w'}.$$

Hence, if

$$\psi(xyzw) \equiv a_{1,1}x^2 + 2a_{1,2}xy + \ldots ;$$

the distance δ between the points is given by,

$$\begin{aligned} -\delta^2.\mathrm{K} &= a_{1,1}(x-x')^2 + 2a_{1,2}(x-x')(y-y') + \ldots \\ &= \psi\{x-x',\ y-y',\ z-z',\ w-w'\}. \qquad \ldots\ldots\ldots \quad (88) \end{aligned}$$

Thus, if δs be a small element of arc, we shall have

$$-\mathrm{K}.(\delta s)^2 = \psi\{\delta x,\ \delta y,\ \delta z,\ \delta w\}. \qquad \ldots\ldots\ldots \quad (89)$$

61. If P, Q, R be any three points, then by equation (15) we have for the area of the triangle

$$-16\{\Delta(\mathrm{P, Q, R})\}^2 = \Pi\begin{pmatrix}\theta, \mathrm{P, Q, R}\\ \theta, \mathrm{P, Q, R}\end{pmatrix};$$

and by § 8

$$\Pi\begin{pmatrix}\theta, \mathrm{P, Q, R}\\ \theta, \mathrm{P, Q, R}\end{pmatrix} \times \Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix} = \left\{\Pi\begin{pmatrix}\theta, \mathrm{P, Q, R}\\ 1, 2, 3, 4\end{pmatrix}\right\}^2.$$

Hence if $(x_1y_1z_1w_1)$, $(x_2y_2z_2w_2)$, $(x_3y_3z_3w_3)$ be the coordinates of the points P, Q, R referred to any system of circles (1, 2, 3, 4) we shall have

$$\Delta(\mathrm{P, Q, R}) = \mu \begin{vmatrix} x_1, & x_2, & x_3, & k_1 \\ y_1, & y_2, & y_3, & k_2 \\ z_1, & z_2, & z_3, & k_3 \\ w_1, & w_2, & w_3, & k_4 \end{vmatrix}; \qquad \ldots\ldots \quad (90)$$

where,

$$-4\mu^2 k_1^2 k_2^2 k_3^2 k_4^2 . \Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix} = 1.$$

Coordinate Systems of Reference.—§§ 62–66.

62. There are two systems of circles which are convenient as systems of reference—(i.) a system consisting of four circles cutting one another orthogonally,* (ii.) a system of two circles cutting orthogonally, and their two points of intersection. The former has been called the "orthogonal" system, and was first used by DARBOUX, 'Sur une Classe remarquable de Courbes et de Surfaces algébriques' (Note X., 1873). The latter system might be called the "semi-orthogonal" system; it is mentioned by Mr. HOMERSHAM COX in the paper "On Systems of Circles and Bicircular Quartics" ('Quart. Journ. Math.,' vol. 19, 1883, p. 116).

* [CASEY uses five orthogonal spheres—"Cyclides" (1871), p. 600. But the first use of four mutually orthotomic circles was, I believe, by CLIFFORD in a series of questions proposed by him in the 'Educational Times' for 1865–6. See Reprint, vol. 6.—October, 1886.]

63. In the case of the orthogonal system it is most convenient to take the constants k_1, k_2, k_3, k_4 equal to the reciprocals of the four radii, so that the equation to the absolute is

$$\left.\begin{aligned} \psi(x, y, z, w) &\equiv x^2+y^2+z^2+w^2=0 \\ \text{and} \qquad \mathrm{K} &= -4 \\ \Psi(a, b, c, d) &= a^2+b^2+c^2+d^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (91)$$

64. In the case of the semi-orthogonal system (see § 29), if r_1, r_2 be the radii of the circles, e the distance between their points of intersection, it is convenient to take $k_1=\frac{1}{r_1}$, $k_2=\frac{1}{r_2}$, $k_3=k_4=\frac{1}{e}$.

We shall have

$$\left.\begin{aligned} \psi(x, y, z, w) &\equiv x^2+y^2-4zw \\ \mathrm{K} &= -4 \\ \Psi(a, b, c, d) &= a^2+b^2-cd \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (92)$$

65. Thus the angle ϕ at which the loci

$$\begin{aligned} ax+by+cz+dw&=0, \\ a'x+b'y+c'z+d'w&=0, \end{aligned}$$

intersect, is given by

$$\cos\phi=-\frac{aa'+bb'+cc'+dd'}{\sqrt{(a^2+b^2+c^2+d^2)(a'^2+b'^2+c'^2+d'^2)}},$$

or

$$\cos\phi=-\frac{aa'+bb'-\frac{1}{2}(cd'+c'd)}{\sqrt{(a^2+b^2-cd)(a'^2+b'^2-c'd')}};$$

according as the system of reference is the "orthogonal" or the "semi-orthogonal" system.

66. Occasionally it may not be convenient to take for system of reference either of the systems just considered. In some cases, however, the equations may be simplified by referring the coordinates of a point to one system of circles, and the coordinates of any line or circle to the system cutting the former system orthogonally. Thus, if the system of reference be (1, 2, 3, 4), and (5, 6, 7, 8) denote the system orthogonal to this, then taking k_1, k_2, k_3, k_4 equal to unity, the equation of the circle (or line), whose coordinates referred to (5, 6, 7, 8) are ξ, η, ζ, ω, referred to the system (1, 2, 3, 4) is

$$\frac{x\xi}{\pi_{1,5}}+\frac{\eta y}{\pi_{2,6}}+\frac{z\zeta}{\pi_{3,7}}+\frac{w\omega}{\pi_{4,8}}=0.$$

The equation

$$ax+by+cz+dw=0$$

will represent a circle whose coordinates are

$$\xi=\frac{a\pi_{1,5}}{a+b+c+d},\ \eta=\&c.;$$

unless $a+b+c+d=0$, in which case the equation represents a straight line.

Inversion.—§§ 67, 68.

67. Let $xyzw$ be the power-coordinates of any point P with respect to the system of circles (1, 2, 3, 4); let P′ be the inverse of P with respect to any point O; then since by § 3, $\frac{\pi_{1,P}}{\sqrt{\pi_{0,1}\cdot\pi_{0,P}}}$ is unaltered by inversion, it follows that, if XYZW be the coordinates of P′ referred to the system which is the inverse of (1, 2, 3, 4), we must have

$$x=\alpha X,\ y=\beta Y,\ z=\gamma Z,\ w=\delta W,$$

where α, β, γ, δ are some constants.

Thus if the equation in power-coordinates of any curve be $f(xyzw)=0$, the equation to the inverse curve will be $f(\alpha X,\ \beta Y,\ \gamma Z,\ \delta W)=0$.

68. The system consisting of two rectangular axes, the point of intersection, and the line at infinity, is clearly the inverse of a system of two orthogonal circles, and their two points of intersection, the centre of inversion being one of these points.

For instance, the equation of a parabola expressed in power-coordinates is clearly

$$X^2=2\alpha YZ.$$

Hence the equation to the inverse of a parabola is of the form

$$x^2=2ayz,$$

x, y having reference to the orthogonal circles, z, w to their two points of intersection.

Similarly the equation to the inverse of a central conic must be of the form

$$\alpha x^2+\beta y^2=\gamma z^2,$$

α, β having the same or different signs according as the conic is an ellipse or hyperbola.

Chapter VI.—General Equation of the Second Degree in Power-Coordinates.

Nature of the Curve.—§§ 69, 70.

69. The most general equation of the second degree in power-coordinates may be written

$$\phi(x, y, z, w) \equiv ax^2+by^2+cz^2+dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw=0, \quad (93)$$

$(xyzw)$ being the coordinates of a point on the curve, referred to some system of circles, and therefore satisfying the equation of the absolute ψ, which is also of the second degree. Consequently the form (93) contains only eight arbitrary constants.

Now $xyzw$ may be expressed as linear functions of X^2+Y^2, X, Y, 1; (X, Y) being the Cartesian coordinates of the point; and substituting, it is easily seen that (93) may be expressed in the form

$$(X^2+Y^2)^2+U_1(X^2+Y^2)+U_2=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (94)$$

U_1, U_2 being of the first and second degree; this equation contains eight constants, and since (94) represents a curve having nodes at each of the circular points at infinity, it appears that (93) is a form to which every bicircular quartic can be reduced.

70. It is otherwise evident that, since the equation of a straight line is of the first degree, every straight line cuts ϕ in four points, unless ϕ is satisfied by the coordinates of the line at infinity; for these coordinates satisfy the equation of every straight line, and therefore in this case ϕ must represent a circular cubic and the line at infinity.

Equation to Tangent at any Point.—§§ 71, 72.

71. Let $(\xi\eta\zeta\omega)$ be the coordinates of any circle touching the curve ϕ at the point $(x'y'z'w')$. We must have, by equation (74)

$$\frac{\partial\psi}{\partial\xi}x'+\frac{\partial\psi}{\partial\eta}y'+\frac{\partial\psi}{\partial\zeta}z'+\frac{\partial\psi}{\partial\omega}w'=0,$$

and since this passes through the point $(x'+\delta x', y'+\delta y', z'+\delta z', w'+\delta w)$ we must have

$$\frac{\partial\psi}{\partial\xi}\delta x'+\frac{\partial\psi}{\partial\eta}\delta y'+\frac{\partial\psi}{\partial\zeta}\delta z'+\frac{\partial\psi}{\partial\omega}\delta w'=0;$$

also

$$\frac{\partial\psi}{\partial x'}\delta x'+\frac{\partial\psi}{\partial y'}\delta y'+\frac{\partial\psi}{\partial z'}\delta z'+\frac{\partial\psi}{\partial w'}\delta w'=0,$$

$$\frac{\partial\phi}{\partial x'}\delta x'+\frac{\partial\phi}{\partial y'}\delta y'+\frac{\partial\phi}{\partial z'}\delta z'+\frac{\partial\phi}{\partial w'}\delta w'=0.$$

Hence we must have

$$\frac{\dfrac{\partial\psi}{\partial\xi}}{\dfrac{\partial\phi}{\partial x'}+k\dfrac{\partial\psi}{\partial x'}}=\frac{\dfrac{\partial\psi}{\partial\eta}}{\dfrac{\partial\phi}{\partial y'}+k\dfrac{\partial\psi}{\partial y'}}=\frac{\dfrac{\partial\psi}{\partial\zeta}}{\dfrac{\partial\phi}{\partial z'}+k\dfrac{\partial\psi}{\partial z'}}=\frac{\dfrac{\partial\psi}{\partial\omega}}{\dfrac{\partial\phi}{\partial w'}+k\dfrac{\partial\psi}{\partial w'}}; \quad . \quad . \quad . \quad . \quad . \quad (95)$$

and every circle whose coordinates satisfy these equations must touch the curve $\phi=0$, at the point $(x'y'z'w')$.

72. Let $(\lambda\mu\nu\rho)$ be the coordinates of the tangent at the point $(x'y'z'w')$, then we must have

$$\frac{\dfrac{\partial\psi}{\partial\lambda}}{\dfrac{\partial\phi}{\partial x'}+k\dfrac{\partial\psi}{\partial x'}}=\frac{\dfrac{\partial\psi}{\partial\mu}}{\dfrac{\partial\phi}{\partial y'}+k\dfrac{\partial\psi}{\partial y'}}=\frac{\dfrac{\partial\psi}{\partial\nu}}{\dfrac{\partial\phi}{\partial z'}+k\dfrac{\partial\psi}{\partial z'}}=\frac{\dfrac{\partial\psi}{\partial\rho}}{\dfrac{\partial\phi}{\partial w'}+k\dfrac{\partial\psi}{\partial w'}}; \quad . \quad . \quad . \quad . \quad . \quad (96)$$

but, by § 56, we must have

$$k_1\frac{\partial\psi}{\partial\lambda}+k_2\frac{\partial\psi}{\partial\mu}+k_3\frac{\partial\psi}{\partial\nu}+k_4\frac{\partial\psi}{\partial\rho}=0,$$

where (k_1, k_2, k_3, k_4) are the coordinates of the line at infinity; hence, if k be determined by the equation

$$\left(k_1\frac{\partial}{\partial x'}+k_2\frac{\partial}{\partial y'}+k_3\frac{\partial}{\partial z'}+k_4\frac{\partial}{\partial w'}\right)(\phi+k\psi)=0; \quad . \quad . \quad . \quad . \quad . \quad (97)$$

then the coordinates of the tangent to the curve are given by (96), and the equation to it is

$$\left(x\frac{\partial}{\partial x'}+y\frac{\partial}{\partial y'}+z\frac{\partial}{\partial z'}+w\frac{\partial}{\partial w'}\right)(\phi+k\psi)=0,$$

or

$$\left(k_1\frac{\partial\phi}{\partial x'}+k_2\frac{\partial\phi}{\partial y'}+k_3\frac{\partial\phi}{\partial z'}+k_4\frac{\partial\phi}{\partial w'}\right)\left(x\frac{\partial\psi}{\partial x'}+y\frac{\partial\psi}{\partial y'}+z\frac{\partial\psi}{\partial z'}+w\frac{\partial\psi}{\partial w'}\right)$$
$$=\left(k_1\frac{\partial\psi}{\partial x'}+k_2\frac{\partial\psi}{\partial y'}+k_3\frac{\partial\psi}{\partial z'}+k_4\frac{\partial\psi}{\partial w'}\right)\left(x\frac{\partial\phi}{\partial x'}+y\frac{\partial\phi}{\partial y'}+z\frac{\partial\phi}{\partial z'}+w\frac{\partial\phi}{\partial w'}\right). \quad . \quad (98)$$

Bitangent Circles.—§§ 73–75.

73. A circle

$$\left(x\frac{\partial}{\partial x'}+y\frac{\partial}{\partial y'}+z\frac{\partial}{\partial z'}+w\frac{\partial}{\partial w'}\right)(\phi+k\psi)=0,$$

will clearly touch the curve ϕ at the point x'', y'', z'', w'' if

$$\frac{\frac{\partial\phi}{\partial x'}+k\frac{\partial\psi}{\partial x'}}{\frac{\partial\phi}{\partial x''}+k\frac{\partial\psi}{\partial x''}}=\frac{\frac{\partial\phi}{\partial y'}+k\frac{\partial\psi}{\partial y'}}{\frac{\partial\phi}{\partial y''}+k\frac{\partial\psi}{\partial y''}}=\frac{\frac{\partial\phi}{\partial z'}+k\frac{\partial\psi}{\partial z'}}{\frac{\partial\phi}{\partial z''}+k\frac{\partial\psi}{\partial z''}}=\frac{\frac{\partial\phi}{\partial w'}+k\frac{\partial\psi}{\partial w'}}{\frac{\partial\phi}{\partial w''}+k\frac{\partial\psi}{\partial w''}};$$

i.e., if k satisfy the equation

$$\mathrm{H}(\phi+k\psi)=0, \quad \ldots\ldots\ldots\ldots \quad (99)$$

where $\mathrm{H}(u)$ denotes the Hessian of u.

Since this equation is of the fourth degree in k, we infer that there are in general four systems of bitangent circles, each circle belonging to any system cutting a certain fixed circle orthogonally, the coordinates of this circle being proportional to the minors of the constituents of any row of the determinant $\mathrm{H}(\phi+k\psi)$.

74. If the coordinates of a bitangent circle satisfy the condition which must be satisfied by coordinates of any straight line, the corresponding equation will represent the double tangents from the centre of the corresponding circle. In general, then, there are eight double tangents.

75. It is clear that, if by any linear transformation of coordinates the equations $\phi=0$, $\psi=0$ become respectively $\Phi=0$, $\Psi=0$, then the same value of k must satisfy both

$$\mathrm{H}(\phi+k\psi)=0 \text{ and } \mathrm{H}(\Phi+k\Psi)=0.$$

Hence the coefficients of the powers of k in equation (99) are invariants.

Equation to Normal at any Point.—§§ 76–79.

76. Let $(\xi\eta\zeta\omega)$ be the coordinates of any circle which cuts the curve $\phi(xyzw)=0$, orthogonally at the point $(x'y'z'w')$, then by equation (75) we must have

$$\left(\xi\frac{\partial}{\partial x'}+\eta\frac{\partial}{\partial y'}+\zeta\frac{\partial}{\partial z'}+\omega\frac{\partial}{\partial w'}\right)(\phi+k\psi)=0, \quad \ldots\ldots \quad (100)$$

for all values of k.

77. It follows that if $(\lambda, \mu, \nu, \rho)$ be the coordinates of the normal at $(x'y'z'w')$ we must have

$$\lambda\frac{\partial\phi}{\partial x'}+\mu\frac{\partial\phi}{\partial y'}+\nu\frac{\partial\phi}{\partial z'}+\rho\frac{\partial\phi}{\partial w'}=0,$$

$$\lambda\frac{\partial\psi}{\partial x'}+\mu\frac{\partial\psi}{\partial y'}+\nu\frac{\partial\psi}{\partial z'}+\rho\frac{\partial\psi}{\partial w'}=0,$$

and

$$\lambda\frac{\partial\psi}{\partial k_1}+\mu\frac{\partial\psi}{\partial k_2}+\nu\frac{\partial\psi}{\partial k_3}+\rho\frac{\partial\psi}{\partial k_4}=0\ ;$$

(k_1, k_2, k_3, k_4) being the coordinates of the line at infinity. Hence the equation to the normal is

$$\begin{vmatrix} \frac{\partial\psi}{\partial x'} & \frac{\partial\psi}{\partial y'} & \frac{\partial\psi}{\partial z'} & \frac{\partial\psi}{\partial w} \\ \frac{\partial\phi}{\partial x''} & \frac{\partial\phi}{\partial y''} & \frac{\partial\phi}{\partial z''} & \frac{\partial\phi}{\partial w'} \\ \frac{\partial\psi}{\partial x''} & \frac{\partial\psi}{\partial y''} & \frac{\partial\psi}{\partial z'} & \frac{\partial\psi}{\partial w'} \\ \frac{\partial\psi}{\partial k_1'} & \frac{\partial\psi}{\partial k_2'} & \frac{\partial\psi}{\partial k_3'} & \frac{\partial\psi}{\partial k_4} \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101)$$

78. We can easily deduce from equation (101) that normals can be drawn from the point $(x'y'z'w')$ to the curve $\phi=0$, at its points of intersection with the curve

$$\begin{vmatrix} \frac{\partial\phi}{\partial x'} & \frac{\partial\phi}{\partial y'} & \frac{\partial\phi}{\partial z'} & \frac{\partial\phi}{\partial w} \\ \frac{\partial\psi}{\partial x'} & \frac{\partial\psi}{\partial y'} & \frac{\partial\psi}{\partial z'} & \frac{\partial\psi}{\partial w} \\ \frac{\partial\psi}{\partial x''} & \frac{\partial\psi}{\partial y''} & \frac{\partial\psi}{\partial z'} & \frac{\partial\psi}{\partial w'} \\ \frac{\partial\psi}{\partial k_1'} & \frac{\partial\psi}{\partial k_2'} & \frac{\partial\psi}{\partial k_3'} & \frac{\partial\psi}{\partial k_4} \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102)$$

This curve is clearly of the second degree, but since it is satisfied by (k_1, k_2, k_3, k_4) the coordinates of the line at infinity, it represents a circular cubic. Hence, in general, eight normals can be drawn from any given point to the curve.

79. In the case of a circle cutting the curve $\phi=0$, normally at the point $(x'y'z'w')$, we shall have

$$x'\frac{\partial\phi}{\partial\xi}+y'\frac{\partial\phi}{\partial\eta}+z'\frac{\partial\phi}{\partial\zeta}+w'\frac{\partial\phi}{\partial\omega}=0,$$

$$x'\frac{\partial\psi}{\partial\xi}+y'\frac{\partial\psi}{\partial\eta}+z'\frac{\partial\psi}{\partial}+w'\frac{\partial\psi}{\partial\omega}=0.$$

If then $(\xi\eta\zeta\omega)$ be chosen so that

$$\frac{\partial\phi}{\partial\xi}\Big/\frac{\partial\psi}{\partial\xi}=\frac{\partial\phi}{\partial\eta}\Big/\frac{\partial\psi}{\partial\eta}=\frac{\partial\phi}{\partial\zeta}\Big/\frac{\partial\psi}{\partial\zeta}=\frac{\partial\phi}{\partial\omega}\Big/\frac{\partial\psi}{\partial\omega}=-k \text{ say},$$

the circle $(\xi\eta\zeta\omega)$ will cut the curve ϕ orthogonally in four points, and we see at once, that k must satisfy the equation

$$\mathrm{H}(\phi+k\psi)=0\,;$$

and then $(\xi\eta\zeta\omega)$ are proportional to the minors of the constituents of any row in the determinant $\mathrm{H}(\phi+k\psi)$.

Thus it appears there are four circles which cut the curve $\phi=0$ orthogonally: and these circles are identical with the four which are mentioned in § 73, as being orthogonal respectively to the four systems of bitangent circles.

The Principal Circles.—§§ 80–82.

80. The four circles considered in § 79 have been called by MOUTARD the principal circles of the curve. And the curve may be considered as the envelope of a system of circles, which cut one of these principal circles orthogonally: it follows then that the curve is its own inverse with respect to any one of the principal circles (hence the principal circles must cut orthogonally); also the four points in which any principal circle cuts the curve must be cyclic points; so that there are in general sixteen cyclic points.

81. We may prove independently that the principal circles cut orthogonally, thus; taking for our system of reference an orthogonal system, so that the equation of the absolute is

$$\psi\equiv x^2+y^2+z^2+w^2=0\,;$$

then the coordinates of any principal circle being $(\xi\eta\zeta\omega)$ we must have, if

$$\phi\equiv ax^2+by^2+cz^2+dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw=0\,;$$

$$\left.\begin{aligned} a\xi+h\eta+g\zeta+l\omega&=-k\xi\\ h\xi+b\eta+f\zeta+m\omega&=-k\eta\\ g\xi+f\eta+c\zeta+n\omega&=-k\zeta\\ l\xi+m\eta+n\zeta+d\omega&=-k\omega \end{aligned}\right\}, \quad \ldots\ldots\ldots \quad (103)$$

where k is a root of the equation

$$\begin{vmatrix} a+k, & h, & g, & l \\ h, & b+k, & f, & m \\ g, & f, & c+k, & n \\ l, & m, & n, & d+k \end{vmatrix} = 0.$$

Let now $(\xi'\eta'\zeta'\omega')$ be the coordinates of the principal circle corresponding to another root k' of this equation; then multiplying equations (102) respectively by $(\xi'\eta'\zeta'\omega')$ and adding, we obtain at once

$$(k-k')(\xi\xi'+\eta\eta'+\zeta\zeta'+\omega\omega')=0.$$

Hence, if k, k' be unequal, the two circles must cut orthogonally.

82. If the curve have four principal circles—*i.e.*, if the roots of the discriminating quartic $\mathrm{H}(\phi+k\psi)=0$ are all different, the curve cannot have a third double point—for, inverting with respect to any principal circle, the inverse must also be a double point, unless the point lies on the principal circle; since then a quartic curve can have but three double points, in the case of a bicircular quartic, the third double point must lie on each principal circle. Hence two of the roots of the discriminating quartic must be equal, and there are only two principal circles.

Similarly, if this third double point be a cusp, it is easy to see, by inverting with respect to a principal circle, that any circle touching the tangent to the cusp at the cusp must touch the principal circle; and hence there is only one principal circle, and the discriminating quartic must have three equal roots.

Reduction of General Equation.—§§ 83, 84.

83. If one of the principal circles be a circle of reference (say $x=0$), then it is clear that the terms involving xy, xz, xw, must be absent from the equation. Supposing, then, that the equation $\mathrm{H}(\phi+k\psi)=0$ has all its roots unequal, then there are four principal circles, and taking these for circles of reference the equation must reduce to the form

$$ax^2+by^2+cz^2+dw^2=0.$$

Suppose, now, that two of the roots of the discriminant are equal; then taking the circles corresponding to the unequal roots, and two circles cutting them orthogonally as circles of reference, the equation will be of the form

$$ax^2+by^2+cz^2+dw^2+2nzw=0;$$

and the system of reference being orthogonal we have for the absolute

$$x^2+y^2+z^2+w^2=0;$$

and, therefore, the discriminant is

$$(k+a)(k+b)\{(k+c)(k+d)-n^2\}=0,$$

which can only have equal roots when

$$(c+d)^2-4cd+n^2=0.$$

If n be real this can only be satisfied by $n=0$, $c=d$.

But of our four circles of reference one must be imaginary; if one of the circles (x, y) is imaginary n will be real, and our equation will reduce to

$$ax^2+by^2+cz^2+cw^2=0,$$

which represents a pair of circles, and need not be considered.

If both circles (x, y) are real, then it is simplest to take as system of reference, these circles and their two points of intersection. So that the absolute is of the form

$$x^2+y^2=4zw,$$

and the discriminant becomes

$$(k+a)(k+b)\{cd-(2k-n)^2\}=0;$$

and, then, if this has equal roots, either $c=0$ or $d=0$; and the equation takes the form

$$ax^2+by^2+cz^2+2nzw=0,$$

which, by means of the absolute, may be written

$$ax^2+by^2+cz^2=0.$$

Let us suppose now that three of the roots of the discriminant are equal; referring our coordinates to the circle corresponding to the unequal root, and any three circles cutting it and one another orthogonally, the equation of the curve will reduce to

$$ax^2+by^2+cz^2+dw^2+2fyz+2myw+2nzw=0;$$

and the discriminating cubic is

$$(k+a)\begin{vmatrix} k+b, & f, & m \\ f, & k+c, & n \\ m, & n, & k+d \end{vmatrix}=0,$$

which we can easily prove can only have three equal roots when

$$f=m=n=0,\ b=c=d;$$

provided that f, m, n are all real, in which case the circle $x=0$ is imaginary.

In this case the curve takes the form

$$ax^2+by^2+bz^2+bw^2=0,$$

which represents a point.

If, however, $x=0$ be real, we may take as system of reference this circle, a circle cutting it orthogonally, and their two points of intersection; then, since the absolute is of the form

$$x^2+y^2=4zw,$$

the discriminant becomes

$$(k+a)\begin{vmatrix} k+b, & f, & m \\ f, & c, & n-2k \\ m, & n-2k, & d \end{vmatrix}=0;$$

which can only have three equal roots when $2b=-n$, $m=d=0$; in which case the equation of the curve takes the form

$$ax^3+by^3-4bzw+2fyz+cz^2=0,$$

and by taking instead of $y=0$, the circle $y+\lambda z$, which clearly cuts x orthogonally, we can get rid of the term cz^2.

And since

$$x^2+y^2=4zw,$$

this equation can be further reduced to the form

$$ax^3+2fyz=0.$$

84. Thus we see that the equation of a bi-circular quartic can be reduced to one of three forms:—

(A.) $$ax^2+by^2+cz^2+dw^2=0,$$

in which case there are four principal circles, the equation of the absolute being

$$x^2+y^2+z^2+w^2=0.$$

(B.) $$ax^2+by^2+cz^2=0,$$

the equation of the absolute being

$$x^2+y^2=4zw,$$

in which case there are two principal circles, which must be real, and a node which is one of the points of intersection of these circles.

(C.) $$ax^2+2fyz=0,$$

the equation of the absolute being

$$x^2+y^2=4zw.$$

In this case there is only one principal circle ($x=0$); the curve passes through the two common points of (x, y), and the point $(x=0, y=0, z=0)$ is a cusp on the curve.

It is also clear that circular cubics can be reduced to one of these three forms: since we have seen that the equation of the second degree represents a cubic when it is satisfied by the coordinates of the line at infinity.

Chapter VII.—Classification of Bicircular Quartics.

i. *Method.*—§ 85.

85. It will be convenient to take as the basis of our classification the nature of the roots of the discriminating quartic; we shall thus have three species, each of which may be subdivided into two—according as the double points at infinity are nodes or cusps. We shall then have three similar species of circular cubics.

Using the notation employed by Salmon ('Higher Plane Curves,' § 82), we shall denote the characteristics of a curve by $(m, n, \delta, \tau, \kappa, \iota)$, and we see that we shall have the following cases :—

	m	n	δ	τ	κ	ι	Name.
i.	4	8	2	8	0	12	
ii.	4	6	0	1	2	8	Cartesian
iii.	4	6	3	4	0	6	
iv.	4	4	1	1	2	2	Limaçon
v.	4	5	2	2	1	4	
vi.	4	3	0	0	3	0	Cardioid
vii.	3	6	0	0	0	9	
viii.	3	4	1	0	0	3	
ix.	3	3	0	0	1	1	

(i.) may be called the general bicircular quartic; (iii.) is the general inverse of a conic; (v.) is the inverse of a parabola; (vii.) may be called the general circular cubic; (viii.) is the inverse of a conic with respect to a point on the curve; and (ix.) is the inverse of a parabola with respect to a point on the curve.

General Bicircular Quartic.—§§ 86–92.

86. The equation of the curve may be written

$$ax^2+by^2+cz^2+dw^2=0;$$

and if we write the absolute

$$\psi\equiv x^2+y^2+z^2+w^2=0,$$

the coordinates of the line at infinity will be $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{r_4}$, *i.e.*, the reciprocals of the radii of the principal circles.

87. The coordinates of any tangent circle at the point $(x'y'z'w')$ will be proportional to

$$(a+k)x',\ (b+k)y',\ (c+k)z',\ (d+k)w'.$$

The equation to the tangent at the point is, by equation (98),

$$(x'x+y'y+z'z+w'w)\left(\frac{ax'}{r_1}+\frac{by'}{r_2}+\frac{cz'}{r_3}+\frac{dw'}{r_4}\right)$$
$$=(ax'x+by'y+cz'z+dw'w)\left(\frac{x'}{r_1}+\frac{y'}{r_2}+\frac{z'}{r_3}+\frac{w'}{r_4}\right) \quad . \quad . \quad . \quad . \quad . \quad (104)$$

The equation to the normal at the point $(x'y'z'w')$ is, by equation (101),

$$\begin{vmatrix} x, & y, & z, & w \\ x', & y', & z', & w' \\ ax', & by', & cz', & dw' \\ \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4} \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105)$$

88. The coordinates of any bitangent circle being $(\xi, \eta, \zeta, 0)$ we must have

$$\frac{\xi}{(a-d)x'}=\frac{\eta}{(b-d)y'}=\frac{\zeta}{(c-d)z'}$$

Hence we must have

$$\frac{\xi^2}{a-d}+\frac{\eta^2}{b-d}+\frac{\zeta^2}{c-d}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (106)$$

89. The pair of double tangents which belong to this system of bitangent circles are given by

$$\frac{\lambda^2}{a-d}+\frac{\mu^2}{b-d}+\frac{\nu^2}{c-d}=0,$$

where

$$\frac{\lambda}{r_1}+\frac{\mu}{r_2}+\frac{\nu}{r_3}=0.$$

If ϕ be the angle between them, we can deduce at once from § 65, remembering that

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}=0\,;$$

$$\tan\phi=\frac{\frac{2}{r_4}\left\{\frac{1}{(a-d)(b-d)(c-d)}\left(\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}+\frac{d}{r_4^2}\right)\right\}^{\frac{1}{2}}}{\frac{1}{a-d}\left(\frac{1}{r_2^2}+\frac{1}{r_3^2}\right)+\frac{1}{b-d}\left(\frac{1}{r_3^2}+\frac{1}{r_1^2}\right)+\frac{1}{c-d}\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)} \quad . \quad . \quad . \quad . \quad (107)$$

90. Since the foci may be considered as bitangent circles whose radii are indefinitely

small, the coordinates of the four foci belonging to the system, given by equation (106), will be given by

$$\frac{x^2}{a-d}+\frac{y^2}{b-d}+\frac{z^2}{c-d}=0,$$

and

$$x^2+y^2+z^2=0\,;$$

whence

$$\frac{x^2}{(b-c)(a-d)}=\frac{y^2}{(c-a)(b-d)}=\frac{z^2}{(a-b)(c-d)}. \quad \ldots\ldots\ldots \quad (108)$$

91. From the form of these equations it follows, that all curves whose equations are of the form

$$\frac{x^2}{\alpha^2+k}+\frac{y^2}{\beta^2+k}+\frac{z^2}{\gamma^2+k}+\frac{w^2}{\delta^2+k}=0, \quad \ldots\ldots\ldots \quad (109)$$

and

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}+\frac{w^2}{\delta^2}=0, \quad \ldots\ldots\ldots \quad (110)$$

will have the same foci.

Subtracting these equations, we have

$$\frac{x^2}{\alpha^2(\alpha^2+k)}+\frac{y^2}{\beta^2(\beta^2+k)}+\frac{z^2}{\gamma^2(\gamma^2+k)}+\frac{w^2}{\delta^2(\delta^2+k)}=0.$$

Hence the circles whose coordinates are respectively proportional to

$$\frac{x}{\alpha^2},\quad \frac{y}{\beta^2},\quad \frac{z}{\gamma^2},\quad \frac{w}{\delta^2}\,;$$

$$\frac{x}{\alpha^2+k},\quad \frac{y}{\beta^2+k},\quad \frac{z}{\gamma^2+k},\quad \frac{w}{\delta^2+k}\,;$$

must cut orthogonally; but these circles touch the curves given by (109), (110) at their common points; hence confocal bicircular quartics cut orthogonally.

Through any point, two quartics can be drawn confocal with a given bicircular quartic, since the equation (109) is a quadratic in k. We see, too, that two circular cubics can be drawn confocal with a given bicircular quartic.

92. Let $(\xi\eta\zeta\omega)$ be the coordinates of any circle S; this will cut orthogonally one of the bitangent circles, at the point $(x'y'z'w')$ on the curve

$$ax^2+by^2+cz^2+dw^2=0,$$

if

$$(a-d)x'\xi+(b-d)y'\eta+(c-d)z'\zeta=0.$$

It follows that two bitangent circles belonging to this system can be drawn to cut S orthogonally; and their four points of contact lie on the circle

$$(a-d)\xi x+(b-d)\eta y+(c-d)\zeta z=0.$$

This circle intersects S in points lying on the circle

$$a\xi x+b\eta y+c\zeta z+d\omega w=0.$$

Hence, given any circle S, four pairs of bitangent circles can be drawn to any bicircular quartic, cutting S orthogonally; and their points of contact lie on four circles, which have with S a common radical axis.

ii. *Cartesian Oval.*—§§ 93, 94.

93. If one of the principal circles has its radius infinite, the curve will be symmetrical with respect to the axis, which will pass through the centres of the other three principal circles. If the foci which lie on this axis coincide with these centres, the curve must have cusps at the circular points at infinity. Let us suppose the circle, whose radius was r_4 in § 86, to become the axis; then by § 90, the coordinates of the foci on the axis will be

$$\frac{x^2}{(b-c)(a-d)}=\frac{y^2}{(c-a)(b-d)}=\frac{z^2}{(a-b)(c-d)}.$$

If these points are the centres of the principal circles we must have

$$r_1^2(b-c)(a-d)=r_2^2(c-a)(b-d)=r_3^2(a-b)(c-d); \quad . \quad . \quad . \quad (111)$$

which is equivalent to only one relation between the coefficients, viz.:—

$$\frac{1}{a-d}\cdot\frac{1}{r_1^2}+\frac{1}{b-d}\cdot\frac{1}{r_2^2}+\frac{1}{c-d}\cdot\frac{1}{r_3^2}=0,$$

since

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=0.$$

Again the double tangents at right angles to the axis are given by

$$\frac{\lambda^2}{a-d}+\frac{\mu^2}{b-d}+\frac{\nu^2}{c-d}=0,$$

$$\frac{\lambda}{r_1}+\frac{\mu}{r_2}+\frac{\nu}{r_3}=0;$$

which are clearly satisfied by taking

$$\lambda r_1=\mu r_2=\nu r_3;$$

thus one of them coincides with the line at infinity, and so there is but one proper double tangent.

94. There are two finite foci on each principal circle, their coordinates being respectively proportional to

$$\begin{array}{cccc} 0, & \frac{1}{r_3}, & \frac{1}{r_2}, & \pm\frac{1}{r_1}, \\ \frac{1}{r_3}, & 0, & \frac{1}{r_1}, & \pm\frac{1}{r_2}, \\ \frac{1}{r_2}, & \frac{1}{r_1}, & 0, & \pm\frac{1}{r_3}. \end{array}$$

But they are all imaginary.

iii. *Bicircular Quartics having a Third Node.*—§§ 95–102.

95. The equation of the curve may be reduced (by § 83) to the form

$$ax^2+by^2+cz^2=0,$$

the equation of the absolute being

$$x^2+y^2=4zw\,;$$

and if r_1, r_2 are the radii of the two principal circles, e the distance between their points of intersection, then the coordinates of the line at infinity are $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{e}, \frac{1}{e}$.

96. The coordinates of any circle touching the curve at the point $(x'y'z'w')$ must be proportional to $(\xi, \eta, \zeta, \omega)$, where

$$\frac{\xi}{(a+k)x'}=\frac{\eta}{(b+k)y'}=\frac{-2\omega}{cz'-2kw'}=\frac{-2\zeta}{-2kz'}; \quad \ldots\ldots\ldots \quad (112)$$

and, by § 72, the equation to the tangent line at $(x'y'z'w')$ will be

$$\left(\frac{ax'}{r_1}+\frac{by'}{r_2}+\frac{cz'}{e}\right)(xx'+yy'-2zw'-2wz')=\left(\frac{x'}{r_1}+\frac{y'}{r_2}-2\frac{w'+z'}{e}\right)(axx'+byy'+czz')\,; \quad (113)$$

also the equation to the normal at the point $(x'y'z'w')$ will be, by equation (101),

$$\begin{vmatrix} x, & y, & -2w, & -2z \\ x', & y', & -2w', & -2z' \\ ax', & by', & cz', & 0 \\ \frac{1}{r_1}, & \frac{1}{r_2}, & -\frac{2}{e}, & -\frac{2}{e} \end{vmatrix} =0. \quad \ldots\ldots\ldots \quad (114)$$

97. The circle $(\xi\eta\zeta\omega)$ given by (112) will be a bitangent circle, if $k=-a$ or $-b$. In the former case we have

$$\frac{\eta}{(b-a)y'}=\frac{-2\zeta}{+2az'}=\frac{-2\omega}{cz'+2aw'}=\frac{c\zeta-2a\omega}{2a^2w'},$$

Hence

$$\frac{\eta^2}{b-a}-\frac{c\zeta^2}{a^2}+\frac{4\omega\zeta}{a}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

98. The double tangents which belong to this system of bitangent circles are given by

$$\frac{\mu^2}{b-a}-\frac{c\nu^2}{a^2}+\frac{4\nu\rho}{a}=0,$$

and

$$\frac{\mu}{r_3}-\frac{2\nu}{e}-\frac{2\rho}{e}=0.$$

If ϕ be the angle between them, we shall have, since

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}=\frac{4}{e^2},$$

$$\tan\phi=\frac{\frac{2}{ar_1}\left\{\frac{1}{b-a}\left(\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{e^2}\right)\right\}^{\frac{1}{2}}}{\frac{2}{ar_2^2}-\frac{c}{a^2e^2}-\frac{4}{ae^2}+\frac{4}{b-a}\frac{1}{e^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116)$$

Hence these double tangents cannot coincide unless

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{e_2^2}=0,$$

in which case, the double tangents from the centre of the other principal circle coincide also; but this equation is the condition that the curve should be a cubic.

99. If in (112) we take $k=0$, we have a series of tangent circles passing through the node, two of these circles will reduce to straight lines, which will be the tangents from the node; in this case we shall have

$$\frac{\lambda}{ax'}=\frac{\mu}{by'}=\frac{\nu}{0}=\frac{-2\rho}{cz'},$$

whence

$$\frac{\lambda^2}{a}+\frac{\mu^2}{b}+\frac{4\rho^2}{c}=0;$$

where

$$\frac{\lambda}{r_1}+\frac{\mu}{r_2}-\frac{2\rho}{e}=0;$$

and the angle between them is given by

$$\tan\phi=\frac{\left\{\frac{-4e}{abc}\left(\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{e^2}\right)\right\}^{\frac{1}{2}}}{\frac{4}{c}+\frac{1}{a}+\frac{1}{b}}.$$

These also coincide if the curve is a cubic.

100. The foci corresponding to the principal circle ($x=0$) are given by

$$\frac{y^2}{b-a}-\frac{cz^2}{a^2}+\frac{4wz}{a}=0,$$

and

$$y^2=4zw.$$

Hence we have

$$\frac{y^2}{c(b-a)}=\frac{z^2}{ab}=\frac{4zw}{c(b-a)} \quad \ldots\ldots\ldots \quad (117)$$

Similarly the coordinates of the two foci on the circle $y=0$, may be written down.

101. From the form of equations (117), it appears that all curves given by the equation

$$\frac{x^2}{a^2+\kappa}+\frac{y^2}{b^2+\kappa}+\frac{z^2}{c^2}=0,$$

are confocal with the curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0.$$

Subtracting, we have

$$\frac{x^2}{a^2(a^2+\kappa)}+\frac{y^2}{b^2(b^2+\kappa)}=0.$$

Hence the circles whose coordinates are respectively proportional to

$$\frac{x}{a^2}, \quad \frac{y}{b^2}, \quad 0, \quad \frac{-z}{2c^2};$$

$$\frac{x}{a^2+\kappa}, \quad \frac{y}{b^2+\kappa}, \quad 0, \quad \frac{-z}{2c^2};$$

must cut orthogonally, but these circles touch the curves at their common points; hence confocal curves cut orthogonally.

Since we have a quadratic to determine κ when (xyz) are given, through any point two curves can be drawn confocal with a given curve; and two nodal circular cubics can also be drawn with the same node and confocal with a given nodal bicircular quartic.

102. The equation

$$ax^2+by^2+cz^2=0$$

represents in general a nodal bicircular quartic, and by inverting with respect to the node, we see that it is the inverse curve of an ellipse or hyperbola, according as a and b have the same or opposite signs, with respect to some point in the same plane.

Such a curve then has two principal circles, with two single foci on each: it has also four double tangents, two from the centre of each principal circle.

If one of the principal circles becomes a straight line, it divides the curve

symmetrically; corresponding, in fact, to the case where a conic is inverted on a point in one of its axes.

If the radii of both principal circles are infinite the centre of inversion is the centre of the conic: *e.g.*, the lemniscate, whose equation would be of the form $z^2=a^2(x^2+y^2)$.

iv. *The Limaçon.*—§ 103.

103. If one of the principal circles, in the last section, becomes a straight line, and one of the two foci coincide with the centre of the principal circle, the nodes at infinity become cusps. This case corresponds to inversion of a conic on a focus.

Suppose that r_2 is infinite in § 95, then the condition that the curve should be a Limaçon is, from equation (117),

$$4r_1^2c(a-b)=4e^2ab=e^2c(a-b)\ ;$$

which, since we must have

$$\frac{1}{r_1^2}=\frac{4}{e^2},$$

becomes

$$4ab=c(a-b). \qquad \ldots\ldots\ldots\ldots \quad (118)$$

The double tangents perpendicular to the axis are given by

$$\frac{\lambda^2}{a-b}-\frac{\sigma^2}{b^2}+\frac{4\nu\rho}{b}=0,$$

and

$$\frac{\lambda}{r_1}-\frac{2\nu}{e}-\frac{2\rho}{e}=0\ ;$$

which equations are satisfied by $\lambda r_1=\nu e=\rho e$: so that there is only one double tangent.

v. *Bicircular Quartic having a Cusp.*—§§ 104–109.

104. The equation of the curve may be reduced (by § 83) to the form

$$x^2=2ayz.$$

The system of reference being the principal circle ($x=0$); the circle ($y=0$) passing through the cusp, and the other point in which the curve cuts the principal circle; the cusp ($z=0$); and the other point ($w=0$) common to the curve and principal circle.

Let r_1, r_2 be the radii of the two circles, e the distance between their points of

intersection; $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{e}, \frac{1}{e}$ the coordinates of the line at infinity; then the equation to the absolute will be

$$x^2+y^2=4zw.$$

The curve is clearly the inverse of a parabola.

105. If $(\xi\eta\zeta\omega)$ be the coordinates of any circle touching the curve at the point $(x'y'z'w')$, we must have

$$\frac{\xi}{(1+k)x'}=\frac{\eta}{-az'+ky'}=\frac{-2\omega}{-ay'-2kw'}=\frac{-2\zeta}{-2kz'} \quad . \quad . \quad . \quad . \quad . \quad . \quad (119)$$

The equation to the tangent line at the point $(x'y'z'w')$ will be, by § 72,

$$\left(\frac{1}{r_1}x'-\frac{a}{r_2}z'-\frac{a}{e}y'\right)(xx'+yy'-2zw'-2wz')$$
$$=\left(\frac{1}{r_1}x'+\frac{1}{r_2}y'-2\frac{z'+w'}{e}\right)(xx'-az'y-azy') \; ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (120)$$

the equation to the normal being

$$\begin{vmatrix} x, & y, & -2w, & -2z \\ x', & y', & -2w', & -2z' \\ x', & -az', & -ay', & 0 \\ \frac{1}{r_1}, & \frac{1}{r_2}, & -\frac{2}{e}, & -\frac{2}{e} \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad (121)$$

106. The circle $(\xi\eta\zeta\omega)$ will be a bitangent circle, if $k=-1$, in which case

$$\frac{\eta}{az'+y'}=\frac{\zeta}{z'}=\frac{2\omega}{2w'-ay'}=\frac{\eta-a\zeta}{y'}=\frac{2\omega+a\eta-a^2\zeta}{2w'} \; ;$$

whence

$$(\eta-a\zeta)^2=4\zeta\omega. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122)$$

107. The two double tangents are given by

$$\mu^2-2a\mu\nu+a^2\nu^2-4\nu\rho=0,$$
$$\frac{\mu}{r_2}-\frac{2\nu}{e}-\frac{2\rho}{e}=0 \; ;$$

the angle ϕ between them being given by

$$\tan\phi=\frac{\frac{2}{r_1}\left\{\left(\frac{2a}{er_2}-\frac{1}{r_1^2}\right)\right\}^{\frac{1}{2}}}{\frac{a^2+8}{e^2}-\frac{2a}{er_2}-\frac{2}{r_2^2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123)$$

These can only coincide if

$$\frac{2a}{er_2}=\frac{1}{r_1^2};$$

in which case the curve is a circular cubic.

108. Taking $k=0$ in (119), we have a series of tangent circles passing through the cusp; to find the tangents from the cusp to the curve we shall have

$$\frac{\lambda}{x'}=\frac{\mu}{-az'}=\frac{-2\rho}{-ay'},$$

whence

$$a\lambda^2+4\mu\rho=0;$$

where

$$\frac{\lambda}{r_1}+\frac{\mu}{r_2}-\frac{2\rho}{e}=0.$$

The angle between these is given by

$$\tan\phi=\frac{2\left\{\frac{1}{r_1^2}-\frac{2a}{er_2}\right\}^{\frac{1}{2}}}{\frac{a}{e}+\frac{2}{r_2}}.$$

109. The focus of the curve is given by

$$(y-az)^2=4wz=y^2,$$

whence

$$\frac{y}{a}=\frac{z}{2}=\frac{8w}{a^2}. \quad \ldots\ldots\ldots\ldots \quad (124)$$

vi. *The Cardioid.*—§ 110.

110. If the radius r_1 of the principal circle of the curve

$$x^2=2ayz$$

become infinite, the curve is symmetrical with respect to the axis; and if the single focus on this axis is at an infinite distance, the curve has cusps at infinity, and is called the cardioid; being the inverse of a parabola with respect to its focus.

Referring to equation (124), we see that the condition is

$$a=4;$$

so that the equation to a cardioid is

$$x^2=8yz. \quad \ldots\ldots\ldots\ldots \quad (125)$$

vii. *General Circular Cubic.*—§§ 111, 112.

111. We have already seen that the equation

$$ax^2+by^2+cz^2+dw^2=0,$$

represents a circular cubic when

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}+\frac{d}{r_4^2}=0, \quad \ldots\ldots\ldots \quad (126)$$

where r_1, r_2, r_3, r_4 are the radii of the four principal circles. The curve also passes through the centre of each of these circles.

By equation (104) we see that the equation to the asymptote (*i.e.*, tangent at the point $\left(\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{r_4}\right)$ is

$$\frac{ax}{r_1}+\frac{by}{r_2}+\frac{cz}{r_3}+\frac{dw}{r}=0; \quad \ldots\ldots\ldots \quad (127)$$

the equation to the tangent at the centre of the circle ($x=0$) being

$$\frac{a-b}{r_2}y+\frac{a-c}{r_3}z+\frac{a-d}{r_4}w=0,$$

which is clearly parallel to the asymptote.

Hence the tangents to the curve at the centres of the principal circles are all parallel to the asymptote.

112. As in the case of the general bicircular quartic, there will be four systems of bitangent circles, and on each principal circle there will be four single foci. There are clearly no double tangents. And if one of the principal circles degenerates into a straight line, the asymptote is perpendicular to it.

viii. *Nodal Circular Cubic.*—§ 113.

113. The equation considered in § 95,

$$ax^2+by^2+cz^2=0,$$

is a circular cubic, when

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{e^2}=0.$$

This curve is the inverse of a conic with respect to a point on the curve. The curve passes through the centres of the principal circles, the tangents being respectively, by equation (113),

$$(a-b)\frac{y}{r_2}-\frac{2a+c}{c}z-2\frac{a}{c}w=0,$$

$$\frac{(b-a)}{r_1}x-\frac{2b+c}{c}z-2\frac{b}{c}w=0\ ;$$

which are parallel to the straight line

$$\frac{ax}{r_1}+\frac{by}{r_2}+\frac{cz}{c}=0,$$

which is the line joining the node to the point in which the line at infinity cuts the curve, so that these three lines are parallel to the asymptote.

As in the case of the nodal bicircular quartic there will be two single foci on each principal circle, and two corresponding systems of bitangent circles.

ix. *Cuspidal Circular Cubic.*—§ 114.

114. The equation

$$x^3=2ayz$$

represents a circular cubic, having the point $z=0$ for a cusp, when

$$\frac{1}{r_1^3}=\frac{2a}{cr_2}\ ;$$

r_1 being the radius of its principal circle, r_2 that of a circle cutting this orthogonally, and passing through the cusp, and the other point common to the curve and its principal circle.

The curve clearly passes through the centre of its principal circle, the tangent at the point being

$$y\left(\frac{1}{r_2}+\frac{a}{c}\right)-\frac{cz}{2r_2^2}-\frac{2w}{r_2}=0,$$

which is parallel to

$$\frac{x}{r_1}-\frac{ay}{c}-\frac{az}{r_2}=0,$$

the line joining the cusp to the third point in which the curve cuts the line at infinity; hence these are parallel to the asymptote.

The curve has one syttem of bitangent circles, and one focus which lies on the principal circle.

Chapter VIII.—Miscellaneous Theorems.

Equation of an Anallagmatic Curve referred to Three Circles Orthogonal to the same Principal Circle.—§§ 115–121.

115. If the system of circles (1, 2, 3, 4) be such that (4) is orthogonal to (1, 2, 3), the equation of the absolute must be of the form

$$w^2+f(xyz)=0;$$

also if (4) be a principal circle of an anallagmatic curve, its equation must be of the same form; by subtraction we see that the equation,

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (128)$$

may be considered as the general equation of such a curve referred to any three circles cutting one of its principal circles orthogonally.

Thus, for any theorem proved in the case of conics we can easily derive an analogous theorem for bicircular quartics or circular cubics.

116. Any bitangent circle of the system which cuts the given principal circle orthogonally, must have for its equation

$$\alpha x+\beta y+\gamma z=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129)$$

and since it touches (128), we shall have

$$\begin{vmatrix} a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (130)$$

Referring to § 24, equation (22), we see that α, β, γ are proportional to the areal coordinates of the centre of the circle (129) referred to the triangle formed by joining the centres of the circles (1, 2, 3), provided that x, y, z are proportional to the powers of a point with respect to these circles.

We see, then, by equation (130), that the locus of the centres of all bitangent circles of the same system is a conic; which is called by Dr. Casey the focal conic of the system.

117. Suppose now the circles (1, 2, 3) to be the other principal circles, then the equation to the curve must be of the form

$$ax^2+by^2+cz^2=0,$$

and the corresponding focal conic is

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}=0.$$

And we see that the focal conic corresponding to one principal circle is self-conjugate with respect to the triangle formed by the centres of the other three.

118. Or, again, in the case of a nodal curve; let (1, 2, 3) denote the other principal circle, and its two points of intersection with (4); the equation of the curve is of the form

$$ax^2+by^2+2fyz=0;$$

and the corresponding focal conic

$$f^2\alpha^2-ab\gamma^2+2af\beta\gamma=0,$$

which also passes through the node.

119. If the system of reference be three bitangent circles, the equation to the curve must be of the form

$$a\sqrt{x}+b\sqrt{y}+c\sqrt{z}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (131)$$

and in that case the focal conic is

$$\frac{\alpha^2}{a}+\frac{\beta^2}{\beta}+\frac{\gamma^2}{\gamma}=0.$$

In particular we see that if A, B, C be three foci on the same principal circle of the quartic, or cubic, and P any point on the curve, we must have

$$a.\mathrm{AP}+b.\mathrm{BP}+c.\mathrm{CP}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132)$$

120. Suppose that the curve is a Cartesian, having cusps at infinity, then the focal conics become circles.

It follows that equation (131) will represent a Cartesian if a, b, c are proportional to the sides of the triangle formed by the centres of the three bitangent circles (xyz). Thus we have the theorem that, the sum of the products of the tangents, from any point on a Cartesian to any three bitangent circles of the same system, into the corresponding sides of the triangle, formed by the centres of the circles, is zero.

121. Let the circles (2, 3) be any two bitangent circles, and let (1) be the circle passing through their four points of contact with the curve; then the equation of the quartic must take the form

$$x^2=2fyz,$$

and then the focal conic is

$$f\alpha^2=2\beta\gamma.$$

If (y, z) be foci, (x) might be called their directrix; and we see that the product of the distances of any point on a bicircular quartic from two foci on the same principal circle, is proportional to the square of the tangent from the point to their directrix.

Again, if (yz) denote the double tangents from the centre of the principal circle of the system, we see that the centre of the focal conic coincides with the centre of the polar circle of the centre of the principal circle, and the asymptotes of the focal conic are perpendicular to the double tangents. It follows, then, that the focal conic is an hyperbola or ellipse, according as these tangents are real or imaginary.

It follows, also, that the focal conic of a circular cubic is a parabola, whose axis is perpendicular to the asymptote of the cubic.

Circle of Curvature at any Point of an Anallagmatic Curve.—§§ 122–124.

122. Let the equation to the circle of curvature at the point $(x'y'z'w')$ be

$$\xi x+\eta y+\zeta z+\omega w=0.$$

Then we must have ξ, η, ζ, ω, proportional to the minors of x, y, z, w, in the determinant

$$\begin{vmatrix} x, & y, & z, & w \\ x', & y', & z', & w' \\ \delta x', & \delta y', & \delta z', & \delta w' \\ \delta^2 x', & \delta^2 y', & \delta^2 z', & \delta^2 w' \end{vmatrix}.$$

And as we are merely concerned with the ratios of the coordinates (x, y, z, w), we may take w constant; so we shall have

$$\frac{\xi}{\delta y'\delta^2 z'-\delta z'\delta^2 y'}=\frac{\eta}{\delta z'\delta^2 x'-\delta x'\delta^2 z'}=\frac{\zeta}{\delta x'\delta^2 y'-\delta y'\delta^2 x'}.$$

123. If the equation of the curve referred to its principal circles be

$$ax^2+by^2+cz^2+dw^2=0,$$

where the equation to the absolute is

$$x^2+y^2+z^2+w^2=0;$$

we shall have

$$ax\delta x+by\delta y+cz\delta z=0,$$
$$x\delta x+y\delta y+z\delta z=0;$$

whence

$$\frac{x\delta x}{b-c}=\frac{y\delta y}{c-a}=\frac{z\delta z}{a-b}=u \text{ say};$$

hence

$$\delta^2 x=(b-c)\left(\frac{\delta u}{x}-\frac{u\delta x}{x^2}\right),$$

$$\delta^2 y=(c-a)\left(\frac{\delta u}{y}-\frac{u\delta y}{y^2}\right),$$

$$\delta^2 z=(a-b)\left(\frac{\delta u}{z}-\frac{u\delta z}{z^2}\right);$$

so that

$$\xi=(c-a)(a-b)\frac{u^3}{y'z'}\left(\frac{\delta x'}{x'}-\frac{\delta y'}{y'}\right)$$

$$=(c-a)(a-b)\frac{u^3}{y'z'}\left(\frac{a-b}{x'^2}-\frac{c-a}{y'^2}\right);$$

or since

$$(a-b)y'^2+(a-c)x'^2+(a-d)w'^2=0,$$

$$\xi=(a-b)(a-c)\frac{u^3}{y'^2z'^2}(d-a)w'^2.$$

Hence

$$\frac{\xi}{(a-b)(a-c)(a-d)x'^2}=\frac{\eta}{(b-a)(b-c)(b-d)y'^2}=\frac{\zeta}{(c-a)(c-b)(c-d)z'^2}=\frac{\omega}{(d-a)(d-b)(d-c)w'^2}.$$

So that the equation of the circle of curvature at the point $(x'y'z'w')$ on the curve

$$ax^2+by^2+cz^2+dw^2=0,$$

is

$$(a-b)(a-c)(a-d)x'^2x+(b-a)(b-c)(b-d)y'^2y$$
$$+(c-a)(c-b)(c-d)z'^2z+(d-a)(d-b)(d-c)w'^2w=0; \quad . \quad . \quad . \quad (133)$$

and the points of inflexion of the curve lie on the tricircular sextic,

$$\frac{(a-b)(a-c)(a-d)}{r_1}x^3+\frac{(b-a)(b-c)(b-d)}{r_2}y^3+\frac{(c-a)(c-b)(c-d)}{r_3}z^3$$
$$+\frac{(d-a)(d-b)(d-c)}{r_4}w^3=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134)$$

124. If R be the radius of curvature at the point $(x'y'z'w')$ of the curve

$$ax^2+by^2+cz^2+dw^2=0,$$

we shall have

$$R^2=\frac{\xi^2+\eta^2+\zeta^2+\omega^2}{\left(\frac{\xi}{r_1}+\frac{\eta}{r_2}+\frac{\zeta}{r_3}+\frac{\omega}{r_4}\right)^2}.$$

And it may be easily verified that

$$\xi^3+\eta^3+\zeta^3+\omega^3=(a^2x'^2+b^2y'^2+c^2z'^2+d^2w'^2)^3;$$

so that

$$\frac{(a^2x'^2+b^2y'^2+c^2z'^2+d^2w'^2)^3}{\mathrm{R}}$$
$$=\frac{(a-b)(a-c)(a-d)}{r_1}x'^3+\frac{(b-a)(b-c)(b-d)}{r_2}y'^3+\frac{(c-a)(c-b)(c-d)}{r_3}z'^3$$
$$+\frac{(d-a)(d-b)(d-c)}{r_4}w'^3. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (135)$$

PART II.—SYSTEMS OF CIRCLES ON THE SURFACE OF A SPHERE.

CHAPTER I.—GENERAL SYSTEMS OF CIRCLES.

The Equation of a Small Circle on a Sphere.—§§ 125, 126.

125. Let ABC be a spherical triangle, having all its angles right angles: then, if we denote the sines of the perpendiculars from any point P on the sides of the triangle by x, y, z, we have at once

$$x^2+y^2+z^2=1;$$

and again, if (xyz), $(x'y'z')$ be any two points, ϕ the angular distance between them,

$$xx'+yy'+zz'=\cos\phi.$$

So that the equation of a small circle is of the form

$$ax+by+cz=1,$$

the coordinates of its centre, and its radius, being given by

$$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{\cos r}{1};$$

and if (xyz) be any point, whose angular distance from the centre of the circle is ϕ, we have

$$ax+by+cz=\cos\phi\sec r.$$

It follows from this that, if the angle of intersection of the circles

$$ax+by+cz=1,$$
$$a'x+b'y+c'z=1,$$

be ω, and the distance between the centres ϕ, then

$$\begin{aligned}\tan r \tan r' \cos\omega &= 1 - \cos\phi \sec r \sec r' \\ &= 1 - aa' - bb' - cc' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136)\end{aligned}$$

126 The equation of a great circle will be of the form

$$ax + by + cz = 0\,;$$

and the angle of intersection of this with the circle

$$a'x + b'y + c'z = 1,$$

will be given by

$$\begin{aligned}\tan r' \cos\omega &= -\cos\phi \sec r' \\ &= -(aa' + bb' + cc')\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (137)\end{aligned}$$

and for two great circles we shall have

$$\cos\omega = -(aa' + bb' + cc').$$

The Power of Two Circles on a Sphere.—§§ 127, 128.

127. If r, r' be the angular radii of two small circles on a sphere, ϕ the angular distance between their poles, and ω their angle of intersection, then either of the expressions

$$\tan r \tan r' \cos\omega, \quad 1 - \cos\phi \sec r \sec r',$$

may be defined as the power of the circles. Denoting the power by π (or if S, S′ are any circles by $\pi_{S,S'}$), we see that $\pi_{S,S'} = 0$ if the circles cut at right angles, and if they touch $\pi_{S,S'}^2 = \pi_{S,S} \cdot \pi_{S',S'}$. In the case of a pair of points P, P′: $\pi_{P,P'} = 1 - \cos\phi = 2\sin^2 \frac{1}{2}\phi$, where ϕ is the distance between them.

It will be convenient to define the power of a great circle, with respect to a small circle, as the product of the tangent of the radius of the small circle and the cosine of the angle of intersection: thus

$$\pi = \tan r \cos\omega = -\cos\phi \sec r\,;$$

and the power of two great circles, as the cosine of the angle between them.

128. If O be the pole of a small circle, radius r, and P any point on the sphere, then if Q be taken in the arc OP so that

$$\tan \tfrac{1}{2}\mathrm{OP} \,.\, \tan \tfrac{1}{2}\mathrm{OQ} = \tan^2 \tfrac{1}{2} r,$$

Q may be called the inverse point of P with respect to the circle whose radius is r, or simply the inverse of P with respect to O.

It may be easily shown, if S, S′ be any two small circles, by forming the equations of the inverse circles with respect to A (§ 125), that the expression

$$\frac{\pi_{S,S'}}{\sqrt{\pi_{O,S}\cdot\pi_{O,S'}}}$$

is invariable.

General Theorems.—§§ 129–132.

129. If (1, 2, 3, 4, 5), (6, 7, 8, 9, 10) denote any two systems of circles on the surface of a sphere, the powers of the former are connected with those of the latter by the identical relation

$$\Pi\begin{pmatrix}1,2,3,4,5\\6,7,8,9,10\end{pmatrix}=0.$$

This is at once proved by multiplying together the matrices,

$$\begin{vmatrix} 1, & a_1, & b_1, & c_1 \\ 1, & a_2, & b_2, & c_2 \\ 1, & a_3, & b_3, & c_3 \\ 1, & a_4, & b_4, & c_4 \\ 1, & a_5, & b_5, & c_5 \end{vmatrix}, \quad \begin{vmatrix} 1, & -a_6, & -b_6, & -c_6 \\ 1, & -a_7, & -b_7, & -c_7 \\ 1, & -a_8, & -b_8, & -c_8 \\ 1, & -a_9, & -b_9, & -c_9 \\ 1, & -a_{10}, & -b_{10}, & -c_{10} \end{vmatrix}.$$

Whence we get

$$\begin{vmatrix} \pi_{1,6}, & \pi_{1,7}, & \pi_{1,8}, & \pi_{1,9}, & \pi_{1,10} \\ \pi_{2,6}, & \pi_{2,7}, & \pi_{2,8}, & \pi_{2,9}, & \pi_{2,10} \\ \pi_{3,6}, & \pi_{3,7}, & \pi_{3,8}, & \pi_{3,9}, & \pi_{3,10} \\ \pi_{4,6}, & \pi_{4,7}, & \pi_{4,8}, & \pi_{4,9}, & \pi_{4,10} \\ \pi_{5,6}, & \pi_{5,7}, & \pi_{5,8}, & \pi_{5,9}, & \pi_{5,10} \end{vmatrix} = 0,$$

i.e.

$$\Pi\begin{pmatrix}1,2,3,4,5\\6,7,8,9,10\end{pmatrix}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (138)$$

130. It is evident that this result is true if one, or more, of the circles are great circles, provided that we interpreted the meaning of the symbol π in accordance with the definition in § 127. Again, it is true, if the radius of any of the circles is zero. And we also see it is true if any circle of either system is such that the coordinates of its centre are zero; *i.e.*, any circles of either system may be replaced by the

imaginary circle at infinity, which we will denote by θ; and we see that we must take $\pi_{\theta,\theta}=1$, and

$$\pi_{S,\theta}=1 \text{ if S be any small circle,}$$
$$\pi_{P,\theta}=1 \text{ if P be any point,}$$
$$\pi_{S,\theta}=0 \text{ if S be any great circle.}$$

131. If then $\omega_{S,S'}$ be the angle of intersection of the circles S, S′ we may deduce at once from equation (138):—

$$\begin{vmatrix} 1, & \cot r_5, & \cot r_6, & \cot r_7, & \cot r_8 \\ \cot r_1, & \cos\omega_{1,5}, & \cos\omega_{1,6}, & \cos\omega_{1,7}, & \cos\omega_{1,8} \\ \cot r_2, & \cos\omega_{2,5}, & \cos\omega_{2,6}, & \cos\omega_{2,7}, & \cos\omega_{2,8} \\ \cot r_3, & \cos\omega_{3,5}, & \cos\omega_{3,6}, & \cos\omega_{3,7}, & \cos\omega_{3,8} \\ \cot r_4, & \cos\omega_{4,5}, & \cos\omega_{4,6}, & \cos\omega_{4,7}, & \cos\omega_{4,8} \end{vmatrix} = 0. \quad . \quad . \quad . \quad (139)$$

132. Exactly as in § 8, we can prove that

$$\left\{\Pi\begin{pmatrix}1,2,3,4\\5,6,7,8\end{pmatrix}\right\}^2=\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}\times\Pi\begin{pmatrix}5,6,7,8\\5,6,7,8\end{pmatrix}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

Chapter II.—Special Systems of Circles.

Circle Cutting Three Circles Orthogonally.—§§ 133, 134.

133. Let the circle cutting the system (1, 2, 3) be denoted by (x), then since

$$\Pi\begin{pmatrix}\theta,x,1,2,3\\\theta,x,1,2,3\end{pmatrix}=0,$$

we have

$$\pi_{x,x}.\Pi\begin{pmatrix}\theta,1,2,3\\\theta,1,2,3\end{pmatrix}=\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix};$$

and if the equations of the circles (1, 2, 3) be of the form

$$a_1x+b_1y+c_1z=1,$$

we have at once

$$\Pi\begin{pmatrix}\theta,1,2,3\\\theta,1,2,3\end{pmatrix}=\begin{vmatrix} 1, & 0, & 0, & 0 \\ 1, & -a_1, & -b_1, & -c_1 \\ 1, & -a_2, & -b_2, & -c_2 \\ 1, & -a_3, & -b_3, & -c_3 \end{vmatrix}\times\begin{vmatrix} 1, & 0, & 0, & 0 \\ 1, & a_1, & b_1, & c_1 \\ 1, & a_2, & b_2, & c_2 \\ 1, & a_3, & b_3, & c_3 \end{vmatrix}$$
$$=-\frac{36}{R^6}\sec^2 r_1\sec^2 r_2\sec^2 r_3.\{\nabla(1,2,3)\}^2; \quad . \quad . \quad . \quad . \quad . \quad . \quad (141)$$

where R is the radius of the sphere, and V(1, 2, 3) denotes the volume of the tetrahedron formed by the centre of the sphere and the poles of the circles (1, 2, 3).

Hence the radius of the orthogonal circle of the system (1, 2, 3) is given by

$$\tan^2 r = \frac{\sin^2 r_1 \sin^2 r_2 \sin^2 r_3}{36} \frac{R^6}{\{V(1,2,3)\}^2} \times \begin{vmatrix} -1, & \cos\omega_{1,2}, & \cos\omega_{1,3} \\ \cos\omega_{2,1}, & -1, & \cos\omega_{2,3} \\ \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1 \end{vmatrix} . \quad (142)$$

134. If the three circles meet in a point, r must be zero, hence

$$\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} = 0.$$

Four Circles having a Common or Orthogonal Circle.—§§ 135, 136.

135. Let (x) denote the common orthogonal circle of the system (1, 2, 3, 4), then, since

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 1, 2, 3, 4 \end{pmatrix} = 0,$$

it follows that we must have

$$\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 1, 2, 3, 4 \end{pmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (143)$$

This is clearly the necessary and sufficient condition that the system (1, 2, 3, 4) may have a common orthogonal circle.

136. If (5, 6, 7, 8) be any other system of circles, we must also have

$$\Pi\begin{pmatrix} 1, 2, 3, 4 \\ 5, 6, 7, 8 \end{pmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

As a particular case, we have

$$\Pi\begin{pmatrix} 1, 2, 3, 4 \\ x, 1, 2, 3 \end{pmatrix} = 0,$$

where (x) denotes any other circle.

We deduce that

$$\pi_{x,1}.\Pi\begin{pmatrix} 2, 3, 4 \\ 1, 2, 3 \end{pmatrix} + \pi_{x,2}.\Pi\begin{pmatrix} 1, 4, 3 \\ 1, 2, 3 \end{pmatrix} + \pi_{x,3}.\Pi\begin{pmatrix} 1, 2, 4 \\ 1, 2, 3 \end{pmatrix} = \pi_{x,4}.\Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} . \quad . \quad (145)$$

But from (144) we can deduce, as in § 24, that,

$$\left\{\Pi\begin{pmatrix} 1, 2, 3 \\ 5, 6, 7 \end{pmatrix}\right\}^2 = \Pi\begin{pmatrix} 1, 2, 3 \\ 1, 2, 3 \end{pmatrix} \times \Pi\begin{pmatrix} 5, 6, 7 \\ 5, 6, 7 \end{pmatrix};$$

so that (145) may be written,

$$\pi_{x,4}.\left\{\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}\right\}^{\frac{1}{2}}=\pi_{x,1}\left\{\Pi\begin{pmatrix}2,3,4\\2,3,4\end{pmatrix}\right\}^{\frac{1}{2}}+\pi_{x,2}\left\{\Pi\begin{pmatrix}1,4,3\\1,4,3\end{pmatrix}\right\}^{\frac{1}{2}}+\pi_{x,3}\left\{\Pi\begin{pmatrix}1,2,4\\1,2,4\end{pmatrix}\right\}^{\frac{1}{2}}.$$

But if r be the radius of the common orthogonal circle, we have, by § 133,

$$\frac{36}{R^6}\tan^2 r^2=\cos^2 r_1\cos^2 r_2\cos^2 r_3\frac{\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}}{\{V(1,2,3)\}^2}=\cos^2 r_2\cos^2 r_3\cos^2 r_4\frac{\Pi\begin{pmatrix}2,3,4\\2,3,4\end{pmatrix}}{\{V(2,3,4)\}^2}=\&c.\ .$$

Hence we have

$$\pi_{x,4}.\cos r_4.V(1,2,3)=\pi_{x,1}.\cos r_1.V(2,3,4)+\pi_{x,2}.\cos r_2.V(1,4,3)+\pi_{x,3}.\cos r_3.V(1,2,4)\ ; \quad . \quad (146)$$

which result may also be written

$$\pi_{x,4}.\cos r_4=\alpha.\cos r_1.\pi_{x,1}+\beta.\cos r_2.\pi_{x,2}+\gamma.\cos r_3.\pi_{x,3}\ ; \quad . \quad . \quad . \quad (147)$$

where α, β, γ may be defined as the areal coordinates of the pole of the circle (4) with respect to the triangle formed by the poles of the circles (1, 2, 3).

Thus, if A, B, C be the triangle, P the pole of (4), then

$$\alpha=\frac{V(P,B,C)}{V(A,B,C)}=\frac{\sin(\text{perp. from P on BC})}{\sin(\text{perp. from A on BC})}.$$

As a particular case of (147), let x be a point, O say, then A, B, C being the centres of (1, 2, 3), and P being a point on the circle which cuts them orthogonally, we shall have

$$1-\cos OP=\alpha(\cos r_1-\cos OA)+\beta(\cos r_2-\cos OB)+\gamma(\cos r_3-\cos OC),$$

or more generally, P being the pole of a circle which, with (1, 2, 3), has a common orthogonal circle,

$$\cos r_4-\cos OP=\alpha(\cos r_1-\cos OA)+\beta(\cos r_2-\cos OB)+\gamma(\cos r_3-\cos OC).$$

Orthogonal Systems.—§§ 137–139.

137. Four circles may be said to form an orthogonal system if each one cuts the other three orthogonally. It is clear that the pole of any one of four such circles must be the orthocentre of triangle formed by the poles of the other three.

Let (1, 2, 3, 4) be such a system, then if (x, y) be any other circles we must have from the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 1, 2, 3, 4 \end{pmatrix}=0,$$

$$\pi_{x,y}=\frac{\pi_{x,1}\pi_{y,1}}{\pi_{1,1}}+\frac{\pi_{x,2}\pi_{y,2}}{\pi_{2,2}}+\frac{\pi_{x,3}\pi_{y,3}}{\pi_{3,3}}+\frac{\pi_{x,4}\pi_{y,4}}{\pi_{4,4}}; \quad \ldots\ldots (148)$$

whence we have as particular cases

$$\text{(i.)}\ \pi_{x,1}\cot^2 r_1+\pi_{x,2}\cot^2 r_2+\pi_{x,3}\cot^2 r_3+\pi_{x,4}\cot^2 r_4=-1,$$
$$\text{(ii.)}\ \pi_{x,1}^2\cot^2 r_1+\pi_{x,2}^2\cot^2 r_2+\pi_{x,3}^2\cot^2 r_3+\pi_{x,4}^2\cot^2 r_4=\tan^2 r_x,$$

where x denotes any circle, radius r_x;

$$\text{(iii.)}\ \pi_{z,1}\cot^2 r_1+\pi_{z,2}\cot^2 r_2+\pi_{z,3}\cot^2 r_3+\pi_{z,4}\cot^2 r_4=0,$$
$$\text{(iv.)}\ \pi_{z,1}^2\cot^2 r_1+\pi_{z,2}^2\cot^2 r_2+\pi_{z,3}^2\cot^2 r_3+\pi_{z,4}^2\cot^2 r_4=1,$$

where z denotes any great circle;

$$\text{(v.)}\ \cot^2 r_1+\cot^2 r_2+\cot^2 r_3+\cot^2 r_4=-1; \quad \ldots\ldots\ldots (149)$$

so that one of the circles must be imaginary.

138. If the circles (1, 2, 3, 4) form a system not having a common orthogonal circle, we may find four other circles, (5, 6, 7, 8) say, such that each of the latter is orthogonal to three of the former. One such system may be called the "orthogonal system" of the other.

Let x, y denote any two circles, then since

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 5, 6, 7, 8 \end{pmatrix}=0,$$

we shall have

$$\pi_{x,y}=\frac{\pi_{x,5}\pi_{y,1}}{\pi_{1,5}}+\frac{\pi_{x,6}\pi_{y,2}}{\pi_{2,6}}+\frac{\pi_{x,7}\pi_{y,3}}{\pi_{3,7}}+\frac{\pi_{x,8}\pi_{y,4}}{\pi_{4,8}}; \quad \ldots\ldots (150)$$

whence we obtain as particular cases

$$\frac{1}{\pi_{1,5}}+\frac{1}{\pi_{2,6}}+\frac{1}{\pi_{3,7}}+\frac{1}{\pi_{4,8}}=\pi_{\theta,\theta}=1, \quad \ldots\ldots\ldots (151)$$

and, x denoting any small circle,

$$\frac{\pi_{x,5}}{\pi_{1,5}}+\frac{\pi_{x,6}}{\pi_{2,6}}+\frac{\pi_{x,7}}{\pi_{3,7}}+\frac{\pi_{x,8}}{\pi_{4,8}}=1.$$

139. The system of circles (1, 2, 3, 4) may be called a "semi-orthogonal" system, if (1, 2) cut orthogonally in the points (3, 4). Then x, y denoting any circles we have by

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 1, 2, 3, 4 \end{pmatrix}=0,$$

the equation

$$\pi_{x,y}=\frac{\pi_{x,1}\pi_{y,1}}{\pi_{1,1}}+\frac{\pi_{x,2}\pi_{y,2}}{\pi_{2,2}}+\frac{\pi_{x,3}\pi_{y,4}+\pi_{x,4}\pi_{y,3}}{\pi_{3,4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

If $2e$ denote the arc between (3, 4) we have as a particular case,

$$-1=\cot^2 r_1+\cot^2 r_2-\operatorname{cosec}^2 e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (153)$$

Circles touching one another.—§§ 140–144.

140. If the four circles (1, 2, 3, 4) touch one another externally, we shall have from the equation

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4 \\ \theta, 1, 2, 3, 4 \end{pmatrix}=0,$$

$$\begin{aligned} 4+\cot^2 r_1+\cot^2 r_2+\cot^2 r_3+\cot^2 r_4 \\ =2\{\cot r_1\cot r_2+\cot r_1\cot r_3+\cot r_1\cot r_4 \\ +\cot r_2\cot r_3+\cot r_2\cot r_4+\cot r_3\cot r_4\}; \end{aligned}$$

whence

$$\cot r_4=\cot r_1+\cot r_2+\cot r_3\pm 2\{\cot r_2\cot r_3+\cot r_3\cot r_1+\cot r_1\cot r_2-1\}^{\frac{1}{2}}. \quad (154)$$

141. We may also easily extend the formulæ (39) and (40) in § 31. Thus, let two circles (1, 2) be described, with angular radii α, γ, and let another circle radius r be described touching these internally, and having its pole on their common diameter. Let S_1 be a circle touching this circle internally, and (1, 2) externally; and let a series of circles S_2, S_3, S_4, &c. be described touching externally (1, 2) and the preceding one in the series; and let the radii of these circles be r_1, r_2, r_3, &c.

We shall have, since S_{n-1} and S_{n+1} touch S_n,

$$\cot r_{n+1}-2\cot r_n+\cot r_{n-1}=2(\cot\alpha+\cot\gamma);$$

whence, exactly as in § 31,

$$\begin{aligned} \cot r_n &= n^2(\cot\alpha+\cot\gamma)-\cot r \\ &= \frac{n^2\sin r}{\sin\alpha\sin\gamma}-\cot r. \end{aligned}$$

So that

$$\tan r_s = \frac{\sin\alpha \sin\gamma \sin r}{n^2 \sin^2 r - \cos r \sin\alpha \sin\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (155)$$

Similarly if S'_1, S'_2, S'_3, &c. be a series of circles touching (1, 2) and one another externally, and S'_Γ touching the common diameter of (1, 2), we shall find

$$\tan r'_s = \frac{4 \sin\alpha \sin\gamma \sin r}{(2n-1)^2 \sin^2 r - 4 \cos r \sin\alpha \sin\gamma} \quad . \quad . \quad . \quad . \quad . \quad (156)$$

142. Since

$$\pi_{x,y} + \sqrt{\pi_{x,x}.\pi_{y,y}} = \tan r_x \tan r_y (\cos\omega_{x,y} + 1),$$

we infer from § 32 that if the four circles (1, 2, 3, 4) are all touched by another circle externally, then we must have

$$\cos\tfrac{1}{2}\omega_{1,2}.\cos\tfrac{1}{2}\omega_{3,4} \pm \cos\tfrac{1}{2}\omega_{1,3}.\cos\tfrac{1}{2}\omega_{2,4} \pm \cos\tfrac{1}{2}\omega_{1,4}.\cos\tfrac{1}{2}\omega_{2,3} = 0,$$

$\omega_{1,2}$ being the angle of intersection of the circles (1, 2).

This formula must be slightly modified if the tangent circle does not touch them all externally: if, for instance, the circles (1, 2) have contact with the tangent circle of opposite nature, then $\cos\frac{1}{2}\omega_{1,2}$ must be replaced by $\sin\frac{1}{2}\omega_{1,2}$.

143. If this condition be satisfied the radius of the circle touching the circles (1, 2, 3, 4) may be easily found by means of § 138. Thus, let the orthogonal system of (1, 2, 3, 4) be (5, 6, 7, 8); and let the contact be external in each case.

Then since

$$\frac{\pi_{x,1}}{\pi_{1,5}} + \frac{\pi_{x,2}}{\pi_{2,6}} + \frac{\pi_{x,3}}{\pi_{3,7}} + \frac{\pi_{x,4}}{\pi_{4,8}} = 1,$$

we shall have

$$\cot r_x = \frac{\tan r_1}{\pi_{1,5}} + \frac{\tan r_2}{\pi_{2,6}} + \frac{\tan r_3}{\pi_{3,7}} + \frac{\tan r_4}{\pi_{4,8}} \quad . \quad . \quad . \quad . \quad (157)$$

144. If the system of circles (1, 2, 3, 4) be such that four other circles (5, 6, 7, 8) can be drawn to touch them all, symmetrically, say let each of the latter touch one of the former internally and the others externally; *e.g.*, let (5) touch (2, 3, 4) externally: then since

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ y, 5, 6, 7, 8 \end{pmatrix} = 0,$$

where (x, y) denote any other circles, we have

$$-4\pi_{x,y} = 2\Sigma\pi_{y,1}.\pi_{x,5} \cot r_1 \cot r_5 - (\Sigma\pi_{y,1}.\cot r_1)(\Sigma\pi_{x,5}.\cot r_5);$$

whence as a particular case,

$$(\cot r_1+\cot r_2+\cot r_3+\cot r_4)(\cot r_5+\cot r_6+\cot r_7+\cot r_8)$$
$$=4+2(\cot r_1\cot r_5+\cot r_2\cot r_6+\cot r_3\cot r_7+\cot r_4\cot r_8). \quad . \quad . \quad (158)$$

An example of this would be, when (1, 2, 3, 4) are the inscribed and escribed circles of a spherical triangle, and (5, 6, 7, 8) are the corresponding nine-points circles.

Chapter III.—Circles connected with a Spherical Triangle.

Regarding a spherical triangle as formed by the arcs of three small circles, most of the theorems concerning the three species of circles, connected with a triangle formed by great circle-arcs, may be readily extended. It will be seen that there is a much greater resemblance than there is between the corresponding formulæ for plane triangles formed by arcs of circles and straight lines. We shall suppose that the circles intersect in the points P, Q, R, P′, Q′, R′, the former points lying within the triangle formed by arcs joining the poles of the circles; and we will call the angles of the triangle P, Q, R—α, β, γ; then the formulæ for any other of the eight triangles which make up the whole figure may be at once written down by changing two of the angles into their supplements. We shall use r_1, r_2, r_3 to denote angular radii of the circles, and r to denote the angular radius of their orthogonal circle.

The Circum-circle of a Triangle.—§§ 145, 146.

145. If x denote the circle which passes through the points P, Q, R, the points of intersection of the circles (1, 2, 3); and if (4) denote the orthogonal circle of the system (1, 2, 3), we shall have, since

$$\Pi\begin{pmatrix}S, 1, 2, 3, \theta\\ S, 1, 2, 3, 4\end{pmatrix}=0,$$

exactly as in § 39,

$$\frac{\pi_{4,4}-\pi_{4,x}}{\pi_{4,x}}\cdot\left\{\frac{\Pi\begin{pmatrix}1, 2, 3\\ 1, 2, 3\end{pmatrix}}{-\pi_{4,4}}\right\}^{\frac{1}{2}}=\left\{\Pi\begin{pmatrix}2, 3\\ 2, 3\end{pmatrix}\right\}^{\frac{1}{2}}+\left\{\Pi\begin{pmatrix}3, 1\\ 3, 1\end{pmatrix}\right\}^{\frac{1}{2}}+\left\{\Pi\begin{pmatrix}1, 2\\ 1, 2\end{pmatrix}\right\}^{\frac{1}{2}}.$$

But

$$\Pi\begin{pmatrix}1, 2, 3\\ 1, 2, 3\end{pmatrix}=-\frac{36}{R^6}\pi_{4,4}\{V(1, 2, 3)\}^2\sec^2 r_1\sec^2 r_2\sec^2 r_3,$$

$$\Pi\begin{pmatrix}2, 3\\ 2, 3\end{pmatrix}=\frac{36}{R^4}\sec^2 r_2\sec^2 r_3.\{V(P, 2, 3)\}^2,$$

where R is the radius of the sphere, and V(1, 2, 3) denotes the volume of the tetrahedron formed by the poles of the circles (1, 2, 3) and the centre of the sphere.

Hence we obtain

$$\frac{\pi_{4,4}-\pi_{4,s}}{\pi_{4,s}}=\frac{V(P,2,3)\cos r_1+V(P,3,1)\cos r_2+V(P,1,2)\cos r_3}{V(1,2,3)};$$

or if ω be the angle of intersection of the circle PQR with the orthogonal circle to (1, 2, 3), we may write

$$\frac{\tan r_s}{\tan r\sec\omega}=\frac{V(1,2,3)}{V(1,2,3)+V(P,2,3)\cos r_1+V(Q,3,1)\cos r_2+V(R,1,2)\cos r_3}. \quad (159)$$

146. Again, if $\mu_{1,1}$, $\mu_{1,2}$, &c., denote the minors of $\pi_{1,1}$, $\pi_{1,2}$ &c., in $\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}$, we shall have, as in § 39,

$$\frac{\pi_{s,s}\pi_{4,4}-\pi^2_{4,s}}{\pi^2_{4,s}}\cdot\left\{\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}\right\}^2=\begin{vmatrix}0, & \sqrt{\mu_{1,1}}, & \sqrt{\mu_{2,2}}, & \sqrt{\mu_{3,3}}\\ \sqrt{\mu_{1,1}}, & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3}\\ \sqrt{\mu_{2,2}}, & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3}\\ \sqrt{\mu_{3,3}}, & \mu_{3,1}, & \mu_{3,2}, & \mu_{3,3}\end{vmatrix}. \quad (160)$$

But

$$\mu_{1,1}=\Pi\begin{pmatrix}2,3\\2,3\end{pmatrix}=\frac{36}{R^6}\sec^2 r_2\sec^2 r_3\{V(P,2,3)\}^2;$$

hence

$$\frac{36}{R^6}\tan^2\omega\sec^2 r_1\sec^2 r_2\sec^2 r_3\tan^4 r^2.\{V(1,2,3)\}^4$$

$$=\begin{vmatrix}0, & V(P,2,3)\cos r_1, & V(Q,3,1)\cos r_2, & V(R,1,2)\cos r_3\\ V(P,2,3)\cos r_1, & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3}\\ V(Q,3,1)\cos r_2, & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3}\\ V(R,1,2)\cos r_3, & \mu_{3,1}, & \mu_{2,3}, & \mu_{3,3}\end{vmatrix};$$

whence may be deduced,

$$\cos^2\omega=\sec s.\cos(s-\alpha).\cos(s-\beta).\cos(s-\gamma), \quad (161)$$

where

$$2s=\alpha+\beta+\gamma.$$

If the three given circles are great circles, then the imaginary circle θ will be their orthogonal circle, in this case equation (160) reduces to,

$$-\cot^2 r_s=\sec s.\cos(s-\alpha).\cos(s-\beta).\cos(s-\gamma);$$

the ordinary formula for finding the radius of the circum-circle of a spherical triangle

The Inscribed and Escribed Circles of a Triangle.—§§ 147, 148.

147. If the inscribed and escribed circles of the triangle PQR cut the orthogonal circle at angles ω, ω_1, ω_2, ω_3, we have, as in § 43,

$$\left.\begin{aligned} \mathrm{K}.\cos^2\omega &= 2(1+\cos\alpha)(1+\cos\beta)(1+\cos\gamma) \\ \mathrm{K}.\cos^2\omega_1 &= 2(1+\cos\alpha)(1-\cos\beta)(1-\cos\gamma) \\ \mathrm{K}.\cos^2\omega_2 &= 2(1-\cos\alpha)(1+\cos\beta)(1-\cos\gamma) \\ \mathrm{K}.\cos^2\omega_3 &= 2(1-\cos\alpha)(1-\cos\beta)(1+\cos\gamma) \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad (162)$$

where

$$\begin{aligned} \mathrm{K} &= \cos^2\alpha+\cos^2\beta+\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma-1 \\ &= 4\cos s.\cos(s-\alpha).\cos(s-\beta).\cos(s-\gamma). \end{aligned}$$

In these formulæ $\cos^2\omega$ has been written for $\frac{\pi_{4,s}{}^2}{\pi_{4,4}.\pi_{s,s}}$, and so, if the given triangle be an ordinary spherical triangle, $\cos^2\omega$ must be replaced by $-\cot^2 r_s$; thus the above formulæ correspond, in the case of an ordinary spherical triangle, to the formulæ,

$$\left.\begin{aligned} \mathrm{N}\cot r &= 2\cos\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma \\ \mathrm{N}\cot r_1 &= 2\cos\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma \\ \mathrm{N}\cot r_2 &= 2\sin\tfrac{1}{2}\alpha\cos\tfrac{1}{2}\beta\sin\tfrac{1}{2}\gamma \\ \mathrm{N}\cot r_3 &= 2\sin\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\beta\cos\tfrac{1}{2}\gamma \end{aligned}\right\};$$

where

$$\mathrm{N}^2 = -\cos s.\cos(s-\alpha).\cos(s-\beta).\cos(s-\gamma).$$

In our present case, the radii will be given by formulæ similar to

$$\mathrm{K}(\cot r_s+\cos\omega\cot r)+\begin{vmatrix} 0, & \cot r_1, & \cot r_2, & \cot r_3 \\ 1, & -1, & \cos\gamma, & \cos\beta \\ 1, & \cos\gamma, & -1, & \cos\alpha \\ 1, & \cos\beta, & \cos\alpha, & -1 \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad (163)$$

148. In exactly the same way as in § 45 we may show that, associated with every triangle, there are eight circles analogous to the nine-points circle of a plane triangle, each of them touching four of the circles, which touch the sides of the spherical triangle; that is, taking any one of the eight associated triangles formed by three circles, say P Q R, the inscribed and escribed circles are touched by another circle. If this circle cut the orthogonal circle of the triangle at the angle ϖ, we shall have, as in § 46,

$$\cos^2\varpi = 4\sec s.\cos(s-\alpha).\cos(s-\beta).\cos(s-\gamma), \quad . \quad . \quad . \quad . \quad (164)$$

and we also infer, from § 46, that this circle cuts the sides of the triangle PQR at the angles $\beta-\gamma$, $\gamma-\alpha$, $\alpha-\beta$.

Comparing equation (164) with equation (161) we see that if the circle PQR cut the orthogonal circle at the angle ω, then $\cos\varpi=2\cos\omega$. Whence we infer that in a spherical triangle formed by great circle arcs, the radius ρ of the nine-points circle, and the radius R of the circum-circle, are connected by the formula

$$\cot\rho=2\cot\mathrm{R}.$$

In the case of a general spherical triangle, this is replaced by the formula

$$\cot\rho-\cot\rho'=2(\cot\mathrm{R}-\cot\mathrm{R}'),$$

where ρ, R are the radii of the analogous circles connected with the triangle, and ρ', R′ the radii of the corresponding circles connected with the inverse triangle, with respect to the orthogonal circle of the triangle.

Chapter IV.—Power-Coordinates.

Definition.—§§ 149–151.

149. Given any system of circles, say (1, 2, 3, 4), on the surface of a sphere, then any circle (great or small), or any point, is completely determinate when its powers with respect to the system (1, 2, 3, 4) are known, provided that this system be not a system having a common orthogonal circle.

If then P be any point, we may define the coordinates of P referred to the system (1, 2, 3, 4) as any multiples, the same or different, of the powers of P with respect to these circles; thus denoting the coordinates of P by $(xyzw)$, then k_1, k_2, k_3, k_4 being any constant multiples, we may take

$$x=k_1.\pi_{\mathrm{P},1},\quad y=k_2.\pi_{\mathrm{P},2},\quad z=k_3.\pi_{\mathrm{P},3},\quad w=k_4.\pi_{\mathrm{P},4}.$$

Since $\pi_{\mathrm{P},\mathrm{P}}=0$, and $\pi_{\mathrm{P},4}=1$, we see at once that the coordinates of any point must satisfy a homogeneous quadric relation, viz.,

$$\Pi\begin{pmatrix}\mathrm{P},1,2,3,4\\ \mathrm{P},1,2,3,4\end{pmatrix}=0,$$

and a non-homogeneous linear relation,

$$\Pi\begin{pmatrix}\mathrm{P},1,2,3,4\\ \theta,1,2,3,4\end{pmatrix}=0.$$

The former is called the equation of the Absolute, and will be usually denoted by $\psi(x, y, z, w)$, and then the latter may be written

$$\left.\begin{aligned} k_1\frac{\partial\psi}{\partial x}+k_2\frac{\partial\psi}{\partial y}+k_3\frac{\partial\psi}{\partial z}+k_4\frac{\partial\psi}{\partial w}&=\mathrm{K}\\ \text{or}\qquad x\frac{\partial\psi}{\partial k_1}+y\frac{\partial\psi}{\partial k_2}+z\frac{\partial\psi}{\partial k_3}+w\frac{\partial\psi}{\partial k_4}&=\mathrm{K}\end{aligned}\right\};\quad\ldots\ldots\quad(164^*)$$

where $\frac{\partial\psi}{\partial k_1}$, as usual, means the partial differential coefficient of ψ with respect to x, (k_1, k_2, k_3, k_4) being afterwards put for (x, y, z, w), and where K is some constant.

150. If S be any small circle, we may define the coordinates of S with respect to the system (1, 2, 3, 4) as constant multiples of the powers; thus, denoting them by ξ, η, ζ, ω, we will take

$$\xi=k_1\pi_{s,1},\ \eta=k_2\pi_{s,2},\ \zeta=k_3\pi_{s,3},\ \omega=k_4\pi_{s,4};$$

k_1, k_2, k_3, k_4 having the same values as in § 149.

Since $\pi_{s,s}=1$, we see that the coordinates of any small circle must satisfy the non-homogeneous linear relation

$$k_1\frac{\partial\psi}{\partial\xi}+k_2\frac{\partial\psi}{\partial\eta}+k_3\frac{\partial\psi}{\partial\zeta}+k_4\frac{\partial\psi}{\partial\omega}=\mathrm{K}.\quad\ldots\ldots\quad(165)$$

151. If, however, S be a great circle, we shall have, since $\pi_{s,s}=0$, $\pi_{s,s}=-1$, the homogeneous linear relation,

$$k_1\frac{\partial\psi}{\partial\xi}+k_2\frac{\partial\psi}{\partial\eta}+k_3\frac{\partial\psi}{\partial\zeta}+k_4\frac{\partial\psi}{\partial\omega}=0,\quad\ldots\ldots\quad(166)$$

and the non-homogeneous quadric relation,

$$2\psi(\xi, \eta, \zeta, \omega)=-\mathrm{K}.\quad\ldots\ldots\quad(167)$$

The Small Circle.—§§ 152–157.

152. If P be any point on the circle S whose coordinates are $(\xi, \eta, \zeta, \omega)$, we shall have by the equation

$$\Pi\begin{pmatrix}\mathrm{P}, 1, 2, 3, 4\\ \mathrm{S}, 1, 2, 3, 4\end{pmatrix}=0,$$

since $\pi_{P,S}=0$,

$$\frac{\partial\psi}{\partial\xi}x+\frac{\partial\psi}{\partial\eta}y+\frac{\partial\psi}{\partial\zeta}z+\frac{\partial\psi}{\partial\omega}w=0.\quad\ldots\ldots\quad(168)$$

Thus the equation of a small circle is of the first degree.

It follows that the equation of the first degree, say

$$ax+by+cz+dw=0,$$

will represent in general a small spherical circle, whose coordinates are given by

$$\frac{\partial\psi}{\partial\xi}=\frac{\partial\psi}{\partial\eta}=\frac{\partial\psi}{\partial\zeta}=\frac{\partial\psi}{\partial\omega}=\frac{K}{ak_1+bk_2+ck_3+dk_4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169)$$

by equation (165).

153. Given any two circles whose coordinates are $(\xi, \eta, \zeta, \omega)$, $(\xi', \eta', \zeta', \omega')$, their power π is given by

$$\Pi\begin{pmatrix}S, & 1, & 2, & 3, & 4\\ S', & 1, & 2, & 3, & 4\end{pmatrix}=0,$$

or

$$\pi.K=\xi'\frac{\partial\psi}{\partial\xi}+\eta'\frac{\partial\psi}{\partial\eta}+\zeta'\frac{\partial\psi}{\partial\zeta}+\omega'\frac{\partial\psi}{\partial\omega}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170)$$

and the radius r of the circle $(\xi\eta\zeta\omega)$ will consequently be given by,

$$-\tan^2 r.K=2\psi(\xi, \eta, \zeta, \omega). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (171)$$

154. It follows that the radius of the circle

$$ax+by+cz+dw=0,$$

will be given by

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a\\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b\\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c\\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d\\ a, & b, & c, & d, & M\tan^2 r \end{vmatrix}=0; \quad . \quad . \quad . \quad . \quad (172)$$

where

$$M=-\frac{2}{K}(ak_1+bk_2+ck_3+dk_4)^2,$$

and where $a_{1,1}$, $a_{1,2}$, &c., are the coefficients in the equation of the absolute, so that

$$\psi(xyzw)\equiv a_{1,1}x^2+2a_{1,2}xy+\ldots$$

155. Again, the power of the circle $(\xi\eta\zeta\omega)$ with respect to the circle

$$ax+by+cz+dw=0,$$

is clearly given by

$$\pi=\frac{a\xi+b\eta+c\zeta+d\omega}{ak_1+bk_2+ck_3+dk_4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (173)$$

156. And further, the power of the two circles,

$$ax+by+cz+dw=0,$$
$$a'x+b'y+c'z+d'w=0,$$

will be given by

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d \\ a', & b', & c', & d', & \mathrm{M}\pi \end{vmatrix} = 0 ; \quad . \quad . \quad . \quad . \quad . \quad (174)$$

where

$$\mathrm{M}=\frac{2}{\mathrm{K}}(ak_1+bk_2+ck_3+dk_4)(a'k_1+b'k_2+c'k_3+d'k_4).$$

Whence, if the two circles cut at an angle ϕ, we have

$$\cos\phi=-\frac{a'\frac{\partial\Psi}{\partial a}+b'\frac{\partial\Psi}{\partial b}+c'\frac{\partial\Psi}{\partial c}+d'\frac{\partial\Psi}{\partial d}}{2\sqrt{\Psi(a,\,b,\,c,\,d).\Psi(a,\,b,\,c,\,d)}} ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

where

$$\Psi(a,\,b,\,c,\,d)=-\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & d \\ a, & b, & c, & d, & 0 \end{vmatrix}.$$

157. The coordinates of θ, the imaginary circle at infinity, are evidently k_1, k_2, k_3, k_4; and the equation of this circle is

$$x\frac{\partial\Psi}{\partial k_1}+y\frac{\partial\Psi}{\partial k_2}+z\frac{\partial\Psi}{\partial k_3}+w\frac{\partial\Psi}{\partial k_4}=0 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

its radius is equal to $\tan^{-1}\sqrt{-1}$.

The Great Circle.—§§ 158, 159.

158. The equation of the first degree

$$ax+by+cz+dw=0,$$

will represent a great circle on the sphere, when

$$ak_1+bk_2+ck_3+dk_4=0 ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177)$$

and if this condition be satisfied, its coordinates will be given by

$$\frac{\partial\Psi}{\partial\xi}_{a}=\frac{\partial\Psi}{\partial\eta}_{b}=\frac{\partial\Psi}{\partial\zeta}_{c}=\frac{\partial\Psi}{\partial\omega}_{d}=\sqrt{\frac{2\mathrm{K}\Delta}{-\Psi}}, \quad \ldots\ldots\ldots \quad (178)$$

by equation (167), where Ψ has the same meaning as in § 156, and where

$$\Delta=\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix}.$$

159. The power of the great circle $(\xi\eta\zeta\omega)$ with respect to the small circle

$$ax+by+cz+dw=0,$$

is given by

$$\pi\mathrm{K}=\frac{a\xi+b\eta+c\zeta+d\omega}{ak_1+bk_2+ck_3+dk_4},$$

as in § 155.

But if the equation $ax+by+cz+dw=0$ represent a great circle, then the power of any other circle $(\xi\eta\zeta\omega)$ with respect to it is

$$=\sqrt{\frac{2\mathrm{K}\Delta}{-\Psi}}.(a\xi+b\eta+c\zeta+d\omega). \quad \ldots\ldots\ldots \quad (179)$$

The angle between two great circles whose equations are given is the same as that given by equation (175).

The Point.—§§ 160–162.

160. The power of the point $(xyzw)$ with respect to the circle

$$ax+by+cz+dw=0,$$

is equal to

$$\frac{ax+by+cz+dw}{ak_1+bk_2+ck_3+dk_4},$$

or

$$(ax+by+cz+dw)\sqrt{\frac{2\Delta\mathrm{K}}{-\Psi}};$$

according as the equation represents a small circle or a great circle.

161. The power of two points $(xyzw)$, $(x'y'z'w')$ is, by equation (168), given by

$$\pi K = x'\frac{\partial\psi}{\partial x}+y'\frac{\partial\psi}{\partial y}+z'\frac{\partial\psi}{\partial z}+w'\frac{\partial\psi}{\partial w};$$

which, since

$$\psi(xyzw)=0, \quad \psi(x'y'z'w')=0,$$

may be written

$$-2\pi K=\psi\{(x-x'),\ y-y',\ z-z',\ w-w'\}. \quad . \quad . \quad . \quad . \quad . \quad (180)$$

162. If R be the radius of the sphere, and (A, B, C) any three points, we have by equation (141),

$$\Pi\begin{pmatrix}\theta, A, B, C\\ \theta, A, B, C\end{pmatrix}=-\frac{36}{R^6}\cdot\{V(A, B, C)\}^2;$$

and by § 132

$$\Pi\begin{pmatrix}\theta, A, B, C\\ \theta, A, B, C\end{pmatrix}\times\Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix}=\left\{\Pi\begin{pmatrix}\theta, A, B, C\\ 1, 2, 3, 4\end{pmatrix}\right\}^2.$$

Hence, if $(x_1y_1z_1w_1)$, $(x_2y_2z_2w_2)$, $(x_3y_3z_3w_3)$ be the coordinates of A, B, C, referred to the system (1, 2, 3, 4), we shall have

$$V(ABC)=\mu\begin{vmatrix} x_1, & x_2, & x_3, & k_1\\ y_1, & y_2, & y_3, & k_2\\ z_1, & z_2, & z_3, & k_3\\ w_1, & w_2, & w_3, & k_4\end{vmatrix}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (181)$$

where

$$36\mu^2k_1^2k_2^2k_3^2k_4^2\cdot\Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix}+R^6=0.$$

Coordinate Systems of Reference.—§§ 163, 164.

163. There are two convenient systems of reference: (i.) four mutually orthotomic circles, called the orthogonal system; (ii.) two orthogonal circles and their two points of intersection, called the semi-orthogonal system. A particular case of the former would be three great circles cutting orthogonally and the imaginary circle at infinity.

(i.) If the system (1, 2, 3, 4) be an orthogonal system, we shall find it most convenient to take k_1, k_2, k_3, k_4 equal respectively to the cotangents of the radii of the circles (1, 2, 3, 4). So that the equation of the absolute will be

$$\psi(x, y, z, w)=x^2+y^2+z^2+w^2=0;$$

and referring to § 137, we see that we shall have

$$K=-2,$$

and

$$\Psi(a, b, c, d)=a^2+b^2+c^2+d^2;$$

also we have

$$\cot^2 r_1+\cot^2 r_2+\cot^2 r_3+\cot^2 r_4=-1.$$

(ii.) If the system (1, 2, 3, 4) be a semi-orthogonal system, we may take $k_1 = \cot r_1$, $k_2 = \cot r_2$, $k_3 = k_4 = \operatorname{cosec} e$, where r_1, r_2 are the radii of the circles, $2e$ their common chord. We shall have by § 149,

$$\psi(xyzw) = x^2 + y^2 - zw,$$
$$K = -2,$$
$$\Psi(abcd) = a^2 + b^2 - \tfrac{1}{2}cd;$$

and also

$$\cot^2 r_1 + \cot^2 r_2 - \cot^2 e = 0.$$

164. By § 128, if two circles S, S′ be inverted with respect to any point (O) on the sphere, then the expression

$$\frac{\pi_{S,S'}}{\sqrt{\pi_{O,S} \cdot \pi_{O,S'}}}$$

is unaltered. Hence, if $(xyzw)$ be the coordinates of any point referred to a system (1, 2, 3, 4), and (XYZW) be the coordinates of the inverse point, with respect to any point (O), referred to the inverse system with respect to the same point, we must have

$$x = \alpha X, \quad y = \beta Y, \quad z = \gamma Z, \quad w = \gamma W;$$

and if ξ, η, ζ, ω be the coordinates of any circle, the coordinates of the corresponding circle referred to the new system will be $\alpha\xi, \beta\eta, \gamma\zeta, \delta\omega$: $\alpha\beta\gamma\delta$ being some constants.

Chapter V.—General Equation of the Second Degree in Power-Coordinates.

Nature of the Curve.—§ 165.

165. The most general form of the equation of the second degree may be written

$$\phi(x, y, z, w) = ax^2 + by^2 + cz^2 + dw^2 + 2fyz + 2gzx + 2hxy + 2lxw + 2myw + 2nzw = 0, \quad (182)$$

$(xyzw)$ being the coordinates of a point, and therefore satisfying the equation of the absolute ψ, which is also of the second degree; it follows, then, that the general equation of the second degree contains only eight arbitrary constants.

Let P be any point on the curve, and let the Cartesian coordinates of P referred to rectangular axes through the centre of the sphere be X, Y, Z; and let R be the radius of the sphere, then by § 125 we see that we may put

$$x = X^2 + Y^2 + Z^2 - a_1X - b_1Y - c_1Z,$$
$$y = X^2 + Y^2 + Z^2 - a_2X - b_2Y - c_2Z,$$
$$z = X^2 + Y^2 + Z^2 - a_3X - b_3Y - c_3Z,$$
$$w = X^2 + Y^2 + Z^2 - a_4X - b_4Y - c_4Z.$$

Substituting for (x, y, z, w) in equation (182), we see that the curve in question is the curve of intersection of the surfaces

$$X^2+Y^2+Z^2=R^2,$$

and

$$(X^2+Y^2+Z^2)^2+U_1(X^2+Y^2+Z^2)+U_2=0,$$

U_1 and U_2 being homogeneous expressions of the first and second degree respectively in (X, Y, Z).

So that the curve is the complete intersection of a sphere with a quadric surface, and therefore may be called a "spheri-quadric." These curves have been extensively studied. CASEY calls them sphero-quartics ("Cyclides and Sphero-quartics," (1871). 'Phil. Trans.,' vol. 161). DARBOUX calls them spherical cyclics ('Sur une Classe remarquable de Courbes et de Surfaces Algébriques,' 1873). Mr. H. M. JEFFERY ('London Math. Soc. Proc.,' vol. 20, 1885, p. 102) has proposed to call them sphero-cyclides. The name spheri-quadric is due to Professor CAYLEY.

Equation to Tangent at any Point.—§§ 166–171.

166. Let $(\xi\eta\zeta\omega)$ be the coordinates of any circle touching the spheri-quadric $\phi(xyzw)=0$ at the point $(x'y'z'w')$, then $\psi(xyzw)=0$ being the equation of the absolute, we must have

$$\frac{\partial\psi}{\partial\xi}x'+\frac{\partial\psi}{\partial\eta}y'+\frac{\partial\psi}{\partial\zeta}z'+\frac{\partial\psi}{\partial\omega}w'=0,$$

$$\frac{\partial\psi}{\partial\xi}\delta x'+\frac{\partial\psi}{\partial\eta}\delta y'+\frac{\partial\psi}{\partial\zeta}\delta z'+\frac{\partial\psi}{\partial\omega}\delta w'=0,$$

$$\frac{\partial\phi}{\partial x'}\delta x'+\frac{\partial\phi}{\partial y'}\delta y'+\frac{\partial\phi}{\partial z'}\delta z+\frac{\partial\phi}{\partial w'}\delta w'=0,$$

$$\frac{\partial\psi}{\partial x'}\delta x'+\frac{\partial\psi}{\partial y'}\delta y'+\frac{\partial\psi}{\partial z'}\delta z+\frac{\partial\psi}{\partial w'}\delta w'=0.$$

And hence we must have

$$\frac{\dfrac{\partial\psi}{\partial\xi}}{\dfrac{\partial\phi}{\partial x'}+k\dfrac{\partial\psi}{\partial x'}}=\frac{\dfrac{\partial\psi}{\partial\eta}}{\dfrac{\partial\phi}{\partial y'}+k\dfrac{\partial\psi}{\partial y'}}=\frac{\dfrac{\partial\psi}{\partial\zeta}}{\dfrac{\partial\phi}{\partial z'}+k\dfrac{\partial\psi}{\partial z'}}=\frac{\dfrac{\partial\psi}{\partial\omega}}{\dfrac{\partial\phi}{\partial w'}+k\dfrac{\partial\psi}{\partial w'}}, \quad \ldots\ldots (183)$$

where k is indeterminate.

Hence every circle which touches the curve at the point $(x'y'z'w')$ has its equation of the form

$$\left(x\frac{\partial}{\partial x'}+y\frac{\partial}{\partial y'}+z\frac{\partial}{\partial z'}+w\frac{\partial}{\partial w'}\right)(\phi+k\psi)=0. \quad \ldots\ldots (184)$$

167. To determine the equation of the tangent (*i.e.*, tangent great circle) to the curve at the point $(x'y'z'w')$, we must determine k in equation (184), so that the equation may be satisfied by $(k_1k_2k_3k_4)$ the coordinates of the circle at infinity. Thus the equation of the tangent is

$$\left(k_1\frac{\partial\phi}{\partial x'}+k_2\frac{\partial\phi}{\partial y'}+k_3\frac{\partial\phi}{\partial z'}+k_4\frac{\partial\phi}{\partial w'}\right)\left(x\frac{\partial\psi}{\partial x'}+y\frac{\partial\psi}{\partial y'}+z\frac{\partial\psi}{\partial z'}+w\frac{\partial\psi}{\partial w'}\right)$$
$$=\left(k_1\frac{\partial\psi}{\partial x'}+k_2\frac{\partial\psi}{\partial y'}+k_3\frac{\partial\psi}{\partial z'}+k_4\frac{\partial\psi}{\partial w'}\right)\left(x\frac{\partial\phi}{\partial x'}+y\frac{\partial\phi}{\partial y'}+z\frac{\partial\phi}{\partial z'}+w\frac{\partial\phi}{\partial w'}\right). \quad . \quad . \quad . \quad (185)$$

168. The circle given by equations (183) or (184) will touch the curve at the point $(x''y''z''w'')$ if

$$\frac{\dfrac{\partial\phi}{\partial x'}+k\dfrac{\partial\psi}{\partial x'}}{\dfrac{\partial\phi}{\partial x''}+k\dfrac{\partial\psi}{\partial x''}}=\frac{\dfrac{\partial\phi}{\partial y'}+k\dfrac{\partial\psi}{\partial y'}}{\dfrac{\partial\phi}{\partial y''}+k\dfrac{\partial\psi}{\partial y''}}=\frac{\dfrac{\partial\phi}{\partial z'}+k\dfrac{\partial\psi}{\partial z'}}{\dfrac{\partial\phi}{\partial z''}+k\dfrac{\partial\psi}{\partial z''}}=\frac{\dfrac{\partial\phi}{\partial w'}+k\dfrac{\partial\psi}{\partial w'}}{\dfrac{\partial\phi}{\partial w''}+k\dfrac{\partial\psi}{\partial w''}};$$

i.e., if k satisfy the quartic equation

$$\begin{vmatrix} \frac{\partial^2}{\partial x^2}, & \frac{\partial^2}{\partial x\partial y}, & \frac{\partial^2}{\partial x\partial z}, & \frac{\partial^2}{\partial x\partial w} \\ \frac{\partial^2}{\partial x\partial y}, & \frac{\partial^2}{\partial y^2}, & \frac{\partial^2}{\partial y\partial z}, & \frac{\partial^2}{\partial y\partial w} \\ \frac{\partial^2}{\partial x\partial z}, & \frac{\partial^2}{\partial y\partial z}, & \frac{\partial^2}{\partial z^2}, & \frac{\partial^2}{\partial z\partial w} \\ \frac{\partial^2}{\partial x\partial w}, & \frac{\partial^2}{\partial y\partial w}, & \frac{\partial^2}{\partial z\partial w}, & \frac{\partial^2}{\partial w^2} \end{vmatrix}(\phi+k\psi)=0,$$

i.e.,

$$\mathrm{H}(\phi+k\psi)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

where $\mathrm{H}(u)$ denotes the Hessian of u.

We infer then that there are, in general, four systems of bitangent circles, each circle belonging to a particular system cutting a certain circle orthogonally; the coordinates of these four circles being proportional to the minors of the constituents of any row of the above determinants, corresponding to the four values of k.

169. If the coordinates of a bitangent circle satisfy the condition

$$k_1\frac{\partial\psi}{\partial\xi}+k_2\frac{\partial\psi}{\partial\eta}+k_3\frac{\partial\psi}{\partial\zeta}+k_4\frac{\partial\psi}{\partial\omega}=0,$$

the circle is a great circle; there will, in general, be eight such great circles, two belonging to each system of bitangent circles.

170. If the coordinates of a bitangent circle satisfy the equation of the absolute, the circle reduces to a point and corresponds to a focus of a plane bicircular quartic—there are clearly sixteen such foci, four on each of the circles which cut the bitangent systems orthogonally. Dr. CASEY ("Cyclides and Sphero-Quartics") calls these single-foci.

171. It is clear that, if by any linear transformation of coordinates the equations $\phi=0$, $\psi=0$ become respectively $\Phi=0$, $\Psi=0$, then the same value of k which satisfies $\mathrm{H}(\phi+k\psi)=0$ must also satisfy $\mathrm{H}(\Phi+k\Psi)=0$.

Hence the coefficients of powers of k in the equation (186) are invariants.

Equation of the Normal at any Point.—§§ 172–174.

172. Let $(\xi\eta\zeta\omega)$ be the coordinates of any circle which cuts the curve $\phi(xyzw)=0$ orthogonally at the point $(x'y'z'w')$, then we must have

$$\left(\xi\frac{\partial}{\partial x'}+\eta\frac{\partial}{\partial y'}+\zeta\frac{\partial}{\partial z'}+\omega\frac{\partial}{\partial w'}\right)(\phi+k\psi)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (187)$$

for all values of k.

173. It follows that the coordinates of the normal (*i.e.*, great circle) at $(x'y'z'w')$ must satisfy

$$\xi\frac{\partial\phi}{\partial x'}+\eta\frac{\partial\phi}{\partial y'}+\zeta\frac{\partial\phi}{\partial z'}+\omega\frac{\partial\phi}{\partial w'}=0,$$

$$\xi\frac{\partial\psi}{\partial x'}+\eta\frac{\partial\psi}{\partial y'}+\zeta\frac{\partial\psi}{\partial z'}+\omega\frac{\partial\psi}{\partial w'}=0,$$

$$\xi\frac{\partial\psi}{\partial k_1}+\eta\frac{\partial\psi}{\partial k_2}+\zeta\frac{\partial\psi}{\partial k_3}+\omega\frac{\partial\psi}{\partial k_4}=0.$$

Hence the equation of the normal is

$$\begin{vmatrix} \frac{\partial\psi}{\partial x}, & \frac{\partial\psi}{\partial y}, & \frac{\partial\psi}{\partial z}, & \frac{\partial\psi}{\partial w} \\ \frac{\partial\phi}{\partial x'}, & \frac{\partial\phi}{\partial y'}, & \frac{\partial\phi}{\partial z'}, & \frac{\partial\phi}{\partial w'} \\ \frac{\partial\psi}{\partial x'}, & \frac{\partial\psi}{\partial y'}, & \frac{\partial\psi}{\partial z'}, & \frac{\partial\psi}{\partial w'} \\ \frac{\partial\psi}{\partial k_1}, & \frac{\partial\psi}{\partial k_2}, & \frac{\partial\psi}{\partial k_3}, & \frac{\partial\psi}{\partial k_4} \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (188)$$

174. From equation (187) we can deduce

$$x'\frac{\partial\phi}{\partial\xi}+y'\frac{\partial\phi}{\partial\eta}+z'\frac{\partial\phi}{\partial\zeta}+w'\frac{\partial\phi}{\partial\omega}=0,$$

$$x'\frac{\partial\psi}{\partial\xi}+y'\frac{\partial\psi}{\partial\eta}+z'\frac{\partial\psi}{\partial\zeta}+w'\frac{\partial\psi}{\partial\omega}=0.$$

If then $(\xi\eta\zeta\omega)$ be so chosen that

$$\frac{\partial\phi}{\partial\xi}\Big/\frac{\partial\psi}{\partial\xi}=\frac{\partial\phi}{\partial\eta}\Big/\frac{\partial\psi}{\partial\eta}=\frac{\partial\phi}{\partial\zeta}\Big/\frac{\partial\psi}{\partial\zeta}=\frac{\partial\phi}{\partial\omega}\Big/\frac{\partial\psi}{\partial\omega}=-\mu, \text{ say},$$

then the circle $(\xi\eta\zeta\omega)$ cuts $\phi(xyzw)$ orthogonally in each of the four points in which it meets it; we see at once that μ must satisfy the equation

$$\mathrm{H}(\phi+\mu\psi)=0,$$

which is the same equation as in § 168; the coordinates of the four orthogonal circles corresponding to the four values of μ, being proportional to the minors of the determinant $\mathrm{H}(\phi+\mu\psi)$.

The Principal Circles.—§§ 175–179.

175. The four orthogonal circles found in the last article are usually called the principal circles of the curve. By § 168, we see that a spheri-quadric is the envelope of a series of circles which cut one of the principal circles orthogonally, and it is evident by inversion that the curve must be anallagmatic, *i.e.*, its own inverse with respect to each of its four principal circles; also each point in which a principal circle cuts a spheri-quadric must be a cyclic point on the curve; there are in general sixteen such points. Again, the double great circle tangents are the tangents which can be drawn from the poles of the principal circles.

176. It is easily proved, as in § 81, that any two circles corresponding to different values of k given by $\mathrm{H}(\phi+k\psi)=0$, cut orthogonally; hence, if the four roots of the discriminating quartic be different there are four principal circles which are mutually orthotomic, and the poles of these circles must be such that the arc joining any two is perpendicular to the arc joining the remaining two.

177. If the roots of $\mathrm{H}(\phi+k\psi)=0$ are all different, then we can reduce the equation to the form

$$ax^2+by^2+cz^2+dw^2=0,$$

the system of reference being the four principal circles, and a, b, c, d being the roots of the discriminating quartic.

178. If two roots of the quartic $\mathrm{H}(\phi+k\psi)$ are equal, then taking the two principal circles corresponding to the two other values of k, and any other circles forming with them an orthogonal system, as circles of reference, we can reduce the equation to the form

$$ax^2+by^2+cz^2+dw^2+2nzw=0,$$

and, exactly as in § 83, we see that if one of the two circles (x, y) be imaginary, then the discriminating quartic can have two equal roots only when

$$c=d, \qquad n=0;$$

in which case the equation reduces to

$$ax^2+by^2+cz^2+cw^2=0,$$

which represents two imaginary circles.

But if (x, y) be both real, then by taking for system of reference the two principal circles (x, y), and their two points of intersection (z, w) we can show that the equation may be reduced to the form

$$ax^2+by^2+cz^2=0,$$

the equation of the absolute being

$$x^2+y^2=4zw.$$

A spheri-quadric represented by an equation of this form has a finite node, viz., the point $z=0$.

179. Now, let us suppose the discriminating quartic to have three equal roots, then, as in § 84, we can show that if we take as system of reference the principal circle (x) corresponding to the unequal root, the node (z), and the circle (y), passing through (z), cutting (x) orthogonally, and passing through the other point (w), in which (x) cuts the curve: the equation may be reduced to the form

$$x^2=2fyz.$$

The point z is clearly a cusp, the circle $x=0$ being the cuspidal edge.

Observation.—If we suppose two of our circles of reference to be great circles, the curves degenerate into sphero-conics. As from § 128, it is clear that inversion is merely equivalent to a linear transformation, nodal and cuspidal spheri-quadrics are the inverse curves of sphero-conics.

Chapter VI.—Classification of Spheri-quadrics.

The method followed in Part I. for the classification of bicircular quartics is not suited for a systematic classification of spheri-quadrics, for which see Casey, "On Cyclides and Sphero-Quartics," chap. xi. In this memoir it is only proposed to discuss the chief properties of the curves, following the order of chap. vii., Part I.

General Spheri-quadric.—§§ 180–184*.

180. The equation of the curve is of the form

$$ax^2+by^2+cz^2+dw^2=0,$$

and the equation of the absolute may be taken as

$$x^2+y^2+z^2+w^2=0;$$

where the coordinates of the circle at infinity are $\cot r_1$, $\cot r_2$, $\cot r_3$, $\cot r_4$; r_1, r_2, r_3, r_4 being the radii of the principal circles.

181. The coordinates of any circle touching the curve at the point $(x'y'z'w')$ must be proportional to

$$(a+k)x',\ (b+k)y',\ (c+k)z',\ (d+k)w'.$$

The equation to the tangent at the point will be,

$$\begin{aligned}&(x'x+y'y+z'z+w'w)(ax'\cot r_1+by'\cot r_2+cz'\cot r_3+dw'\cot r_4)\\&=(ax'x+by'y+cz'z+dw'w)(x'\cot r_1+y'\cot r_2+z'\cot r_3+w'\cot r_4).\end{aligned}\quad(189)$$

The equation to the normal at the point $(x'y'z'w')$ will be,

$$\begin{vmatrix} x, & y, & z, & w \\ x', & y', & z', & w' \\ ax', & by', & cz', & dw' \\ \cot r_1, & \cot r_2, & \cot r_3, & \cot r_4 \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad (190)$$

182. The systems of bitangent circles will be given by,

$$\left.\begin{aligned}\xi=0,\ &\frac{\eta^2}{b-a}+\frac{\zeta^2}{c-a}+\frac{\omega^2}{d-a}=0\\ \eta=0,\ &\frac{\xi^2}{a-b}+\frac{\zeta^2}{c-b}+\frac{\omega^2}{d-b}=0\\ \zeta=0,\ &\frac{\xi^2}{a-c}+\frac{\eta^2}{b-c}+\frac{\omega^2}{d-c}=0\\ \omega=0,\ &\frac{\xi^2}{a-d}+\frac{\eta^2}{b-d}+\frac{\zeta^2}{c-d}=0\end{aligned}\right\};\quad . \quad . \quad . \quad . \quad . \quad . \quad (191)$$

and the coordinates of the single foci will be given by,

$$\left.\begin{aligned}x=0,\ &\frac{y^2}{(c-d)(b-a)}=\frac{z^2}{(d-b)(c-a)}=\frac{w^2}{(b-c)(d-a)}\\ y=0,\ &\frac{x^2}{(c-d)(a-b)}=\frac{z^2}{(d-a)(c-b)}=\frac{w^2}{(a-c)(d-b)}\\ z=0,\ &\frac{x^2}{(b-d)(a-c)}=\frac{y^2}{(d-a)(b-c)}=\frac{w^2}{(a-b)(d-c)}\\ w=0,\ &\frac{x^2}{(b-c)(a-d)}=\frac{y^2}{(c-a)(b-d)}=\frac{z^2}{(a-b)(c-d)}\end{aligned}\right\}\quad . \quad . \quad . \quad . \quad . \quad (192)$$

The curve has also six double foci (see CASEY, "Cyclides," § 130), and thus the twenty-eight points of intersection of the eight common tangents of the curve and the circle at infinity are accounted for.

183. From the form of the equations (192), it follows that all curves given by

$$\frac{x^2}{\alpha^2+\kappa}+\frac{y^2}{\beta^2+\kappa}+\frac{z^2}{\gamma^2+\kappa}+\frac{w^2}{\delta^2+\kappa}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (193)$$

must be confocal with the curve

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}+\frac{w^2}{\delta^2}=0.$$

Subtracting these equations, we have

$$\frac{x^2}{\alpha^2(\alpha^2+\kappa)}+\frac{y^2}{\beta^2(\beta^2+\kappa)}+\frac{z^2}{\gamma^2(\gamma^2+\kappa)}+\frac{w^2}{\delta^2(\delta^2+\kappa)}=0.$$

Hence the curves cut orthogonally at their common points.

And since equation (193) may be regarded as a quadratic in κ, we infer that through any point on a sphere two spheri-quadrics can be drawn confocal with a given spheri-quadric, and these two cut orthogonally.

184. We may prove exactly, as in § 123, that the coordinates of the osculating circle at any point $(x'y'z'w')$ on the curve

$$ax^2+by^2+cz^2+dw^2=0,$$

will be proportional to

$$(a-b)(a-c)(a-d)x'^3, \quad (b-a)(b-c)(b-d)y'^3, \quad (c-a)(c-b)(c-d)z'^3,$$
$$(d-a)(d-b)(d-c)w'^3; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (194)$$

and if R be the radius of curvature at the point we shall have by § 154,

$$(a^2x'^2+b^2y'^2+c^2z'^2+d^2w'^2)^{\frac{1}{2}}.\cot \mathrm{R}$$
$$=(a-b)(a-c)(a-d)\cot r_1x'^3+(b-a)(b-c)(b-d)\cot r_2y'^3$$
$$+(c-a)(c-b)(c-d)\cot r_3z'^3+(d-a)(d-b)(d-c)\cot r_3w'^3. \quad . \quad (195)$$

184*. If one of the principal circles is a great circle, the corresponding foci may coincide with the centres of one of the other principal circles; in this case the curve has been called by CASEY a "sphero-Cartesian." Thus suppose $r_4=\frac{\pi}{2}$, and let one of the foci on $w=0$ coincide with the centre of the circle whose radius is r_1, the coordinates of this point are $-\operatorname{cosec} r_1$, $\cot r_2$, $\cot r_3$; and the necessary condition that the curve

$$ax^2+by^2+cz^2+dw^2=0$$

may represent a "Cartesian" is given by

$$\frac{\operatorname{cosec}^2 r_1}{a-d}+\frac{\cot^2 r_2}{b-d}+\frac{\cot^2 r_3}{c-d}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (196)$$

Spheri-quadrics having a Third Node.—§§ 185–189.

185. The equation of the curve is of the form

$$ax^2+by^2+cz^2=0,$$

the equation of the absolute being

$$x^2+y^2=zw;$$

where the coordinates of the circle at infinity are $\cot r_1$, $\cot r_2$, $\operatorname{cosec} e$, $\operatorname{cosec} e$; r_1, r_2 being the radii of the two principal circles, and $2e$ the arc between their points of intersection.

186. The coordinates $(\xi\eta\zeta\omega)$ of any circle which touches the curve at the point $(x'y'z'w')$ are given by

$$\frac{2\xi}{(a+2k)x'}=\frac{2\eta}{(b+2k)y'}=\frac{-\omega}{cz'-kw'}=\frac{-\zeta}{-kz'}.$$

The equation of the tangent at the point will be,

$$\begin{aligned}&(2x'x+2y'y-w'z-z'w)(ax'\cot r_1+by'\cot r_2+cz'\operatorname{cosec} e)\\&\quad=(ax'x+by'y+cz'z)(2x'\cot r_1+2y'\cot r_2-w'\operatorname{cosec} e-z'\operatorname{cosec} e). \quad . \quad (197)\end{aligned}$$

The equation of the normal at $(x'y'z'w')$ will be,

$$\begin{vmatrix} 2x, & 2y, & -w, & -z \\ 2x', & 2y', & -w', & -z' \\ ax', & by', & cz', & 0 \\ 2\cot r_1, & 2\cot r_2, & -\operatorname{cosec} e, & -\operatorname{cosec} e \end{vmatrix}=0. \quad . \quad . \quad (198)$$

187. The systems of bitangent circles will be given by

$$\left.\begin{aligned}\xi=0,\ &\frac{\eta^2}{b-a}+\frac{\zeta\omega}{a}-\frac{c\zeta^2}{a^2}=0\\ \eta=0,\ &\frac{\xi^2}{a-b}+\frac{\zeta\omega}{b}-\frac{c\zeta^2}{b^2}=0\end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (199)$$

and the coordinates of the foci by

$$\left.\begin{aligned} x=0,\quad & y^2=zw=\frac{(b-a)c}{ab}x^2 \\ y=0,\quad & x^2=zw=\frac{(a-b)c}{ab}z^2 \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (200)$$

188. From the form of these equations we see that every curve whose equation is of the form

$$\frac{x^2}{\alpha^2+\kappa}+\frac{y^2}{\beta^2+\kappa}+\frac{z^2}{\gamma^2}=0,$$

is confocal with the curve

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=0\ ;$$

and subtracting we have

$$\frac{x^2}{\alpha^2(\alpha^2+\kappa)}+\frac{y^2}{\beta^2(\beta^2+\kappa)}=0.$$

Hence two such curves intersect orthogonally.

We infer that through any point on a sphere two spheri-quadrics can be drawn confocal with a given nodal-spheri-quadric ; and these curves will cut orthogonally.

189. If $r_3=\frac{\pi}{2}$ and the coefficients a, b, c in the equation

$$ax^2+by^2+cz^2=0,$$

satisfy the relation

$$(a-b)c=ab, \quad \ldots\ldots\ldots\ldots \quad (201)$$

then one of the foci on the principal circle $y=0$ coincides with the centre of the other principal circle, and the curve becomes a Cartesian, having a third node.

Cuspidal Spheri-quadrics.—§§ 190–192.

190. The equation of the curve is of the form

$$x^2=2ayz,$$

the system of reference being the principal circle, the circle orthogonal to it through the cusp and the other point, in which the principal circle cuts the curve, the cusp and the other point common to the two circles.

If r_1, r_2 be the radii of the circles, $2e$ the arc between their points of intersection, we may take the equation of the absolute as

$$x^2+y^2=zw,$$

the coordinates of the circle at infinity being $\cot r_1$, $\cot r_2$, $\operatorname{cosec} e$, $\operatorname{cosec} e$.

191. The coordinates $(\xi\eta\zeta\omega)$ of any circle which touches the curve at the point $(x'y'z'w')$ must satisfy

$$\frac{2\xi}{(1+2k)x'}=\frac{2\eta}{-az'+2ky'}=\frac{-\omega}{-ay'-kw'}=\frac{-\zeta}{-kz'}.$$

The equation of the tangent at $(x'y'z'w')$ will be

$$\begin{aligned}&(2x'x+2y'y-w'z-wz')(x'\cot r_1-az'\cot r_2-ay'\operatorname{cosec} e)\\&=(x'x-az'y-ay'z)(2x'\cot r_1+2y'\cot r_2-(w'+z')\operatorname{cosec} e)\quad\ldots\quad(202)\end{aligned}$$

The equation of the normal at $(x'y'z'w')$ will be,

$$\begin{vmatrix} 2x, & 2y, & w, & z \\ 2x', & 2y', & w', & z' \\ x', & -az', & ay', & 0 \\ 2\cot r_1, & 2\cot r_2, & \operatorname{cosec} e, & \operatorname{cosec} e \end{vmatrix}=0.\quad\ldots\quad(203)$$

192. The system of bitangent circles is given by

$$\xi=0,\quad(\eta-a\zeta)^2=\zeta\omega\quad\ldots\quad(204)$$

The focus of the curve is given by

$$\frac{x}{0}=\frac{y}{a}=\frac{z}{2}=\frac{2w}{a^2}\quad\ldots\quad(205)$$

Equation of a Spheri-quadric Referred to three Circles Orthogonal to one of its Principal Circles.—§§ 193–198.

193. Let $w=0$ be one of the principal circles of a spheri-quadric; then if (x, y, z) are any three circles orthogonal to w, the equation of the spheri-quadric must be of the form

$$w^2+f(x, y, z)=0,$$

and the equation of the absolute will also be of the same form. Hence, by subtraction, we have for the equation of the spheri-quadric

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,\quad\ldots\quad(206)$$

and this is a form to which the equation of any spheri-quadric can be reduced.

194. We shall find it convenient to suppose the coordinates (xyz) to be equal to the

powers of a point with respect to the three circles of reference; then by § 136, equation (147), the equation of any circle orthogonal to w will be given by

$$\alpha x+\beta y+\gamma z=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (207)$$

where α, β, γ are proportional to the triangular coordinates of the pole of the circle referred to the triangle formed by the poles of the circles (x, y, z).

Suppose, now, the circle given by (206) to be a bitangent circle of the spheri-quadric (205), then we must have

$$\begin{vmatrix} a, & h, & g, & \alpha \\ h, & b, & f, & \beta \\ g, & f, & c, & \gamma \\ \alpha, & \beta, & \gamma, & 0 \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (208)$$

Hence it follows that the poles of all bitangent circles belonging to the same system lie on a sphero-conic; or again, the spheri-quadric (205) is the envelope of circles whose poles lie on (208), and which cut a given circle $w=0$ orthogonally.

195. If the circles (x, y, z) are the other three principal circles of the curve (205), we know that the equation of the curve is of the form

$$ax^2+by^2+cz^2=0,$$

hence the equation of the sphero-conic is

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}=0.$$

Thus the sphero-conic corresponding to one principal circle is self-conjugate to the triangle formed by the poles of the other three principal circles.

196. Again, in the case of a nodal-spheri-quadric the equation of the curve referred to its other principal circle, and the two points in which its two principal circles intersect, is of the form

$$ax^2+by^2+2fyz=0,$$

so that the equation of the sphero-conic must be

$$f^2\alpha^2-ab\gamma^2+2af\beta\gamma=0.$$

Thus the sphero-conic must pass through the node.

197. Let (y, z) be any two bitangent circles, (x) the circle which passes through their four points of contact; the equation of the curve takes the form

$$x^2=2fyz,$$

and the sphero-conic must be given by

$$f\alpha^2=2\beta\gamma.$$

As a particular case, we may suppose (y, z) to be the pair of great circles which can be drawn from the pole of (w) to have double contact with the curve, and then it follows that the centre of the sphero-conic coincides with the centre of the circle which passes through the points of contact of these double tangents: *i.e.*, the centre of the sphero-conic coincides with the centre of the polar circle of the centre of w with respect to the spheri-quadric.

198. Taking for circles of reference any three bitangent circles orthogonal to w, the equation of the curve takes the form

$$a\sqrt{x}+b\sqrt{y}+c\sqrt{z}=0,$$

and the equation of the sphero-conic becomes

$$\frac{a^2}{\alpha}+\frac{b^2}{\beta}+\frac{c^2}{\gamma}=0.$$

Hence, if P be any point on the curve, A, B, C three foci on the same principal circle, then

$$a.\sin\tfrac{1}{2}\mathrm{AP}+b.\sin\tfrac{1}{2}\mathrm{BP}+c.\sin\tfrac{1}{2}\mathrm{CP}=0. \quad . \quad . \quad . \quad . \quad . \quad (209)$$

Or, again, if the curve is a sphero-Cartesian, so that the sphero-conic becomes a circle, then A, B, C being the centres of any three bitangent circles of the system, P a point on the curve; $2p$, $2q$, $2r$ the tangents from P to these circles; we have

$$\sin\tfrac{1}{2}a.\sin\tfrac{1}{2}p+\sin\tfrac{1}{2}b.\sin\tfrac{1}{2}q+\sin\tfrac{1}{2}c.\sin\tfrac{1}{2}r=0, \quad . \quad . \quad . \quad . \quad (210)$$

a, b, c being the sides of the triangle ABC.

PART III.—SYSTEMS OF SPHERES.

CHAPTER I.—GENERAL SYSTEMS OF SPHERES.

The Power of two Spheres.—§§ 199–201.

199. The power of two spheres is the square of the distance between their centres less the sum of the squares of their radii.

Thus if any two spheres be denoted by (1, 2) we shall have

$$\pi_{1,2}=d^2_{1,2}-r_1^2-r_2^2=2r_1r_2\cos\omega_{1,2};$$

where $\pi_{1,2}$ denotes their power, r_1, r_2 their radii, $\omega_{1,2}$ their angle of intersection, $d_{1,2}$ the distance between their centres.

The definition is due to DARBOUX ('Annales de l'École Normale Supérieure,' vol. 1, 1872); it is also given in a paper to be found in CLIFFORD'S 'Mathematical Papers,' p. 332; the date of which paper is assigned by the editor as 1868 (see note on p. 332).

200. If the equations of two spheres be

$$x^2+y^2+z^2+2fx+2gy+2hz+c=0,$$
$$x^2+y^2+z^2+2f'x+2g'y+2h'z+c'=0;$$

we have at once for their power

$$\pi=c+c'-2ff'-2gg'-2hh'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (211)$$

Extending the definition given in § 4, the power of a sphere and a plane may be defined as twice the perpendicular distance of the centre from the plane; thus the power of the sphere

$$x^2+y^2+z^2+2fx+2gy+2hz+c=0,$$

and the plane

$$x\cos\alpha+y\cos\beta+z\cos\gamma-p=0,$$

will be

$$\pi=2p-2f\cos\alpha-2g\cos\beta-2h\cos\gamma. \quad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

And similarly the power of two planes may be defined as twice the cosine of the angle between them.

Also if θ denote the plane at infinity, S any sphere, or point (considered as a sphere of indefinitely small radius), we shall have

$$\pi_{\theta,S}=1;$$

and if L be any plane, $\pi_{\theta,L}=0$; and also $\pi_{\theta,\theta}=0$.

201. If we take the inverse spheres, with respect to a sphere whose centre is the origin and radius R, of the spheres

$$x^2+y^2+z^2+2fx+2gy+2hz+c=0,$$
$$x^2+y^2+z^2+2f'x+2g'y+2h'z+c'=0;$$

we see at once, that the power π' of the inverse spheres is connected with the power of the original spheres by the formula

$$\pi'=\pi.\frac{R^4}{cc'};$$

whence we deduce at once, denoting the spheres by S_1, S_2, and the inverse spheres by S'_1, S'_2,

$$\frac{\pi_{S_1,S_2}}{\sqrt{\pi_{0,S_1}\cdot\pi_{0,S_2}}} = \frac{\pi_{S'_1,S'_2}}{\sqrt{\pi_{0,S'_1}\cdot\pi_{0,S'_2}}},$$

(O) denoting the origin.

And generally, if x, y denote either spheres, points, or planes, we infer that the expression

$$\frac{\pi_{x,y}}{\sqrt{\pi_{0,x}\cdot\pi_{0,y}}}$$

is unaltered by inverting on the point (O).

General Theorems.—§§ 202–205.

202. If we have a system of six spheres, say (1, 2, 3, 4, 5, 6), their powers with respect to any other system of six spheres, say (7, 8, 9, 10, 11, 12), are connected by the relation

$$\Pi\begin{pmatrix}1,&2,&3,&4,&5,&6\\7,&8,&9,&10,&11,&12\end{pmatrix}=0.$$

For if we multiply together the matrices

$$\begin{vmatrix} 1, & 2f_1, & 2g_1, & 2h_1, & c_1 \\ 1, & 2f_2, & 2g_2, & 2h_2, & c_2 \\ 1, & 2f_3, & 2g_3, & 2h_3, & c_3 \\ 1, & 2f_4, & 2g_4, & 2h_4, & c_4 \\ 1, & 2f_5, & 2g_5, & 2h_5, & c_5 \\ 1, & 2f_6, & 2g_6, & 2h_6, & c_6 \end{vmatrix}, \qquad \begin{vmatrix} c_7, & -f_7, & -g_7, & -h_7, & 1 \\ c_8, & -f_8, & -g_8, & -h_8, & 1 \\ c_9, & -f_9, & -g_9, & -h_9, & 1 \\ c_{10}, & -f_{10}, & -g_{10}, & -h_{10}, & 1 \\ c_{11}, & -f_{11}, & -g_{11}, & -h_{11}, & 1 \\ c_{12}, & -f_{12}, & -g_{12}, & -h_{12}, & 1 \end{vmatrix};$$

we have at once the equation,

$$\begin{vmatrix} \pi_{1,7}, & \pi_{1,8}, & \pi_{1,9}, & \pi_{1,10}, & \pi_{1,11}, & \pi_{1,12} \\ \pi_{2,7}, & \pi_{2,8}, & \pi_{2,9}, & \pi_{2,10}, & \pi_{2,11}, & \pi_{2,12} \\ \pi_{3,7}, & \pi_{3,8}, & \pi_{3,9}, & \pi_{3,10}, & \pi_{3,11}, & \pi_{3,12} \\ \pi_{4,7}, & \pi_{4,8}, & \pi_{4,9}, & \pi_{4,10}, & \pi_{4,11}, & \pi_{4,12} \\ \pi_{5,7}, & \pi_{5,8}, & \pi_{5,9}, & \pi_{5,10}, & \pi_{5,11}, & \pi_{5,12} \\ \pi_{6,7}, & \pi_{6,8}, & \pi_{6,9}, & \pi_{6,10}, & \pi_{6,11}, & \pi_{6,12} \end{vmatrix} = 0,$$

i.e.,

$$\Pi\begin{pmatrix}1,&2,&3,&4,&5,&6\\7,&8,&9,&10,&11,&12\end{pmatrix}=0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (213)$$

a relation which is clearly true when any of the spheres are replaced by planes, or points, or the plane at infinity.

203. An important case is when the plane at infinity is a member of both systems of spheres; thus taking the two systems as $(\theta, 1, 2, 3, 4, 5)$, $(\theta, 6, 7, 8, 9, 10)$, we have

$$\Pi\begin{pmatrix}\theta, 1, 2, 3, 4, 5\\ \theta, 6, 7, 8, 9, 10\end{pmatrix}=0\,;$$

whence, if the radii of the spheres be all different from zero, and they cut at angles $\omega_{1,6}$, &c., we have

$$\begin{vmatrix} 0, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4}, & \frac{1}{r_5} \\ \frac{1}{r_6}, & \cos\omega_{1,6}, & \cos\omega_{2,6}, & \cos\omega_{3,6}, & \cos\omega_{4,6}, & \cos\omega_{5,6} \\ \frac{1}{r_7}, & \cos\omega_{1,7}, & \cos\omega_{2,7}, & \cos\omega_{3,7}, & \cos\omega_{4,7}, & \cos\omega_{5,7} \\ \frac{1}{r_8}, & \cos\omega_{1,8}, & \cos\omega_{2,8}, & \cos\omega_{3,8}, & \cos\omega_{4,8}, & \cos\omega_{5,8} \\ \frac{1}{r_9}, & \cos\omega_{1,9}, & \cos\omega_{2,9}, & \cos\omega_{3,9}, & \cos\omega_{4,9}, & \cos\omega_{5,9} \\ \frac{1}{r_{10}}, & \cos\omega_{1,10}, & \cos\omega_{2,10}, & \cos\omega_{3,10}, & \cos\omega_{4,10}, & \cos\omega_{5,10} \end{vmatrix} = 0\,. \quad (214)$$

204. If we have two systems of five spheres each, say $(1, 2, 3, 4, 5)$, $(6, 7, 8, 9, 10)$, then we have

$$\Pi\begin{pmatrix}1, 2, 3, 4, 5\\ 6, 7, 8, 9, 10\end{pmatrix}= \begin{vmatrix} 1, & 2f_1, & 2g_1, & 2h_1, & c_1 \\ 1, & 2f_2, & 2g_2, & 2h_2, & c_2 \\ 1, & 2f_3, & 2g_3, & 2h_3, & c_3 \\ 1, & 2f_4, & 2g_4, & 2h_4, & c_4 \\ 1, & 2f_5, & 2g_5, & 2h_5, & c_5 \end{vmatrix} \times \begin{vmatrix} c_6, & -f_6, & -g_6, & -h_6, & 1 \\ c_7, & -f_7, & -g_7, & -h_7, & 1 \\ c_8, & -f_8, & -g_8, & -h_8, & 1 \\ c_9, & -f_9, & -g_9, & -h_9, & 1 \\ c_{10}, & -f_{10}, & -g_{10}, & -h_{10}, & 1 \end{vmatrix};$$

and hence we see that,

$$\left\{\Pi\begin{pmatrix}1, 2, 3, 4, 5\\ 6, 7, 8, 9, 10\end{pmatrix}\right\}^2=\Pi\begin{pmatrix}1, 2, 3, 4, 5\\ 1, 2, 3, 4, 5\end{pmatrix}\times\Pi\begin{pmatrix}6, 7, 8, 9, 10\\ 6, 7, 8, 9, 10\end{pmatrix} \quad . \quad . \quad . \quad . \quad (215)$$

205. Again we have

$$\Pi\begin{pmatrix}\theta, 1, 2, 3, 4\\ \theta, 1, 2, 3, 4\end{pmatrix}=\begin{vmatrix} 0, & 0, & 0, & 0, & 1\\ 1, & 2f_1, & 2g_1, & 2h_1, & c_1\\ 1, & 2f_2, & 2g_2, & 2h_2, & c_2\\ 1, & 2f_3, & 2g_3, & 2h_3, & c_3\\ 1, & 2f_4, & 2g_4, & 2h_4, & c_4\end{vmatrix}\times\begin{vmatrix} 1, & 0, & 0, & 0, & 0\\ c_1, & -f_1, & -g_1, & -h_1, & 1\\ c_2, & -f_2, & -g_2, & -h_2, & 1\\ c_3, & -f_3, & -g_3, & -h_3, & 1\\ c_4, & -f_4, & -g_4, & -h_4, & 1\end{vmatrix}$$

$$=8\begin{vmatrix} -f_1, & -g_1, & -h_1, & 1\\ -f_2, & -g_2, & -h_2, & 1\\ -f_3, & -g_3, & -h_3, & 1\\ -f_4, & -g_4, & -h_4, & 1\end{vmatrix}^2$$

$$=288.\{\mathrm{V}(1, 2, 3, 4)\}^2; \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (216)$$

where V(1, 2, 3, 4) denotes the volume of the tetrahedron, whose vertices are the centres of the spheres (1, 2, 3, 4).

Again, let P be the common point of the spheres (1, 2, 3), then if P be denoted by the symbol (4) we have

$$288.\{\mathrm{V}(1, 2, 3, \mathrm{P})\}^2=\Pi\begin{pmatrix}\theta, 1, 2, 3, 4\\ \theta, 1, 2, 3, 4\end{pmatrix}$$

$$=\begin{vmatrix} 0, & 1, & 1, & 1, & 1\\ 1, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & 0\\ 1, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & 0\\ 1, & \pi_{3,1}, & \pi_{3,3}, & \pi_{3,4}, & 0\\ 1, & 0, & 0, & 0, & 0\end{vmatrix}$$

$$=-\Pi\begin{pmatrix}1, 2, 3\\ 1, 2, 3\end{pmatrix}.$$

Thus, if P be a common point of the system (1, 2, 3),

$$\Pi\begin{pmatrix}1, 2, 3\\ 1, 2, 3\end{pmatrix}=-288.\{\mathrm{V}(1, 2, 3, \mathrm{P})\}^2. \quad \ldots\ldots \quad (217)$$

Chapter II.—Special Systems of Spheres.

Sphere Cutting Four given Spheres Orthogonally.—§§ 206–208.

206. Let x denote the sphere which cuts the four given spheres (1, 2, 3, 4) orthogonally; then, since

$$\Pi\begin{pmatrix}\theta, x, 1, 2, 3, 4\\ \theta, x, 1, 2, 3, 4\end{pmatrix}=0,$$

we have

$$\pi_{x,x}\Pi\begin{pmatrix}\theta, 1, 2, 3, 4\\ \theta, 1, 2, 3, 4\end{pmatrix}=\Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix};$$

whence, by equation (216),

$$\pi_{x,x}=\frac{\Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix}}{288.\{V(1, 2, 3, 4)\}^2}.$$

Hence the radius of the sphere is given by

$$r=\frac{\left\{-\Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix}\right\}^{\frac{1}{2}}}{24.V(1, 2, 3, 4)} \qquad (218)$$

207. If the radii of the spheres (1, 2, 3, 4) be all zero, and the sides of the tetrahedron (1, 2, 3, 4) be denoted by a, b, c, a', b', c', we have at once,

$$-\Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix}=-\begin{vmatrix} 0, & a^2, & b^2, & c^2\\ a^2, & 0, & c'^2, & b'^2\\ b^2, & c'^2, & 0, & a'^2\\ c^2, & b'^2, & a'^2, & 0\end{vmatrix}$$

$$=2b^2b'^2c^2c'^2+2c^2c'^2a^2a'^2+2a^2a'^2b^2b'^2-a^4a'^4-b^4b'^4-c^4c'^4$$

$$=16\,\sigma.(\sigma-aa')(\sigma-bb')(\sigma-cc');$$

where

$$2\sigma=aa'+bb'+cc'.$$

Hence the radius of the sphere circumscribing a tetrahedron is equal to

$$\frac{1}{6}\frac{\{\sigma(\sigma-aa')(\sigma-bb')(\sigma-cc')\}^{\frac{1}{2}}}{V}, \qquad (219)$$

where V denotes the volume of the tetrahedron, which agrees with the known value (Todhunter, 'Spherical Trig.,' § 163).

208. If the four spheres meet in a point, the sphere which cuts them orthogonally will be coincident with this point, and so the radius must be zero. Hence, if the system (1, 2, 3, 4) have a common point, we must have

$$\Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix}=0. \quad \ldots \quad (220)$$

Five Spheres having a Common Orthogonal Sphere.—§§ 209–211.

209. Suppose the system of spheres (1, 2, 3, 4, 5) have a common orthogonal sphere, x say; then, y denoting any other sphere, the equation

$$\Pi\begin{pmatrix}x, 1, 2, 3, 4, 5\\ y, 1, 2, 3, 4, 5\end{pmatrix}=0,$$

leads at once to the condition

$$\Pi\begin{pmatrix}1, 2, 3, 4, 5\\ 1, 2, 3, 4, 5\end{pmatrix}=0; \quad \ldots \quad (221)$$

which is the necessary and sufficient condition that the system may have a common orthogonal sphere.

Similarly, if (6, 7, 8, 9, 10) denote any other system of spheres, we should have, since

$$\Pi\begin{pmatrix}x, 6, 7, 8, 9, 10\\ y, 1, 2, 3, 4, 5\end{pmatrix}=0,$$

$$\Pi\begin{pmatrix}6, 7, 8, 9, 10\\ 1, 2, 3, 4, 5\end{pmatrix}=0;$$

and hence

$$\left\{\Pi\begin{pmatrix}6, 7, 8, 9, 10\\ 1, 2, 3, 4, 5\end{pmatrix}\right\}^2=\Pi\begin{pmatrix}1, 2, 3, 4, 5\\ 1, 2, 3, 4, 5\end{pmatrix}\times\Pi\begin{pmatrix}6, 7, 8, 9, 10\\ 6, 7, 8, 9, 10\end{pmatrix}.$$

210. It is easy to prove, that if the system of spheres (1, 2, 3, 4, 5) be such that the condition (221) is satisfied, then any four of them will be connected with any four other spheres (6, 7, 8, 9) by the relation

$$\left\{\Pi\begin{pmatrix}6, 7, 8, 9\\ 1, 2, 3, 4\end{pmatrix}\right\}^2=\Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix}\times\Pi\begin{pmatrix}6, 7, 8, 9\\ 6, 7, 8, 9\end{pmatrix}.$$

211. Suppose now (x) to denote any sphere, then the equation

$$\Pi\begin{pmatrix}x, 1, 2, 3, 4\\ 1, 2, 3, 4, 5\end{pmatrix}=0$$

gives us

$$\pi_{x,1}.\Pi\begin{pmatrix}1,2,3,4\\2,3,4,5\end{pmatrix}-\pi_{x,2}.\Pi\begin{pmatrix}1,2,3,4\\1,3,4,5\end{pmatrix}+\pi_{x,3}.\Pi\begin{pmatrix}1,2,3,4\\1,2,4,5\end{pmatrix}$$
$$-\pi_{x,4}.\Pi\begin{pmatrix}1,2,3,4\\1,2,3,5\end{pmatrix}+\pi_{x,5}.\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}=0;$$

applying the theorem of § 210, we have

$$\pi_{x,5}.\left\{\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}\right\}^{\frac{1}{4}}=\pi_{x,1}.\left\{\Pi\begin{pmatrix}2,3,4,5\\2,3,4,5\end{pmatrix}\right\}^{\frac{1}{4}}+\pi_{x,2}.\left\{\Pi\begin{pmatrix}3,4,5,1\\3,4,5,1\end{pmatrix}\right\}^{\frac{1}{4}}$$
$$+\pi_{x,3}.\left\{\Pi\begin{pmatrix}4,5,1,2\\4,5,1,2\end{pmatrix}\right\}^{\frac{1}{4}}+\pi_{x,4}.\left\{\Pi\begin{pmatrix}5,1,2,3\\5,1,2,3\end{pmatrix}\right\}^{\frac{1}{4}}.$$

But if r denote the radius of the common orthogonal sphere, we have by equation (218)

$$-576r^2=\frac{\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}}{V(1,2,3,4)}=\frac{\Pi\begin{pmatrix}2,3,4,5\\2,3,4,5\end{pmatrix}}{V(2,3,4,5)}=\frac{\Pi\begin{pmatrix}3,4,5,1\\3,4,5,1\end{pmatrix}}{V(3,4,5,1)}=\frac{\Pi\begin{pmatrix}4,5,1,2\\4,5,1,2\end{pmatrix}}{V(4,5,1,2)}=\frac{\Pi\begin{pmatrix}5,1,2,3\\5,1,2,3\end{pmatrix}}{V(5,1,2,3)};$$

and thus our equation becomes

$$\pi_{x,5}.V(1,2,3,4)=\pi_{x,1}.V(2,3,4,5)+\pi_{x,2}.V(3,4,5,1)$$
$$+\pi_{x,3}.V(4,5,1,2)+\pi_{x,4}.V(5,1,2,3).$$

Thus, if any five spheres have a common orthogonal sphere, and the tetrahedral coordinates of the centre of one of them, (5) say, referred to the tetrahedron formed by the centres of the other four (1, 2, 3, 4), be $\alpha, \beta, \gamma, \delta$, then the powers of any other sphere are connected by the relation

$$\pi_{x,5}=\alpha.\pi_{x,1}+\beta.\pi_{x,2}+\gamma.\pi_{x,3}+\delta.\pi_{x,4}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (222)$$

As a particular case, if A, B, C, D be the centres of (1, 2, 3, 4), P any point on the sphere cutting these orthogonally, and O be any other point,

$$OP^2=\alpha.(OA^2-r_1^2)+\beta.(OB^2-r_2^2)+\gamma.(OC^2-r_3^2)+\delta.(OD^2-r_4^2),$$

$\alpha, \beta, \gamma, \delta$ being the tetrahedral coordinates of P referred to ABCD.

Orthogonal Systems of Spheres.—§§ 212–214.

212. Five spheres may be said to form an orthogonal system if they cut one another orthogonally. It is clear that the centres of any four must form a tetrahedron, such that the perpendiculars from the angular points on the opposite faces meet in a point, viz., the centre of the fifth.

If the system be denoted by (1, 2, 3, 4, 5), then (x, y) denoting any other spheres, we have, since

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ y, 1, 2, 3, 4, 5 \end{pmatrix}=0,$$

the formula

$$\pi_{x,y}=\frac{\pi_{x,1}\pi_{y,1}}{\pi_{1,1}}+\frac{\pi_{x,2}\pi_{y,2}}{\pi_{2,2}}+\frac{\pi_{x,3}\pi_{y,3}}{\pi_{3,3}}+\frac{\pi_{x,4}\pi_{y,4}}{\pi_{4,4}}+\frac{\pi_{x,5}\pi_{y,5}}{\pi_{5,5}};$$

or if the radii of the spheres be r_1, r_2, r_3, r_4, r_5, we have,

$$-2\pi_{x,y}=\frac{\pi_{x,1}\pi_{y,1}}{r_1^2}+\frac{\pi_{x,2}\pi_{y,2}}{r_2^2}+\frac{\pi_{x,3}\pi_{y,3}}{r_3^2}+\frac{\pi_{x,4}\pi_{y,4}}{r_4^2}+\frac{\pi_{x,5}\pi_{y,5}}{r_5^2} \quad . \quad . \quad . \quad . \quad (223)$$

From this we can deduce at once the formulæ—

$$\left.\begin{aligned} -2&=\frac{\pi_{x,1}}{r_1^2}+\frac{\pi_{x,2}}{r_2^2}+\frac{\pi_{x,3}}{r_3^2}+\frac{\pi_{x,4}}{r_4^2}+\frac{\pi_{x,5}}{r_5^2} \\ 4r_x^2&=\frac{\pi_{x,1}^2}{r_1^2}+\frac{\pi_{x,2}^2}{r_2^2}+\frac{\pi_{x,3}^2}{r_3^2}+\frac{\pi_{x,4}^2}{r_4^2}+\frac{\pi_{x,5}^2}{r_5^2} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (224)$$

where x denotes any sphere.

Also

$$\left.\begin{aligned} 0&=\frac{\pi_{x,1}}{r_1^2}+\frac{\pi_{x,2}}{r_2^2}+\frac{\pi_{x,3}}{r_3^2}+\frac{\pi_{x,4}}{r_4^2}+\frac{\pi_{x,5}}{r_5^2} \\ 4&=\frac{\pi_{x,1}^2}{r_1^2}+\frac{\pi_{x,2}^2}{r_2^2}+\frac{\pi_{x,3}^2}{r_3^2}+\frac{\pi_{x,4}^2}{r_4^2}+\frac{\pi_{x,5}^2}{r_5^2} \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (225)$$

where x denotes any plane.

Again, taking x and y as coinciding with the plane at infinity, we have, since $\pi_{\theta,\theta}=0$,

$$0=\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}+\frac{1}{r_5^2}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (226)$$

whence we see that one of the five spheres is imaginary.

213. If any system of spheres, say (1, 2, 3, 4, 5), be given, then the five spheres, each of which is orthogonal to four of the given system, form a system, say (6, 7, 8, 9, 10), which may be called the "orthogonal" system of the former.

If (x, y) denote any other spheres, the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ y, 6, 7, 8, 9, 10 \end{pmatrix}=0$$

gives us

$$\begin{vmatrix} \pi_{x,y}, & \pi_{x,6}, & \pi_{x,7}, & \pi_{x,8}, & \pi_{x,9}, & \pi_{x,10} \\ \pi_{y,1}, & \pi_{1,6}, & 0, & 0, & 0, & 0 \\ \pi_{y,2}, & 0, & \pi_{2,7}, & 0, & 0, & 0 \\ \pi_{y,3}, & 0, & 0, & \pi_{3,8}, & 0, & 0 \\ \pi_{y,4}, & 0, & 0, & 0, & \pi_{4,9}, & 0 \\ \pi_{y,5}, & 0, & 0, & 0, & 0, & \pi_{5,10} \end{vmatrix} = 0 ;$$

which may be written

$$\pi_{x,y} = \frac{\pi_{x,6}\pi_{y,1}}{\pi_{1,6}} + \frac{\pi_{x,7}\pi_{y,2}}{\pi_{2,7}} + \frac{\pi_{x,8}\pi_{y,3}}{\pi_{3,8}} + \frac{\pi_{x,9}\pi_{y,4}}{\pi_{4,9}} + \frac{\pi_{x,10}\pi_{y,5}}{\pi_{5,10}} \quad . \quad . \quad . \quad . \quad (227)$$

Hence x denoting any sphere, radius r_x,

$$\left.\begin{aligned} 1 &= \frac{\pi_{x,6}}{\pi_{1,6}} + \frac{\pi_{x,7}}{\pi_{2,7}} + \frac{\pi_{x,8}}{\pi_{3,8}} + \frac{\pi_{x,9}}{\pi_{4,9}} + \frac{\pi_{x,10}}{\pi_{5,10}} \\ -2r_x^2 &= \frac{\pi_{x,6}\pi_{x,1}}{\pi_{1,6}} + \frac{\pi_{x,7}\pi_{x,2}}{\pi_{2,7}} + \frac{\pi_{x,8}\pi_{x,3}}{\pi_{3,8}} + \frac{\pi_{x,9}\pi_{x,4}}{\pi_{4,9}} + \frac{\pi_{x,10}\pi_{x,5}}{\pi_{5,10}} \end{aligned}\right\} \quad . \quad . \quad . \quad (228)$$

Or again, if x denote any plane,

$$\left.\begin{aligned} 0 &= \frac{\pi_{x,6}}{\pi_{1,6}} + \frac{\pi_{x,7}}{\pi_{2,7}} + \frac{\pi_{x,8}}{\pi_{3,8}} + \frac{\pi_{x,9}}{\pi_{4,9}} + \frac{\pi_{x,10}}{\pi_{5,10}} \\ -2 &= \frac{\pi_{x,6}\pi_{x,1}}{\pi_{1,6}} + \frac{\pi_{x,7}\pi_{x,2}}{\pi_{2,7}} + \frac{\pi_{x,8}\pi_{x,3}}{\pi_{3,8}} + \frac{\pi_{x,9}\pi_{x,4}}{\pi_{4,9}} + \frac{\pi_{x,10}\pi_{x,5}}{\pi_{5,10}} \end{aligned}\right\} \quad . \quad . \quad . \quad (229)$$

Also, taking x and y as coinciding with the plane at infinity,

$$0 = \frac{1}{\pi_{1,6}} + \frac{1}{\pi_{2,7}} + \frac{1}{\pi_{3,8}} + \frac{1}{\pi_{4,9}} + \frac{1}{\pi_{5,10}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230)$$

A particular case would be any five points in space, and the five spheres circumscribing the five tetrahedra; thus by equation (230) we see that the sum of the squares of the reciprocals of the tangents from each of five points to the spheres passing through the remaining four is zero.

214. A system of three spheres and their two points of intersection constitute an important system, which may be called a "semi-orthogonal" system. Denoting the three spheres by (1, 2, 3), and their points of intersection by (4, 5), then if (x, y) denote any other spheres, the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ y, 1, 2, 3, 4, 5 \end{pmatrix}=0$$

becomes

$$\begin{vmatrix} \pi_{x,y}, & \pi_{x,1}, & \pi_{x,2}, & \pi_{x,3}, & \pi_{x,4}, & \pi_{x,5} \\ \pi_{y,1}, & \pi_{1,1}, & 0, & 0, & 0, & 0 \\ \pi_{y,2}, & 0, & \pi_{2,2}, & 0, & 0, & 0 \\ \pi_{y,3}, & 0, & 0, & \pi_{3,3}, & 0, & 0 \\ \pi_{y,4}, & 0, & 0, & 0, & 0, & \pi_{4,5} \\ \pi_{y,5}, & 0, & 0, & 0, & \pi_{4,5}, & 0 \end{vmatrix} =0\ ;$$

which may be written

$$\pi_{x,y}=\frac{\pi_{x,1}\pi_{y,1}}{\pi_{1,1}}+\frac{\pi_{x,2}\pi_{y,2}}{\pi_{2,2}}+\frac{\pi_{x,3}\pi_{y,3}}{\pi_{3,3}}+\frac{\pi_{x,4}\pi_{y,5}+\pi_{x,5}\pi_{y,4}}{\pi_{4,5}} \quad . \quad . \quad . \quad . \quad . \quad (231)$$

Denoting the radii of the spheres by r_1, r_2, r_3, and the distance between their common points by e, we have—

if x denote any sphere, radius r_x,

$$\left.\begin{aligned} -2&=\frac{\pi_{x,1}}{r_1^2}+\frac{\pi_{x,2}}{r_2^2}+\frac{\pi_{x,3}}{r_3^2}-2\frac{\pi_{x,4}+\pi_{x,5}}{e^2} \\ 4r_x^2&=\frac{\pi_{x,1}}{r_1^2}+\frac{\pi_{x,2}}{r_2^2}+\frac{\pi_{x,3}}{r_3^2}-4\frac{\pi_{x,4}\pi_{x,5}}{e^2} \end{aligned}\right\}\ ; \quad . \quad . \quad . \quad . \quad . \quad . \quad (232)$$

if x denote any plane,

$$\left.\begin{aligned} 0&=\frac{\pi_{x,1}}{r_1^2}+\frac{\pi_{x,2}}{r_2^2}+\frac{\pi_{x,3}}{r_3^2}-2\frac{\pi_{x,4}+\pi_{x,5}}{e^2} \\ 4&=\frac{\pi_{x,1}}{r_1^2}+\frac{\pi_{x,2}}{r_2^2}+\frac{\pi_{x,3}}{r_3^2}-4\frac{\pi_{x,4}\pi_{x,5}}{e^2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (233)$$

Also, taking x and y as coincident with the plane at infinity, (231) becomes

$$0=\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}-\frac{4}{e^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (234)$$

Spheres touching one another.—§§ 215, 216.

215. If the system of spheres (1, 2, 3, 4, 5) touch each other externally, the equation

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4, 5 \\ \theta, 1, 2, 3, 4, 5 \end{pmatrix}=0,$$

becomes

$$\begin{vmatrix} 0, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4}, & \frac{1}{r_5} \\ \frac{1}{r_1}, & -1, & 1, & 1, & 1, & 1 \\ \frac{1}{r_2}, & 1, & -1, & 1, & 1, & 1 \\ \frac{1}{r_3}, & 1, & 1, & -1, & 1, & 1 \\ \frac{1}{r_4}, & 1, & 1, & 1, & -1, & 1 \\ \frac{1}{r_5}, & 1, & 1, & 1, & 1, & -1 \end{vmatrix} = 0 ;$$

which may be written

$$\Sigma \frac{1}{r^2} = \Sigma \frac{1}{r_1 r_2} ; \qquad (235)$$

or

$$3\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}+\frac{1}{r_5^2}\right)=\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}+\frac{1}{r_5}\right)^2.$$

216. If the spheres (1, 2, 3, 4, 5) have a common tangent sphere, which we will denote by (x), then the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ x, 1, 2, 3, 4, 5 \end{pmatrix} = 0,$$

gives us

$$\begin{vmatrix} 1, & \sqrt{\pi_{1,1}}, & \sqrt{\pi_{2,2}}, & \sqrt{\pi_{3,3}}, & \sqrt{\pi_{4,4}}, & \sqrt{\pi_{5,5}} \\ \sqrt{\pi_{1,1}}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4}, & \pi_{1,5} \\ \sqrt{\pi_{2,2}}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4}, & \pi_{2,5} \\ \sqrt{\pi_{3,3}}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4}, & \pi_{3,5} \\ \sqrt{\pi_{4,4}}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4}, & \pi_{4,5} \\ \sqrt{\pi_{5,5}}, & \pi_{5,1}, & \pi_{5,2}, & \pi_{5,3}, & \pi_{5,4}, & \pi_{5,5} \end{vmatrix} = 0 ;$$

where the positive or negative sign is to be taken with any expression, such as $\sqrt{\pi_{m,n}}$, according as the spheres (m, n) have contact with (x) of the same or opposite kind.

Now if we write $\pi_{r,s} \pm \sqrt{\pi_{r,r}\pi_{s,s}} = t_{r,s}^2$, in which $t_{r,s}$ will denote the length of the common tangent of the two spheres (r, s), "direct" if the positive sign is to be taken, "indirect" if the negative sign is to be taken; then we can deduce at once from this equation the equation of condition, due to Dr. Casey,

$$\begin{vmatrix} 0, & t_{1,2}^2, & t_{1,3}^2, & t_{1,4}^2, & t_{1,5}^2 \\ t_{2,1}^2, & 0, & t_{2,3}^2, & t_{2,4}^2, & t_{2,5}^2 \\ t_{3,1}^2, & t_{3,2}^2, & 0, & t_{3,4}^2, & t_{3,5}^2 \\ t_{4,1}^2, & t_{4,2}^2, & t_{4,3}^2, & 0, & t_{4,5}^2 \\ t_{5,1}^2, & t_{5,2}^2, & t_{5,3}^2, & t_{5,4}^2, & 0 \end{vmatrix} = 0, \quad \ldots \quad (236)$$

which must be satisfied if the spheres all touch the same sphere. Supposing this condition satisfied, the radius of the tangent sphere is easily found thus. Let (6, 7, 8, 9, 10) denote the system of spheres orthogonal to the spheres (1, 2, 3, 4, 5), then by equation (228) we have

$$\frac{\pi_{x,1}}{\pi_{1,6}} + \frac{\pi_{x,2}}{\pi_{2,7}} + \frac{\pi_{x,3}}{\pi_{3,8}} + \frac{\pi_{x,4}}{\pi_{4,9}} + \frac{\pi_{x,5}}{\pi_{5,10}} = 1 ;$$

hence

$$\frac{1}{r_x} = 2\left\{\frac{r_1}{\pi_{1,6}} + \frac{r_2}{\pi_{2,7}} + \frac{r_3}{\pi_{3,8}} + \frac{r_4}{\pi_{4,9}} + \frac{r_5}{\pi_{5,10}}\right\}, \quad \ldots \quad (237)$$

where the radii r_1, r_2, &c. are to be taken with the positive or negative sign, according as the contact is external or internal.

Chapter III.—Spheres Connected with a System of Four Spheres.

In this chapter it is proposed to extend a few of the results arrived at in Chapter IV., Part I. The formulæ for the spheres which pass through the points of intersection of four spheres, and also for the spheres which touch four spheres, will be seen to be exactly analogous to those proved for circles passing through the points of intersection of, and touching three given circles. In the case of a tetrahedron there does not seem to be any sphere analogous to the nine-points circle of a triangle; a special system of spheres will, however, be mentioned, which have a sphere touching their tangent spheres, but even here there is no formula, connecting the radius of this sphere with that of the corresponding sphere circumscribing the tetrahedron formed by the four spheres, analogous to the formula ($2\rho = R$), connecting the radius of the nine-points circle of a triangle with its circumcircle.

Spheres cutting Four Spheres at given Angles.—§§ 217–219.

217. Let the given spheres be denoted by (1, 2, 3, 4), and let their orthogonal sphere be denoted by the symbol (5). Let x denote any sphere cutting (1, 2, 3, 4) at the angles θ, ϕ, ψ, χ; and let x cut (5) at the angle ω.

From the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ x, 1, 2, 3, 4, 5 \end{pmatrix}=0,$$

we may deduce

$$\begin{vmatrix} -1, & \cos\theta, & \cos\phi, & \cos\psi, & \cos\chi, & \cos\omega \\ \cos\theta, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4}, & 0 \\ \cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4}, & 0 \\ \cos\psi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4}, & 0 \\ \cos\chi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1, & 0 \\ \cos\omega, & 0, & 0, & 0, & 0, & -1 \end{vmatrix} = 0;$$

whence we have

$$\sin^2\omega \begin{vmatrix} -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0, & \cos\theta, & \cos\phi, & \cos\psi, & \cos\chi \\ \cos\theta, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ \cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ \cos\psi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ \cos\chi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix} \quad . \quad . \quad . \quad . \quad (238)$$

Let ρ denote the radius of the sphere (x), then by the equation

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4, 5 \\ \theta, 1, 2, 3, 4, 5 \end{pmatrix}=0,$$

we have

$$\begin{vmatrix} \frac{1}{\rho}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4}, & \frac{1}{r_5} \\ \cos\theta, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4}, & 0 \\ \cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4}, & 0 \\ \cos\psi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4}, & 0 \\ \cos\chi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1, & 0 \\ \cos\omega, & 0, & 0, & 0, & 0, & -1 \end{vmatrix} = 0;$$

or

$$\left(\frac{1}{\rho}+\frac{\cos\omega}{r_5}\right)\begin{vmatrix} -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix}$$

$$+\begin{vmatrix} 0, & \frac{1}{r_1'}, & \frac{1}{r_2'}, & \frac{1}{r_3'}, & \frac{1}{r_4'} \\ \cos\theta, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ \cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ \cos\psi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ \cos\chi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix}=0. \quad . \quad . \quad (239)$$

We infer, then, that two spheres can be drawn to cut the given spheres at the angles $(\theta, \phi, \psi, \chi)$, or else at angles supplementary to them. If the two spheres cut (1, 2, 3, 4) at the same angles, they cut the orthogonal sphere at supplementary angles, and *vice versâ*; and evidently one is the inverse of the other with respect to the orthogonal sphere.

Denoting their radii by ρ, ρ' we have

$$\frac{1}{\rho}-\frac{1}{\rho'}+\frac{2\cos\omega}{r_5}=0,$$

and

$$\frac{1}{\rho}+\frac{1}{\rho'}=\mathrm{F}.\cos\theta+\mathrm{G}.\cos\phi+\mathrm{H}.\cos\psi+\mathrm{K}.\cos\chi, \quad . \quad . \quad . \quad . \quad . \quad (240)$$

where F, G, H, K are independent of $(\theta, \phi, \psi, \chi)$.

We see at once, then, that the two spheres will be real, coincident, or imaginary, according as $\frac{\cos^2\omega}{r_5^2}$ is positive, zero, or negative.

But by equation (218) we see that the sign of r_5^2 is opposite to that of

$$\Pi\begin{pmatrix}1, 2, 3, 4\\ 1, 2, 3, 4\end{pmatrix};$$

i.e., opposite to the sign of

$$\begin{vmatrix} -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix}.$$

Hence, by equation (238), the two spheres which cut (1, 2, 3, 4) at the angles $(\theta, \phi, \psi, \chi)$ will be real, coincident, or imaginary, according as the expression

$$\begin{vmatrix} -1, & \cos\theta, & \cos\phi, & \cos\psi, & \cos\chi \\ \cos\theta, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ \cos\phi, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ \cos\psi, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ \cos\chi, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix}$$

is positive, zero, or negative; *i.e.*, according as

$$\Pi\begin{pmatrix} x, 1, 2, 3, 4 \\ x, 1, 2, 3, 4 \end{pmatrix}$$

is positive, zero, or negative.

218. It is evident that eight pairs of spheres can be drawn to cut the four given spheres at angles whose cosines are $\pm\kappa_1$, $\pm\kappa_2$, $\pm\kappa_3$, $\pm\kappa_4$. If we denote the radii of these pairs by ρ_1, ρ'_1; ρ_2, ρ'_2; &c., we have by equation (240)

$$\frac{1}{\rho}+\frac{1}{\rho'}=\pm \mathrm{F}.\kappa_1 \pm \mathrm{G}.\kappa_2 \pm \mathrm{H}.\kappa_3 \pm \mathrm{K}.\kappa_4.$$

Hence we have the relation

$$\begin{vmatrix} \frac{1}{\rho_1}+\frac{1}{\rho'_1}, & \frac{1}{\rho_2}+\frac{1}{\rho'_2}, & \frac{1}{\rho_3}+\frac{1}{\rho'_3}, & \frac{1}{\rho_4}+\frac{1}{\rho'_4}, & \frac{1}{\rho_5}+\frac{1}{\rho'_5}, & \frac{1}{\rho_6}+\frac{1}{\rho'_6}, & \frac{1}{\rho_7}+\frac{1}{\rho'_7}, & \frac{1}{\rho_8}+\frac{1}{\rho'_8} \\ 1, & -1, & 1, & 1, & 1, & -1, & 1, & 1 \\ 1, & 1, & -1, & 1, & 1, & 1, & -1, & 1 \\ 1, & 1, & 1, & -1, & 1, & 1, & 1, & -1 \\ 1, & 1, & 1, & 1, & -1, & -1, & -1, & -1 \end{vmatrix} = 0. \quad . \quad (241)$$

219. In the case of a tetrahedron formed by four planes, the orthogonal sphere becomes the plane at infinity; and so the pair of spheres which cut the faces at angles $(\theta\phi\psi\chi)$ coincide. Thus eight spheres can be drawn cutting the faces of a tetrahedron at angles equal or supplemental to $(\theta, \phi, \psi, \chi)$; and their radii are connected by the set of equations

$$\begin{vmatrix} \frac{1}{\rho_1}, & \frac{1}{\rho_2}, & \frac{1}{\rho_3}, & \frac{1}{\rho_4}, & \frac{1}{\rho_5}, & \frac{1}{\rho_6}, & \frac{1}{\rho_7}, & \frac{1}{\rho_8} \\ 1, & -1, & 1, & 1, & 1, & -1, & 1, & 1 \\ 1, & 1, & -1, & 1, & 1, & 1, & -1, & 1 \\ 1, & 1, & 1, & -1, & 1, & 1, & 1, & -1 \\ 1, & 1, & 1, & 1, & -1, & -1, & -1, & -1 \end{vmatrix} = 0. \quad . \quad . \quad (242)$$

A particular case is that of the eight tangent spheres, *i.e.*, one inscribed, four escribed to one face, three escribed to two faces, of a tetrahedron.

The Spheres Circumscribing the Tetrahedron Formed by Four Spheres.—§§ 220, 221.

220. Let the four given spheres be denoted by (1, 2, 3, 4), their orthogonal sphere by the symbol (5). It is evident that eight pairs of spheres can be drawn. Let P, Q, R, S be the four points in which they intersect, which lie within the tetrahedron formed by the centre of the spheres. Let x denote the sphere circumscribing PQRS.

We have by equation (220),

$$\Pi\begin{pmatrix}x,2,3,4\\x,2,3,4\end{pmatrix}=\Pi\begin{pmatrix}x,3,4,1\\x,3,4,1\end{pmatrix}=\Pi\begin{pmatrix}x,4,1,2\\x,4,1,2\end{pmatrix}=\Pi\begin{pmatrix}x,1,2,3\\x,1,2,3\end{pmatrix}=0.$$

Hence, by a theorem of determinants,

$$\left.\begin{aligned}\left\{\Pi\begin{pmatrix}x,1,2,3\\x,1,2,4\end{pmatrix}\right\}^2&=-\Pi\begin{pmatrix}x,1,2\\x,1,2\end{pmatrix}\times\Pi\begin{pmatrix}x,1,2,3,4\\x,1,2,3,4\end{pmatrix}\\ \left\{\Pi\begin{pmatrix}x,2,3,4\\1,2,3,4\end{pmatrix}\right\}^2&=-\Pi\begin{pmatrix}2,3,4\\2,3,4\end{pmatrix}\times\Pi\begin{pmatrix}x,1,2,3,4\\x,1,2,3,4\end{pmatrix}\end{aligned}\right\}.$$

But since

$$\Pi\begin{pmatrix}x,1,2,3,4,\theta\\x,1,2,3,4,5\end{pmatrix}=0,$$

we have

$$\pi_{x,5}.\Pi\begin{pmatrix}\theta,1,2,3,4\\x,1,2,3,4\end{pmatrix}-\Pi\begin{pmatrix}x,1,2,3,4\\x,1,2,3,4\end{pmatrix}=0,$$

or

$$\Pi\begin{pmatrix}x,1,2,3,4\\x,1,2,3,4\end{pmatrix}=\pi_{x,5}\left\{\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}-\Pi\begin{pmatrix}1,2,3,4\\x,2,3,4\end{pmatrix}-\Pi\begin{pmatrix}1,2,3,4\\1,x,3,4\end{pmatrix}-\Pi\begin{pmatrix}1,2,3,4\\1,2,x,4\end{pmatrix}-\Pi\begin{pmatrix}1,2,3,4\\1,2,3,x\end{pmatrix}\right\}. \quad (243)$$

Also since

$$\Pi\begin{pmatrix}x,1,2,3,4,5\\x,1,2,3,4,5\end{pmatrix}=0,$$

we have

$$\pi_{5,5}.\Pi\begin{pmatrix}x,1,2,3,4\\x,1,2,3,4\end{pmatrix}=\pi^2_{x,5}.\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}. \quad (244)$$

But by equation (218),

$$\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}=288\pi_{5,5}.\{\nabla(1,2,3,4)\}^2;$$

and by equation (217),

$$\Pi\begin{pmatrix}1,2,3\\1,2,3\end{pmatrix}=-288.\{\nabla(1,2,3,S)\}^2;$$

we deduce then from (243), that

$$\frac{\pi_{5,5}-\pi_{5,x}}{\pi_{5,x}}=\frac{V(P,2,3,4)+V(Q,3,4,1)+V(R,4,1,2)+V(S,1,2,3)}{V(1,2,3,4)};$$

i.e., denoting the radius of the sphere (P, Q, R, S) by ρ, and its angle of intersection with (5) by ω,

$$\frac{\rho\cos\omega}{r_5}=\frac{V(1,2,3,4)}{V(1,2,3,4)+V(P,2,3,4)+V(Q,3,4,1)+V(R,4,1,2)+V(S,1,2,3)}. \quad (245)$$

If P be without the tetrahedron formed by the centres of the spheres (1, 2, 3, 4) the sign of V(P, 2, 3, 4) must be changed, and thus the powers of the spheres with respect to the sphere orthogonal to (1, 2, 3, 4) can be written down.

Also we can easily deduce from (244), the equation

$$\begin{vmatrix} -\tan^2\omega, & \sqrt{\mu_{1,1}}, & \sqrt{\mu_{2,2}}, & \sqrt{\mu_{3,3}}, & \sqrt{\mu_{4,4}} \\ \sqrt{\mu_{1,1}}, & \mu_{1,1}, & \mu_{1,2}, & \mu_{1,3}, & \mu_{1,4} \\ \sqrt{\mu_{2,2}}, & \mu_{2,1}, & \mu_{2,2}, & \mu_{2,3}, & \mu_{2,4} \\ \sqrt{\mu_{3,3}}, & \mu_{3,1}, & \mu_{3,2}, & \mu_{3,3}, & \mu_{3,4} \\ \sqrt{\mu_{4,4}}, & \mu_{4,1}, & \mu_{4,2}, & \mu_{4,3}, & \mu_{4,4} \end{vmatrix} =0, \quad (246)$$

where $\mu_{1,1}$, $\mu_{1,2}$, &c. are the minors of $\pi_{1,1}$, $\pi_{1,2}$, &c. in the determinant $\Pi\begin{pmatrix}1,2,3,4\\1,2,3,4\end{pmatrix}$. Thus $\mu_{1,1}=-288\{V(P,2,3,4)\}^2$, and thus the sign of $\sqrt{\mu_{1,1}}$ is positive or negative according as P is within or without the tetrahedron formed by the centres of the spheres (1, 2, 3, 4).

221. The system of spheres $(1, 2, 3, 4, x)$ is orthogonal to the system (P, Q, R, S, 5), and we get some interesting theorems by aid of § 213.

Thus, by equation (230), we have

$$\frac{1}{\pi_{P,1}}+\frac{1}{\pi_{Q,2}}+\frac{1}{\pi_{R,3}}+\frac{1}{\pi_{S,4}}+\frac{1}{\pi_{x,5}}=0. \quad (247)$$

And if x' denote the sphere passing through P′, Q′, R′, S′, the inverse points of P, Q, R, S with respect to (5), we have

$$\frac{1}{\pi_{P',1}}+\frac{1}{\pi_{Q',2}}+\frac{1}{\pi_{R',3}}+\frac{1}{\pi_{S',4}}+\frac{1}{\pi_{x',5}}=0.$$

But since x, x' are inverse with respect to (5),

$$\frac{1}{\pi_{x,5}}+\frac{1}{\pi_{x',5}}=\frac{2}{\pi_{5,5}};$$

hence

$$\frac{1}{\pi_{P,1}}+\frac{1}{\pi_{P',1}}+\frac{1}{\pi_{Q,2}}+\frac{1}{\pi_{Q',2}}+\frac{1}{\pi_{R,3}}+\frac{1}{\pi_{R',3}}+\frac{1}{\pi_{S,4}}+\frac{1}{\pi_{S',4}}+\frac{2}{\pi_{5,5}}=0; \quad . \quad . \quad . \quad (248)$$

or the sum of the reciprocals of the squares of the tangents from the points of intersection of four spheres to the spheres is equal to the reciprocal of the square of the radius of the sphere which cuts them orthogonally.

Again, from equation (228),

$$\frac{\pi_{x,1}}{\pi_{P,1}}+\frac{\pi_{x,2}}{\pi_{Q,2}}+\frac{\pi_{x,3}}{\pi_{R,3}}+\frac{\pi_{x,4}}{\pi_{S,4}}+\frac{\pi_{x,5}}{\pi_{x,5}}=1;$$

or if (x) cut (1, 2, 3, 4) at angles ϕ_1, ϕ_2, ϕ_3, ϕ_4, we have

$$\frac{1}{\rho}+\frac{1}{r}\sec\omega=\frac{2r_1\cos\phi_1}{\pi_{P,1}}+\frac{2r_2\cos\phi_2}{\pi_{Q,2}}+\frac{2r_3\cos\phi_3}{\pi_{R,3}}+\frac{2r_4\cos\phi_4}{\pi_{S,4}} \quad . \quad . \quad . \quad . \quad (249)$$

The Spheres which Touch Four given Spheres.—§§ 222, 223.

222. Let the given spheres be denoted by (1, 2, 3, 4), their orthogonal sphere by (5). Then x denoting any sphere which touches them, we shall have by the equation

$$\Pi\begin{pmatrix}x,1,2,3,4,5\\x,1,2,3,4,5\end{pmatrix}=0,$$

$$\begin{vmatrix} \sin^2\omega, & \sqrt{\pi_{1,1}}, & \sqrt{\pi_{2,2}}, & \sqrt{\pi_{3,3}}, & \sqrt{\pi_{4,4}} \\ \sqrt{\pi_{1,1}}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4} \\ \sqrt{\pi_{2,2}}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4} \\ \sqrt{\pi_{3,3}}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4} \\ \sqrt{\pi_{4,4}}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4} \end{vmatrix}=0; \quad . \quad . \quad (250)$$

by taking the expression $\sqrt{\pi_{1,1}}$, $\sqrt{\pi_{2,2}}$, &c., with different signs, we obtain the eight values of $\sin^2\omega$ corresponding to the eight pairs of tangent spheres.

The radii are given at once by the formula

$$\Pi\begin{pmatrix}x,1,2,3,4,5\\\theta,1,2,3,4,5\end{pmatrix}=0.$$

Thus, if ρ be the radius of the sphere touching (1, 2, 3, 4) externally, and ω be the angle at which it cuts the orthogonal sphere, we have

$$\begin{vmatrix} \frac{1}{\rho}+\frac{\cos\omega}{r}, & \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4} \\ 1, & -1, & \cos\omega_{1,2}, & \cos\omega_{1,3}, & \cos\omega_{1,4} \\ 1, & \cos\omega_{2,1}, & -1, & \cos\omega_{2,3}, & \cos\omega_{2,4} \\ 1, & \cos\omega_{3,1}, & \cos\omega_{3,2}, & -1, & \cos\omega_{3,4} \\ 1, & \cos\omega_{4,1}, & \cos\omega_{4,2}, & \cos\omega_{4,3}, & -1 \end{vmatrix} = 0 \quad . \quad . \quad (251)$$

223. There is no analogous theorem, in general, to FEUERBACH'S theorem. In the case, however, when the given spheres (1, 2, 3, 4) cut at angles such that $\cos\omega_{1,2}=\cos\omega_{1,3}=\cos\omega_{1,4}=\alpha$; $\cos\omega_{2,4}=\cos\omega_{4,3}=\cos\omega_{3,2}=\beta$; it may be shown that the two spheres touching the given spheres, all externally, and the eight spheres touching three externally, or three internally, may be divided into two groups of five, each sphere of either group touching a certain other sphere.

Thus, let the tangent spheres be denoted by (a, b, c, d, e), let z denote the sphere which touches them. Suppose z touches (a) internally, and (b, c, d, e) externally.

From the equation

$$\Pi\begin{pmatrix} z, 1, 2, 3, 4, 5 \\ x, a, b, c, d, e \end{pmatrix}=0,$$

we deduce

$$A\cos\omega_{x,z}+B\cos\omega_{x,1}+C\cos\omega_{x,2}+D\cos\omega_{x,3}+E\cos\omega_{x,4}+F\cos\omega_{x,5}=0;$$

where

$$\begin{aligned} A&= 2\cos\omega_{a,5}-\cos\omega_{b,5}-\cos\omega_{c,5}-\cos\omega_{d,5}-\cos\omega_{e,5}, \\ B&=-\cos\omega_{a,5}+2\cos\omega_{b,5}-\cos\omega_{c,5}-\cos\omega_{d,5}-\cos\omega_{e,5}, \\ C&=-\cos\omega_{a,5}-\cos\omega_{b,5}+2\cos\omega_{c,5}-\cos\omega_{d,5}-\cos\omega_{e,5}, \\ D&=-\cos\omega_{a,5}-\cos\omega_{b,5}-\cos\omega_{c,5}+2\cos\omega_{d,5}-\cos\omega_{e,5}, \\ E&=-\cos\omega_{a,5}-\cos\omega_{b,5}-\cos\omega_{c,5}-\cos\omega_{d,5}+2\cos\omega_{e,5}, \\ F&=-3. \end{aligned}$$

By taking for x, 1, 2, 3, 4, 5 in succession, we can find $\cos\omega_{z,1}$, $\cos\omega_{z,2}$, $\cos\omega_{z,3}$, $\cos\omega_{z,4}$, $\cos\omega_{z,5}$, and then substituting in the equation obtained by putting z for x, we find the required condition.

By equation (250) we find at once,

$$\cos^2\omega_{a,5}=\frac{3(\alpha+1)^2}{3\alpha^2+2\beta-1}; \quad \cos^2\omega_{b,5}=\frac{3(\alpha-1)^2}{3\alpha^2+2\beta-1};$$

and

$$\cos^2\omega_{c,5}=\cos^2\omega_{d,5}=\cos^2\omega_{e,5}=\frac{(\alpha-1)^2(\beta+1)^2-4(\alpha^2-1)(\beta^2-1)}{(\beta+1)^2(3\alpha^2+2\beta-1)}.$$

Taking $\cos\omega_{a,5}$, $\cos\omega_{b,5}$, $\cos\omega_{c,5}$ with the same sign, we have

$$A=2\cos\omega_{a,5}-\cos\omega_{b,5}-3\cos\omega_{c,5},$$
$$B=2\cos\omega_{b,5}-\cos\omega_{a,5}-3\cos\omega_{c,5},$$
$$C=D=E=-\cos\omega_{a,5}-\cos\omega_{b,5}.$$

Hence

$$A\cos\omega_{x,1}=B-3\alpha C,$$
$$A\cos\omega_{x,2}=A\cos\omega_{x,3}=A\cos\omega_{x,4}=-\alpha B+(1-2\beta)C,$$
$$A\cos\omega_{x,5}=-6,$$

and

$$-A+B\cos\omega_{1,x}+3C\cos\omega_{5,x}+F\cos\omega_{5,x}=0.$$

Substituting, we must have

$$-A^2+B^2-6\alpha BC+3C^2(1-2\beta)+36=0,$$

or

$$12+(\cos\omega_{5,b}-\cos\omega_{5,a})(\cos\omega_{5,a}+\cos\omega_{5,b}-6\cos\omega_{5,c})$$
$$+2\alpha(\cos\omega_{5,a}+\cos\omega_{5,b})(2\cos\omega_{5,b}-\cos\omega_{5,a}-3\cos\omega_{5,c})$$
$$-(2\beta-1)(\cos\omega_{5,b}+\cos\omega_{5,a})^2=0.$$

The coefficient of $\cos\omega_{5,c}=(\alpha+1)\cos\omega_{5,b}+(\alpha-1)\cos\omega_{5,a}$ and choosing the signs of $\cos\omega_{5,a}$, $\cos\omega_{5,b}$, so as to make this vanish, we can easily show that this condition is satisfied.

Thus if τ_1, τ'_1 denote the spheres for which $\cos\omega=\pm\frac{(\alpha+1)\sqrt{3}}{\sqrt{3\alpha^2+2\beta-1}}$, and if τ_2, τ'_2 denote the spheres for which $\cos\omega=\mp\frac{(\alpha-1)\sqrt{3}}{\sqrt{3\alpha^2+2\beta-1}}$, and if (τ_3, τ'_3) (τ_4, τ'_4) (τ_5, τ'_5) denote the other spheres, we see that the groups

$$(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) \qquad (\tau'_1, \tau'_2, \tau'_3, \tau'_4, \tau'_5)$$
$$(\tau_1, \tau_2, \tau'_3, \tau'_4, \tau'_5) \qquad (\tau'_1, \tau'_2, \tau_3, \tau_4, \tau_5)$$

have each a common tangent sphere, which touches τ_1 or τ'_1 in the opposite sense to τ_2, τ_3, &c.

Chapter IV.—Power-Coordinates.

Definition.—§§ 224–227.

224. Since a sphere, plane, or point is completely determinate when its powers are known with respect to any five spheres not having a common orthogonal sphere, we may define the coordinates of a point (sphere or plane), referred to such a system of spheres, as any multiples, the same or different, of the powers of the point with respect to them.

Thus if $(xyzwv)$ be the coordinates of any point, whose Cartesian coordinates are $(\alpha\beta\gamma)$, we shall have

$$x \text{ proportional to } (\alpha-a)^2+(\beta-b)^2+(\gamma-c)^2-r^2,$$

where (abc) is the centre of the sphere of reference, r the radius. Thus $(xyzwv)$ are quadric functions of a particular form of the Cartesian coordinates of the centres of the spheres of reference.

We shall find it convenient to restrict the use of (x, y, z, w, v) to denote the coordinates of a point; the coordinates of a sphere will be denoted by $(\xi, \eta, \zeta, \omega, \varpi)$; and the coordinates of a plane by $(\lambda, \mu, \nu, \rho, \sigma)$.

225. Let us denote the system of reference by $(1, 2, 3, 4, 5)$; then if $(xyzwv)$ be the coordinates of any point P, we see that, θ denoting as usual the plane at infinity, so that $\pi_{\theta,P}=1$, then, since $\pi_{P,P}=0$, the coordinates of P must satisfy a homogeneous quadric relation

$$\Pi\begin{pmatrix}P, 1, 2, 3, 4, 5\\P, 1, 2, 3, 4, 5\end{pmatrix}=0,$$

and a non-homogeneous linear relation

$$\Pi\begin{pmatrix}P, 1, 2, 3, 4, 5\\\theta, 1, 2, 3, 4, 5\end{pmatrix}=0.$$

Let us suppose the coordinates of P defined by the equations

$$x=k_1.\pi_{P,1},\ y=k_2.\pi_{P,2},\ z=k_3.\pi_{P,3},\ w=k_4.\pi_{P,4},\ v=k_5.\pi_{P,5}.$$

Then the quadric relation which $(xyzwv)$ must satisfy is

$$\begin{vmatrix} 0 & \frac{x}{k_1}, & \frac{y}{k_2}, & \frac{z}{k_3}, & \frac{w}{k_4}, & \frac{v}{k_5} \\ \frac{x}{k_1}, & \pi_{1,1}, & \pi_{1,2}, & \pi_{1,3}, & \pi_{1,4}, & \pi_{1,5} \\ \frac{y}{k_2}, & \pi_{2,1}, & \pi_{2,2}, & \pi_{2,3}, & \pi_{2,4}, & \pi_{2,5} \\ \frac{z}{k_3}, & \pi_{3,1}, & \pi_{3,2}, & \pi_{3,3}, & \pi_{3,4}, & \pi_{3,5} \\ \frac{w}{k_4}, & \pi_{4,1}, & \pi_{4,2}, & \pi_{4,3}, & \pi_{4,4}, & \pi_{4,5} \\ \frac{v}{k_5}, & \pi_{5,1}, & \pi_{5,2}, & \pi_{5,3}, & \pi_{5,4}, & \pi_{5,5} \end{vmatrix} = 0 \quad . \quad . \quad . \quad . \quad . \quad (252)$$

This is called the equation of the absolute: we will denote it by

$$\psi(x, y, z, w, v) \equiv (a_{1,1}, a_{1,2}, a_{1,3}, a_{1,4}, \ldots)(x, y, z, w, v)^2 = 0, \quad . \quad (253)$$

where

$$a_{1,1} = \frac{1}{k_1^2} \frac{-\Pi\begin{pmatrix} 2, 3, 4, 5 \\ 2, 3, 4, 5 \end{pmatrix}}{\Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 1, 2, 3, 4, 5 \end{pmatrix}}, \quad a_{1,2} = \frac{1}{k_1 k_2} \frac{-\Pi\begin{pmatrix} 1, 3, 4, 5 \\ 2, 3, 4, 5 \end{pmatrix}}{\Pi\begin{pmatrix} 1, 2, 3, 4, 5 \\ 1, 2, 3, 4, 5 \end{pmatrix}}, \text{ \&c.;}$$

and then the linear relation which $(xyzwv)$ must satisfy, may be expressed thus

$$k_1 \frac{\partial \psi}{\partial x} + k_2 \frac{\partial \psi}{\partial y} + k_3 \frac{\partial \psi}{\partial z} + k_4 \frac{\partial \psi}{\partial w} + k_5 \frac{\partial \psi}{\partial v} = -2. \quad . \quad . \quad . \quad . \quad . \quad (254)$$

226. If $(\xi, \eta, \zeta, \omega, \varpi)$ be the coordinates of any sphere, S, we see that, since $\pi_{S,S} = 1$, the coordinates of S must satisfy a non-homogeneous linear relation

$$\Pi\begin{pmatrix} \theta, 1, 2, 3, 4, 5 \\ S, 1, 2, 3, 4, 5 \end{pmatrix} = 0,$$

which may be written

$$k_1 \frac{\partial \psi}{\partial \xi} + k_2 \frac{\partial \psi}{\partial \eta} + k_3 \frac{\partial \psi}{\partial \zeta} + k_4 \frac{\partial \psi}{\partial \omega} + k_5 \frac{\partial \psi}{\partial \varpi} = -2. \quad . \quad . \quad . \quad . \quad . \quad (255)$$

227. Again, if L denote any plane whose coordinates are $(\lambda, \mu, \nu, \rho, \sigma)$, since $\pi_{L,\theta} = 0$, $\pi_{L,L} = -2$, we see that the coordinates of L must satisfy a linear homogeneous equation and a non-homogeneous quadric relation, viz.,

$$\Pi\begin{pmatrix} L, 1, 2, 3, 4, 5 \\ \theta, 1, 2, 3, 4, 5 \end{pmatrix} = 0,$$

which may be written

$$k_1\frac{\partial\psi}{\partial\lambda}+k_2\frac{\partial\psi}{\partial\mu}+k_3\frac{\partial\psi}{\partial\nu}+k_4\frac{\partial\psi}{\partial\rho}+k_5\frac{\partial\psi}{\partial\sigma}=0\,; \quad \ldots\ldots \quad (256)$$

and

$$\Pi\begin{pmatrix}\mathrm{L},\,1,\,2,\,3,\,4,\,5\\ \mathrm{L},\,1,\,2,\,3,\,4,\,5\end{pmatrix}=0,$$

which may be written

$$\lambda\frac{\partial\psi}{\partial\lambda}+\mu\frac{\partial\psi}{\partial\mu}+\nu\frac{\partial\psi}{\partial\nu}+\rho\frac{\partial\psi}{\partial\rho}+\sigma\frac{\partial\psi}{\partial\sigma}=+2,$$

or

$$\psi(\lambda,\,\mu,\,\nu,\,\rho,\,\sigma)=1. \quad \ldots\ldots \quad (257)$$

The Sphere.—§§ 228–231.

228. Let P be any point on the sphere S whose coordinates are $(\xi\eta\zeta\omega\varpi)$, then since $\pi_{\mathrm{P,S}}=0$, and

$$\Pi\begin{pmatrix}\mathrm{P},\,1,\,2,\,3,\,4,\,5\\ \mathrm{S},\,1,\,2,\,3,\,4,\,5\end{pmatrix}=0,$$

we see that the equation of the sphere is

$$\frac{\partial\psi}{\partial\xi}x+\frac{\partial\psi}{\partial\eta}y+\frac{\partial\psi}{\partial\zeta}z+\frac{\partial\psi}{\partial\omega}w+\frac{\partial\psi}{\partial\varpi}v=0\,;$$

and hence the general equation of the first degree,

$$ax+by+cz+dw+ev=0,$$

will, in general, represent a sphere, whose coordinates are given by

$$\frac{\frac{\partial\psi}{\partial\xi}}{a}=\frac{\frac{\partial\psi}{\partial\eta}}{b}=\frac{\frac{\partial\psi}{\partial\zeta}}{c}=\frac{\frac{\partial\psi}{\partial\omega}}{d}=\frac{\frac{\partial\psi}{\partial\varpi}}{e}=\frac{-2}{ak_1+bk_2+ck_3+dk_4}, \quad \ldots\ldots \quad (258)$$

by equation (255).

229. Given any two spheres $(\xi\eta\zeta\omega\varpi)$ $(\xi'\eta'\zeta'\omega'\varpi')$, their power π is, since

$$\Pi\begin{pmatrix}\mathrm{S},\,1,\,2,\,3,\,4,\,5\\ \mathrm{S},\,1,\,2,\,3,\,4,\,5\end{pmatrix}=0,$$

given by

$$-2\pi=\xi'\frac{\partial\psi}{\partial\xi}+\eta'\frac{\partial\psi}{\partial\eta}+\zeta'\frac{\partial\psi}{\partial\zeta}+\omega'\frac{\partial\psi}{\partial\omega}+\varpi'\frac{\partial\psi}{\partial\varpi}. \quad \ldots\ldots \quad (259)$$

In particular the radius of the sphere $(\xi\eta\zeta\omega\varpi)$ will be given by

$$2r^3=\psi(\xi, \eta, \zeta, \omega, \varpi) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (260)$$

230. Hence the radius of the sphere

$$ax+by+cz+dw+ev=0,$$

is given by

$$\begin{vmatrix} a_{1,1}, & a_{1,2}, & a_{1,3}, & a_{1,4}, & a_{1,5}, & a \\ a_{2,1}, & a_{2,2}, & a_{2,3}, & a_{2,4}, & a_{2,5}, & b \\ a_{3,1}, & a_{3,2}, & a_{3,3}, & a_{3,4}, & a_{3,5}, & c \\ a_{4,1}, & a_{4,2}, & a_{4,3}, & a_{4,4}, & a_{4,5}, & d \\ a_{5,1}, & a_{5,2}, & a_{5,3}, & a_{5,4}, & a_{5,5}, & e \\ a, & b, & c, & d, & e, & u \end{vmatrix} =0, \quad . \quad . \quad . \quad . \quad . \quad (261)$$

where

$$u=2r^3(ak_1+bk_2+ck_3+dk_4+ek_5)^2.$$

We shall find it convenient to denote the bordered Hessian of ψ by Ψ; and suppose the coefficients so determined that we may express equation (261) thus--

$$u=\Psi(a, b, c, d, e) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (262)$$

231. The power of the sphere $(\xi\eta\zeta\omega\varpi)$ with respect to the sphere

$$ax+by+cz+dw+ev=0,$$

is clearly given by

$$\pi=\frac{a\xi+b\eta+c\zeta+d\omega+e\varpi}{ak_1+bk_2+ck_3+dk_4+ek_5}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (263)$$

and the power of the two spheres

$$ax+by+cz+dw+ev=0,$$
$$a'x+b'y+c'z+d'w+e'v=0,$$

is clearly given by

$$\pi=\frac{a'\dfrac{\partial\Psi}{\partial a}+b'\dfrac{\partial\Psi}{\partial b}+c'\dfrac{\partial\Psi}{\partial c}+d'\dfrac{\partial\Psi}{\partial d}+e'\dfrac{\partial\Psi}{\partial e}}{(ak_1+bk_2+ck_3+dk_4+ek_5)(a'k_1+b'k_2+c'k_3+d'k_4+e'k_5)}.$$

And so the angle of intersection of the spheres will be given by

$$\cos\phi=-\frac{a'\dfrac{\partial\Psi}{\partial a}+b'\dfrac{\partial\Psi}{\partial b}+c'\dfrac{\partial\Psi}{\partial c}+d'\dfrac{\partial\Psi}{\partial d}+e'\dfrac{\partial\Psi}{\partial e}}{\sqrt{\Psi(a, b, c, d, e).\Psi(a', b', c', d', e')}} \quad . \quad . \quad . \quad . \quad . \quad (264)$$

The Plane.—§§ 232–235.

232. If $(\lambda\mu\nu\rho\sigma)$ be any plane L, $(xyzwv)$ any point P on it, we have at once, since

$$\Pi\begin{pmatrix}\mathrm{P},\,1,\,2,\,3,\,4,\,5\\ \mathrm{L},\,1,\,2,\,3,\,4,\,5\end{pmatrix}=0,$$

$$x\frac{\partial\Psi}{\partial\lambda}+y\frac{\partial\Psi}{\partial\mu}+z\frac{\partial\Psi}{\partial\nu}+w\frac{\partial\Psi}{\partial\rho}+v\frac{\partial\Psi}{\partial\sigma}=0.$$

But by (256)

$$k_1\frac{\partial\Psi}{\partial\lambda}+k_2\frac{\partial\Psi}{\partial\mu}+k_3\frac{\partial\Psi}{\partial\nu}+k_4\frac{\partial\Psi}{\partial\rho}+k_5\frac{\partial\Psi}{\partial\sigma}=0\,;$$

hence the equation

$$ax+by+cz+dw+ev=0,$$

will represent a plane when

$$ak_1+bk_2+ck_3+dk_4+ek_5=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (265)$$

And if this condition be satisfied, we have to determine its coordinates,

$$\frac{\frac{\partial\Psi}{\partial\lambda}}{a}=\frac{\frac{\partial\Psi}{\partial\mu}}{b}=\frac{\frac{\partial\Psi}{\partial\nu}}{c}=\frac{\frac{\partial\Psi}{\partial\rho}}{d}=\frac{\frac{\partial\Psi}{\partial\sigma}}{e}=\frac{2}{\sqrt{\Psi(a,\,b,\,c,\,d,\,e)}}, \quad . \quad . \quad . \quad . \quad . \quad (266)$$

where $\Psi(a,\,b,\,c,\,d,\,e)$ is defined by equation (262).

233. The power of the sphere $(\xi\eta\zeta\omega\varpi)$ and the plane

$$ax+by+cz+dw+ev=0$$

is given by

$$-2\pi=\xi\frac{\partial\Psi}{\partial\lambda}+\eta\frac{\partial\Psi}{\partial\mu}+\zeta\frac{\partial\Psi}{\partial\nu}+\omega\frac{\partial\Psi}{\partial\rho}+\varpi\frac{\partial\Psi}{\partial\sigma}\,;$$

therefore

$$\pi=\frac{a\xi+b\eta+c\zeta+d\omega+e\varpi}{\sqrt{\Psi(a,\,b,\,c,\,d,\,e)}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (267)$$

234. The angle of intersection of the planes

$$ax+by+cz+dw+ev=0,$$
$$a'x+b'y+c'z+d'w+e'v=0,$$

will be given by

$$\cos\phi=-\tfrac{1}{2}\frac{a\frac{\partial\Psi}{\partial a}+b\frac{\partial\Psi}{\partial b}+c\frac{\partial\Psi}{\partial c}+d\frac{\partial\Psi}{\partial d}+e\frac{\partial\Psi}{\partial e}}{\sqrt{\Psi(a,\,b,\,c,\,d,\,e).\Psi(a',\,b',\,c',\,d',\,e')}}. \quad . \quad . \quad . \quad . \quad . \quad (268)$$

235. The coordinates of the plane at infinity are clearly k_1, k_2, k_3, k_4, k_5, and so the equation of the plane at infinity is

$$x\frac{\partial\psi}{\partial k_1}+y\frac{\partial\psi}{\partial k_2}+z\frac{\partial\psi}{\partial k_3}+w\frac{\partial\psi}{\partial k_4}+v\frac{\partial\psi}{\partial k_5}=0$$

The Point.—§§ 236–238.

236. The power of the point $(x\,y\,z\,w\,v)$ with respect to the surface

$$ax+by+cz+dw+ev=0,$$

will be equal to

$$\frac{ax+by+cz+dw+ev}{ak_1+bk_2+ck_3+dk_4+ek_5},$$

if the surface is a sphere, and will be equal to

$$\frac{ax+by+cz+dw+ev}{\sqrt{\Psi(a,\,b,\,c,\,d,\,e)}},$$

if the surface is a plane.

237. The power of the two points $(x\,y\,z\,w\,v)$, $(x'y'z'w'v')$ will be given by

$$-2\pi=x'\frac{\partial\psi}{\partial x}+y'\frac{\partial\psi}{\partial y}+z'\frac{\partial\psi}{\partial z}+w'\frac{\partial\psi}{\partial w}+v'\frac{\partial\psi}{\partial v}.$$

Hence, since

$$\psi(x,\,y,\,z,\,w,\,v)=\psi(x',\,y',\,z',\,w',\,v')=0,$$

the distance δ between the two points will be given by

$$2\delta^2=\psi\{x-x',\,y-y',\,z-z',\,w-w',\,v-v'\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (269)$$

238. Let P, Q, R, S be any four points, then by equation (216) we have for the volume of the tetrahedron formed by them,

$$288.\{\mathrm{V}(\mathrm{P},\,\mathrm{Q},\,\mathrm{R},\,\mathrm{S})\}^2=\Pi\begin{pmatrix}\theta,\,\mathrm{P},\,\mathrm{Q},\,\mathrm{R},\,\mathrm{S}\\ \theta,\,\mathrm{P},\,\mathrm{Q},\,\mathrm{R},\,\mathrm{S}\end{pmatrix};$$

and by § 204,

$$\Pi\begin{pmatrix}\theta,\,\mathrm{P},\,\mathrm{Q},\,\mathrm{R},\,\mathrm{S}\\ \theta,\,\mathrm{P},\,\mathrm{Q},\,\mathrm{R},\,\mathrm{S}\end{pmatrix}\times\Pi\begin{pmatrix}1,\,2,\,3.\,4,\,5\\ 1,\,2,\,3,\,4,\,5\end{pmatrix}=\left\{\Pi\begin{pmatrix}\theta,\,\mathrm{P},\,\mathrm{Q},\,\mathrm{R},\,\mathrm{S}\\ 1,\,2,\,3,\,4,\,5\end{pmatrix}\right\}^2.$$

If then the coordinates of P, Q, R, S referred to (1, 2, 3, 4, 5) be $(x_1y_1z_1w_1v_1)$ $(x_2y_2z_2w_2v_2)(x_3y_3z_3w_3v_3)(x_4y_4z_4w_4v_4)$, we see that we shall have

$$V(P, Q, R, S)=\mu \begin{vmatrix} x_1, & y_1, & z_1, & w_1, & v_1 \\ x_2, & y_2, & z_2, & w_2, & v_2 \\ x_3, & y_3, & z_3, & w_3, & v_3 \\ x_4, & y_4, & z_4, & w_4, & v_4 \\ k_1, & k_2, & k_3, & k_4, & k_5 \end{vmatrix}; \quad \ldots\ldots\ldots \quad (270)$$

where $288\mu^5 k_1^3 k_2^3 k_3^3 k_4^3 k_5^3 . \Pi\begin{pmatrix}1,2,3,4,5\\1,2,3,4,5\end{pmatrix}=1.$

Coordinate Systems of Reference.—§§ 239, 240.

239. The most convenient system of reference is five spheres which are mutually orthotomic: this may be called the orthogonal system. If r_1, r_2, r_3, r_4, r_5 are the radii of five such spheres, it is simplest to take k_1, k_2, k_3, k_4, k_5, the coordinates of the plane at infinity, as inversely proportional to them. In this case we shall have, by § 212,

$$\left.\begin{aligned} 2.\psi(x, y, z, w, v) &\equiv x^2+y^2+z^2+w^2+v^2 \\ 2\,\Psi(x, y, z, w, v) &\equiv x^2+y^2+z^2+w^2+v^2 \end{aligned}\right\} \quad \ldots\ldots \quad (271)$$

Thus the angle between the spheres

$$ax+by+cz+dw+ev=0,$$
$$a'x+b'y+c'z+d'w+e'v=0,$$

will be given by

$$\cos\phi=-\frac{aa'+bb'+cc'+dd'+ee'}{\sqrt{(a^2+b^2+c^2+d^2+e^2)(a'^2+b'^2+c'^2+d'^2+e'^2)}}.$$

240. It is, however, very often convenient to take as the system of reference three spheres cutting orthogonally, and their two points of intersection. And then if r_1, r_2, r_3 be the radii of the spheres, ϵ the distance between their points of intersection, it is simplest to take k_1, k_2, k_3, k_4, k_5, the coordinates of the plane at infinity, as inversely proportional to r_1, r_2, r_3, ϵ, ϵ: so that we shall have by § 214,

$$\left.\begin{aligned} 2\psi(x, y, z, w, v) &\equiv x^2+y^2+z^2-4vw \\ 2\Psi(x, y, z, w, v) &\equiv x^2+y^2+z^2- \quad vw \end{aligned}\right\}; \quad \ldots\ldots \quad (272)$$

so that the angle between the spheres

$$ax+by+cz+dw+ev=0,$$
$$a'x+b'y+c'z+d'w+e'v=0,$$

will be given by

$$\cos\phi=-\frac{aa'+bb'+cc'-\frac{1}{2}(ed'+e'd)}{\sqrt{(a^2+b^2+c^2-da)(a'^2+b'^2+c'^2-d'e')}}.$$

Chapter V.—General Equation of the Second Degree in Power-Coordinates.

Nature of the Surface.—§§ 241, 242.

241. The most general equation of the second degree in power-coordinates may be written

$$\begin{aligned}\phi(x, y, z, w, v) \equiv & \alpha_{1,1}x^2+\alpha_{2,2}y^2+\alpha_{3,3}z^2+\alpha_{4,4}w^2+\alpha_{5,5}v^2 \\ & +2\alpha_{1,2}xy+2\alpha_{1,3}xz+2\alpha_{1,4}xw+2\alpha_{1,5}xv \\ & +2\alpha_{2,3}yz+2\alpha_{2,4}yw+2\alpha_{2,5}yv \\ & +2\alpha_{3,4}zw+2\alpha_{3,5}zv+2\alpha_{4,5}vw=0;\end{aligned}$$

and since the coordinates of any point must satisfy the equation of the absolute, which is also of the second degree, we see that this equation contains only 14 constants.

Now (X, Y, Z) denoting Cartesian coordinates of a point, we can express the power-coordinates (*xyzwv*) of the point as linear functions of $X^2+Y^2+Z^2$, X, Y, Z, 1; and if we substitute for (*xyzwv*), the equation becomes of the form

$$(X^2+Y^2+Z^2)^2+U_1(X^2+Y^2+Z^2)+U_2=0,$$

where U_1 is of the first degree, U_2 of the second degree in (X, Y, Z); this equation has 14 constants, and represents a surface having the circle at infinity as a nodal curve, and is usually called a cyclide. It thus appears that $\phi=0$, is a form to which the equation of every cyclide can be reduced.

242. It is also evident that, since the equation of every plane is satisfied by the coordinates of the plane at infinity, the surface $\phi=0$ will represent a cubic surface and the plane at infinity, if the equation $\phi=0$ be satisfied by the coordinates of the plane at infinity. Such surfaces have been called cubic cyclides; they intersect the plane at infinity in the imaginary circle, and also a straight line.

Equation of Tangent at any Point.—§§ 243–247.

243. If $(\xi\ \eta\ \zeta\ \omega\ \varpi)$ be the coordinates of any sphere touching the cyclide ϕ at the point $(x'y'z'w'v')$, we must have

$$\frac{\frac{\partial\psi}{\partial\xi}}{\frac{\partial\phi}{\partial x'}+k\frac{\partial\psi}{\partial v'}}=\frac{\frac{\partial\psi}{\partial\eta}}{\frac{\partial\phi}{\partial y'}+k\frac{\partial\psi}{\partial y'}}=\frac{\frac{\partial\psi}{\partial\zeta}}{\frac{\partial\phi}{\partial z'}+k\frac{\partial\psi}{\partial z'}}=\frac{\frac{\partial\psi}{\partial\omega}}{\frac{\partial\phi}{\partial w'}+k\frac{\partial\psi}{\partial w'}}=\frac{\frac{\partial\psi}{\partial\varpi}}{\frac{\partial\phi}{\partial v'}+k\frac{\partial\psi}{\partial v'}} \quad . \quad . \quad . \quad (273)$$

And thus the equation of any tangent sphere is of the form,

$$\left(\frac{\partial\phi}{\partial x'}+k\frac{\partial\psi}{\partial x'}\right)x+\left(\frac{\partial\phi}{\partial y'}+k\frac{\partial\psi}{\partial y'}\right)y+\left(\frac{\partial\phi}{\partial z'}+k\frac{\partial\psi}{\partial z'}\right)z+\left(\frac{\partial\phi}{\partial w'}+k\frac{\partial\psi}{\partial w'}\right)w+\left(\frac{\partial\phi}{\partial v'}+k\frac{\partial\psi}{\partial v'}\right)v=0.$$

Hence the equation of the tangent plane at the point $(x'y'z'w'v')$ is

$$\begin{aligned}&\left(k_1\frac{\partial\phi}{\partial x'}+k_2\frac{\partial\phi}{\partial y'}+k_3\frac{\partial\phi}{\partial z'}+k_4\frac{\partial\phi}{\partial w'}+k_5\frac{\partial\phi}{\partial v'}\right)\left(x\frac{\partial\psi}{\partial x'}+y\frac{\partial\psi}{\partial y'}+z\frac{\partial\psi}{\partial z'}+w\frac{\partial\psi}{\partial w'}+v\frac{\partial\psi}{\partial v'}\right)\\ =&\left(k_1\frac{\partial\psi}{\partial x'}+k_2\frac{\partial\psi}{\partial y'}+k_3\frac{\partial\psi}{\partial z'}+k_4\frac{\partial\psi}{\partial w'}+k_5\frac{\partial\psi}{\partial v'}\right)\left(x\frac{\partial\phi}{\partial x'}+y\frac{\partial\phi}{\partial y'}+z\frac{\partial\phi}{\partial z'}+w\frac{\partial\phi}{\partial w'}+v\frac{\partial\phi}{\partial v'}\right). \quad . \quad . \quad (274)\end{aligned}$$

244. The sphere given by the equation

$$\left(x\frac{\partial}{\partial x'}+y\frac{\partial}{\partial y'}+z\frac{\partial}{\partial z'}+w\frac{\partial}{\partial w'}+v\frac{\partial}{\partial v'}\right)(\phi+k\psi)=0,$$

will clearly touch the cyclide at the point $(x''y''z''w''v'')$ if

$$\frac{\dfrac{\partial\phi}{\partial x'}+k\dfrac{\partial\psi}{\partial x'}}{\dfrac{\partial\phi}{\partial x''}+k\dfrac{\partial\psi}{\partial x''}}=\frac{\dfrac{\partial\phi}{\partial y'}+k\dfrac{\partial\psi}{\partial y'}}{\dfrac{\partial\phi}{\partial y''}+k\dfrac{\partial\psi}{\partial y''}}=\frac{\dfrac{\partial\phi}{\partial z'}+k\dfrac{\partial\psi}{\partial z'}}{\dfrac{\partial\phi}{\partial z''}+k\dfrac{\partial\psi}{\partial z''}}=\frac{\dfrac{\partial\phi}{\partial w'}+k\dfrac{\partial\psi}{\partial w'}}{\dfrac{\partial\phi}{\partial w''}+k\dfrac{\partial\psi}{\partial w''}}=\frac{\dfrac{\partial\phi}{\partial v'}+k\dfrac{\partial\psi}{\partial v'}}{\dfrac{\partial\phi}{\partial v''}+k\dfrac{\partial\psi}{\partial v''}};$$

in which case k must satisfy the quintic

$$\mathrm{H}(\phi+k\psi)=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (275)$$

where $\mathrm{H}(u)$ denotes the Hessian of u.

We infer, then, that there are in general five systems of bitangent spheres of a cyclide; *i.e.*, of the whole number of tangent spheres at any point of the surface five touch the surface elsewhere. Moreover it is evident that the system of bitangent spheres corresponding to a particular root of (275) all cut a certain fixed sphere orthogonally, the coordinates of this sphere being proportional to the minors of the constituents of any row in the determinant $\mathrm{H}(\phi+k\psi)$.

245. If the coordinates of a bitangent sphere satisfy the absolute, *i.e.*, if the radius of the sphere be indefinitely small, the sphere may be considered as a focus. It is evident that the foci belonging to any system of bitangents will form a curve of the second degree on the corresponding orthogonal sphere. Such a curve will be a spheriquadric.

246. If the coordinates of a bitangent sphere satisfy the condition, for a plane, the corresponding equation will represent a double tangent plane; and its coordinates

must satisfy an equation of the second degree, and thus we shall have a double tangent cone corresponding to each system of bitangent spheres, the vertex of the cone being the centre of the sphere which cuts the particular system of bitangents orthogonally.

247. Again, it is clear that if by any transformation the equation of the surface becomes Φ, and the equation of the absolute Ψ, then the same value of k which satisfies $H(\phi+k\psi)=0$ must also satisfy $H(\Phi+k\Psi)=0$; and hence the coefficients of the powers of k in the equation $H(\phi+k\psi)=0$ are invariants.

Equation of Normal at any Point.—§§ 248–250.

248. Let $(\xi\,\eta\,\zeta\,\omega\,\varpi)$ be the coordinates of any sphere which cuts the surface $\phi(x\,y\,z\,w\,v)=0$ orthogonally at the point $(x'y'z'w'v')$, we must have

$$\left(\xi\frac{\partial}{\partial x'}+\eta\frac{\partial}{\partial y'}+\zeta\frac{\partial}{\partial z'}+\omega\frac{\partial}{\partial w'}+\varpi\frac{\partial}{\partial v'}\right)(\phi+k\psi)=0, \quad . \quad . \quad . \quad . \quad . \quad (276)$$

for all values of k.

It follows that if $(\lambda\,\mu\,\nu\,\rho\,\sigma)$ be any plane containing the normal to the cyclide at the point $(x'y'z'w'v')$, we must have

$$\lambda\frac{\partial\phi}{\partial x'}+\mu\frac{\partial\phi}{\partial y'}+\nu\frac{\partial\phi}{\partial z'}+\rho\frac{\partial\phi}{\partial w'}+\sigma\frac{\partial\phi}{\partial v'}=0,$$

$$\lambda\frac{\partial\psi}{\partial x'}+\mu\frac{\partial\psi}{\partial y'}+\nu\frac{\partial\psi}{\partial z'}+\rho\frac{\partial\psi}{\partial w'}+\sigma\frac{\partial\psi}{\partial v'}=0,$$

$$\lambda\frac{\partial\psi}{\partial k_1}+\mu\frac{\partial\psi}{\partial k_2}+\nu\frac{\partial\psi}{\partial k_3}+\rho\frac{\partial\psi}{\partial k_4}+\sigma\frac{\partial\psi}{\partial k_5}=0.$$

249. If we take as our system of reference an orthogonal system of spheres, radii $(r_1, r_2, r_3, r_4, r_5)$, then, by forming equations to planes containing the normal at $(x'y'z'w'v')$ and passing through the centres of the spheres of reference, we easily obtain for the equations of the normal,

$$\left\|\begin{matrix} x, & y, & z, & w, & v \\ \frac{\partial\phi}{\partial x'}, & \frac{\partial\phi}{\partial y'}, & \frac{\partial\phi}{\partial z'}, & \frac{\partial\phi}{\partial w'}, & \frac{\partial\phi}{\partial v'} \\ x', & y', & z', & w', & v' \\ \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4}, & \frac{1}{r_5} \end{matrix}\right\|=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (277)$$

250. Again, from (277), we see that if a sphere cut the cyclide normally at the point $(x'y'z'w'v')$, we must have

$$x'\frac{\partial\phi}{\partial\xi}+y'\frac{\partial\phi}{\partial\eta}+z'\frac{\partial\phi}{\partial\zeta}+w'\frac{\partial\phi}{\partial\omega}+v'\frac{\partial\phi}{\partial\varpi}=0,$$

$$x'\frac{\partial\psi}{\partial\xi}+y'\frac{\partial\psi}{\partial\eta}+z'\frac{\partial\psi}{\partial\zeta}+w'\frac{\partial\psi}{\partial\omega}+v'\frac{\partial\psi}{\partial\varpi}=0.$$

If then $(\xi\,\eta\,\zeta\,\omega\,\varpi)$ be chosen, so that

$$\frac{\partial\phi}{\partial\psi}\Big/\frac{\partial\xi}{\partial\psi}=\frac{\partial\phi}{\partial\psi}\Big/\frac{\partial\eta}{\partial\psi}=\frac{\partial\phi}{\partial\psi}\Big/\frac{\partial\zeta}{\partial\psi}=\frac{\partial\phi}{\partial\psi}\Big/\frac{\partial\omega}{\partial\psi}=\frac{\partial\phi}{\partial\psi}\Big/\frac{\partial\varpi}{\partial\psi}=-\mu \text{ say},$$

then the sphere $(\xi\eta\zeta\omega\varpi)$ will cut the cyclide orthogonally at every point of its curve of intersection.

We see at once that μ must satisfy the equation

$$H(\phi+\mu\psi)=0,$$

and thus there are five such spheres; and the coordinates of the sphere which corresponds to a particular value of μ are proportional to the minors of the constituents of any row in the determinant $H(\phi+\mu\psi)$.

These spheres are evidently the same as those which were mentioned in § 244, as being orthogonal to the five systems of bitangent spheres. We can easily prove by a similar process to that in § 81, that any two of these spheres cut orthogonally. They may be called the principal spheres of the surface.

The Principal Spheres.—§§ 251, 252.

251. By § 244, we see that at every point on the surface of a cyclide can be drawn a tangent sphere cutting one of the principal spheres orthogonally, and touching the surface elsewhere; and hence it follows that the surface must be its own inverse with respect to each principal sphere. Hence these species of surfaces have been called by MOUTARD anallagmatic surfaces, and the principal spheres have been called by CASEY the spheres of inversion.

We have seen that the principal spheres cut the cyclide orthogonally, and it is evident that at points along the curve of section the corresponding bitangent sphere will not touch the cyclide elsewhere, but the curve of section will be a line of curvature on the cyclide.

252. If a cyclide have a node, then, by the principle of inversion, this node must lie on each principal sphere; and thus in this case there can be but three principal spheres, and the node will be one of their points of intersection.

If a cyclide have two nodes, they must be the two points of intersection of the three principal spheres, and any other two spheres forming with these an orthogonal system may be regarded as principal spheres; this case corresponding to that of a quadric of revolution. Similarly if the cyclide have four nodes they occur in pairs, and lie on the only principal sphere; but if we denote the nodes by P, P′, Q, Q′; and the principal sphere by S; then any pair of spheres orthogonal to S and passing through P, P′, which with any pair orthogonal to S and passing through Q, Q′, make up an orthogonal system, may be considered as principal spheres.

But if a cyclide have three nodes, then there are only three principal spheres.

There are also cyclides with only two, and with only one principal sphere.

Reduction of the General Equation to its Simplest Form.—§§ 253–255.

253. We have seen that the number of principal spheres is the same as the number of roots of the discriminating quintic $\mathrm{H}(\phi+k\psi)=0$. Thus in general there are five principal spheres, and we have seen that these cut orthogonally, and it is clear that if we take these spheres as our system of reference, we can express the equation of the cyclide in the form*

$$ax^2+by^2+cz^2+dw^2+ev^2=0;$$

and we might expect, perhaps, that this equation would still be the simplest form, when two or more roots of the equation $\mathrm{H}(\phi+k\psi)=0$ are equal. But five orthogonal spheres cannot be all real, one must be imaginary; and we shall find that if one of the principal spheres corresponding to the unequal roots is imaginary, then this form is the simplest form of the equation; but if all the principal spheres corresponding to the unequal roots are real it is not the simplest form.

254. Let us suppose that two of the roots of $\mathrm{H}=0$ are equal; then, taking for our system of reference the three principal spheres (x, y, z), say corresponding to the unequal roots, and any two other spheres (w, v) forming with them an orthogonal system, we can at once reduce the equation of the surface to the form

$$ax^2+by^2+cz^2+dw^2+ev^2+2nvw=0;$$

and the discriminating quintic is

$$\mathrm{H}\equiv(a+k)(b+k)(c+k)\{(d+k)(e+k)-n^2\}=0,$$

which can only have equal roots provided $d=e$, $n=0$; *i.e.*, supposing n is real. Thus, supposing one of the three spheres (x, y, z) to be imaginary, so that (w, v) must be real, the equation can be reduced to the form

$$ax^2+by^2+cz^2+dw^2+dv^2=0,$$

or since the equation of the absolute is

$$x^2+y^2+z^2+w^2+v^2=0,$$

the equation of the cyclide can be put in the form

$$ax^2+by^2+cz^2=0;$$

and we see that each of the points common to (x, y, z) is a node; thus the surface has two nodes; and, moreover, any two spheres which with (x, y, z) make up an orthogonal system may be taken as principal spheres.

Similarly, if the sphere (x) be imaginary, and the discriminating quintic $\mathrm{H}=0$ has two pairs of equal roots, the equation can be reduced to the form

$$ax^2+b(y^2+z^2)+d(w^2+v^2)=0;$$

* [The equation of a cyclide was first given in this form by Casey (1871) ('Phil. Trans.,' vol. 161, p. 600).—October, 1886.]

and we see that each of the points $x=y=z=0$, and $x=y=w=0$, are nodes; this surface is therefore quadrinodal.

Let us now suppose that each of the spheres (x, y, z) is real, so that the coefficient n must be imaginary. It is simplest to take for system of reference the three principal spheres (x, y, z) and their two points of intersection (w, v), so that the equation of the absolute is

$$x^2+y^2+z^2=4wv.$$

The equation $\mathrm{H}(\phi+k\psi)=0$ becomes now

$$(a+k)(b+k)(c+k)\{de-(2k-n)^2\}=0,$$

which has equal roots if either d or $e=0$, so that the equation may be reduced to the form

$$ax^2+by^2+cz^2+dw^2+2nwv=0;$$

or by means of the absolute to the form

$$ax^2+by^2+cz^2+dw^2=0.$$

The surface represented by this equation has only three principal spheres and it has one node, viz., $w=0$.

Similarly, if the discriminating quintic have two pairs of equal roots, and the sphere x correspond to the unequal root, we can show that x being real, the equation can be reduced to the form

$$ax^2+by^2+bz^2+dw^2=0;$$

which represents a surface having three nodes, viz., the point (w) and the points common to the spheres (x, u, u'); u, u' being such that they form with (x, y, z) an orthogonal system.

255. Let us now suppose that three of the roots of the quintic $\mathrm{H}(\phi+k\psi)=0$ are equal. Taking for spheres of reference (x, y) corresponding to the remaining roots, and (z, w, v) any other spheres forming with (x, y) an orthogonal system, then the equation of the surface must take the form

$$ax^2+by^2+cz^2+dw^2+ev^2+2fwv+2gzv+2hzw=0,$$

and the equation of the absolute the form

$$x^2+y^2+z^2+w^2+v^2=0;$$

so that the discriminating quintic becomes

$$(k+a)(k+b)\begin{vmatrix} k+c, & h, & g \\ h, & k+d, & f \\ g, & f, & k+e \end{vmatrix}=0.$$

This can only have equal roots if the coefficients h, g, f are real, and then we must have

$$h=f=g=0, \quad c=d=e.$$

In this case one of the two spheres (x, y) must be imaginary, and so the equation reduces to

$$ax^2+by^2+c(z^2+w^2+v^2)=0,$$

which clearly represents two spheres.

Similarly, if the quintic $H=0$ has four equal roots, and x be imaginary, the equation will reduce to

$$ax^2+by^3+bz^2+bw^3+bv^3=0,$$

which represents the imaginary sphere $x=0$.

Let us now suppose the spheres (x, y) to be real, then let us take z any sphere cutting them orthogonally, and let (w, v) be the two points of intersection of (x, y, z): so that the equation of the absolute is

$$x^2+y^2+z^2=4wv,$$

and the discriminating quintic H becomes

$$(a+k)(b+k)\begin{vmatrix} k+c, & h, & g \\ h, & d, & f-2k \\ g, & f-2h, & e \end{vmatrix}=0;$$

which can only have equal roots provided that

$$e=g=0, \quad f=-2c;$$

and then the equation takes the form

$$ax^2+by^2+cz^2+dw^2+2hzw-4cwv=0.$$

By taking, instead of $z=0$, the sphere $z-\frac{d}{2h}w=0$, this equation may clearly be reduced to the form

$$ax^2+by^2+2hzw=0.$$

The surface represented by this equation has the point $w=0$ for a *binode*: if a or b is zero, the node is a *unode*.

Again, if the equation $H(\phi+k\psi)=0$ has four equal roots and the sphere corresponding to the remaining root is real: let us take this sphere as $x=0$, and let us take for our system of reference any two spheres (y, z) and the two points (w, v) in which the spheres (x, y, z) intersect, so that the system is semi-orthogonal: then the equation of the surface must be of the form

$$ax^2+by^2+cz^2+dw^2+ev^2+2fzw+2gwy+2hyz+2lyv+2mzv+2nwv=0;$$

and since the absolute is of the form

$$x^2+y^2+z^2=4wv,$$

the discriminating quintic becomes

$$(k+a)\begin{vmatrix} k+b, & h, & g, & l \\ h, & k+c, & f, & m \\ g, & f, & d, & n-2k \\ l, & m, & n-2k, & e \end{vmatrix}=0.$$

This can only have four equal roots if

$$e=l=m=0, \quad 2b=2c=-n.$$

Thus the equation can be reduced to the form

$$ax^2+by^2+bz^2+dw^2+2fwz+2gwy+2hyz-4bwv=0.$$

This may also be written

$$ax^2+dw^2+2fwz+2gwy+2hyz=0\ ;$$

and by suitably choosing for spheres of reference $(y+\lambda w, z+\mu w)$ instead of (y, z) this equation can evidently be reduced to the form

$$ax^2+2hyz+dw^2=0.$$

The point $w=0$ is a node on the surface: it is in general a *cnic*-node; but if $a=0$ it is a *binode*, and if $h=0$ it is a *unode*.

Thus there are four distinct forms of cyclides: their equations can be reduced to one of the four forms

(A.) $$ax^2+by^2+cz^2+dw^2+ev^2=0;$$

the absolute being

$$x^2+y^2+z^2+w^2+v^2=0.$$

There are five principal spheres: if $d=e$ there are two nodes, and if $b=c$, $d=e$, there are four nodes.

(B.) $$ax^2+by^2+cz^2+dw^2=0,$$

the absolute being

$$x^2+y^2+z^2=4wv.$$

This is the general inverse of a quadric surface: there are three principal spheres and one *cnic*-node. If $b=c$ there are three *cnic*-nodes, this is the inverse of a quadric of revolution.

(C.) $$ax^2+by^2+2hwz=0,$$

the absolute being

$$x^2+y^2+z^2=4wv.$$

This surface has two principal spheres and a *binode* ($w=0$). If $a=b$ there will be two other nodes.

(D.) $$ax^2+2hyz+dw^2=0,$$

the absolute being

$$x^2+y^2+z^2=4wv.$$

This surface has only one principal sphere and a *cnic*-node. Also, since the general equation represents a cubic cyclide, when it is satisfied by the coordinates of the plane at infinity, it follows that the equation of a cubic cyclide can always be reduced to one of the four forms given above.

Chapter VI.—Classification of Cyclides.

Cyclides have been classified by Darboux and Casey according to the nature of the focal quadrics. We shall find it more convenient here to discuss the different forms of cyclides in order, according to the number of different roots which the discriminating quintic has. We have seen that there are four distinct forms to which the equation of a cyclide may be reduced; and it is proposed to discuss briefly the different species represented by similar equations, and a few of their properties.

A. *The General Cyclide.*—§§ 256–264.

256. The equation of the surface is of the form

$$ax^2+by^2+cz^2+dw^2+ev^2=0,$$

the equation of the absolute being

$$x^2+y^2+z^2+w^2+v^2=0\ ;$$

and the coordinates of the plane at infinity $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{r_4}, \frac{1}{r_5}$, *i.e.*, the reciprocals of the radii of the five principal spheres.

257. The coordinates $(\xi\eta\zeta\omega\varpi)$ of any tangent sphere at the point $(x'y'z'w'v')$ must satisfy

$$\frac{\xi}{(a+k)x'}=\frac{\eta}{(b+k)y'}=\frac{\zeta}{(c+k)z'}=\frac{\omega}{(d+k)w'}=\frac{\varpi}{(e+k)v'}$$

Hence the equation to the tangent plane at the point $(x'y'z'w'v')$ is

$$(x'x+y'y+z'z+w'w+v'v)\left(\frac{ax'}{r_1}+\frac{by'}{r_2}+\frac{cz'}{r_3}+\frac{dw'}{r_4}+\frac{ev'}{r_5}\right)$$
$$=(ax'x+by'y+cz'z+dw'w+ev'v)\left(\frac{x'}{r_1}+\frac{y'}{r_2}+\frac{z'}{r_3}+\frac{w'}{r_4}+\frac{v'}{r_5}\right), \quad . \quad . \quad . \quad (278)$$

and the equations to the normal at the point $(x'y'z'w'v')$ are

$$\left\| \begin{matrix} x, & y, & z, & w, & v \\ x', & y', & z', & w', & v' \\ ax', & by', & cz', & dw', & ev' \\ \frac{1}{r_1'}, & \frac{1}{r_2'}, & \frac{1}{r_3'}, & \frac{1}{r_4'}, & \frac{1}{r_5} \end{matrix} \right\| =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (279)$$

258. The five systems of bitangent spheres are given by:

$$\left.\begin{aligned} \xi=0,\ \frac{\eta^2}{b-a}+\frac{\zeta^2}{c-a}+\frac{\omega^2}{d-a}+\frac{\varpi^2}{e-a}=0 \\ \eta=0,\ \frac{\xi^2}{a-b}+\frac{\zeta^2}{c-b}+\frac{\omega^2}{d-b}+\frac{\varpi^2}{e-b}=0 \\ \zeta=0,\ \frac{\xi^2}{a-c}+\frac{\eta^2}{b-c}+\frac{\omega^2}{d-c}+\frac{\varpi^2}{e-c}=0 \\ \omega=0,\ \frac{\xi^2}{a-d}+\frac{\eta^2}{b-d}+\frac{\zeta^2}{c-d}+\frac{\varpi^2}{e-d}=0 \\ \varpi=0,\ \frac{\xi^2}{a-e}+\frac{\eta^2}{b-e}+\frac{\zeta^2}{c-e}+\frac{\omega^2}{d-e}=0 \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad . \quad (280)$$

and the five systems of focal spheri-quadrics by

$$\left.\begin{aligned} x=0,\ \frac{y^2}{b-a}+\frac{z^2}{c-a}+\frac{w^2}{d-a}+\frac{v^2}{e-a}=0 \\ y^2+z^2+w^2+v^2=0 \end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad (281)$$

and similar equations. These curves have evidently, for their principal circles on the sphere, the circles in which the sphere cuts the other principal spheres of the surface.

259. From the form of the equations of the focal curves, it is clear that any surface represented by the equation

$$\frac{x^2}{\alpha^2+\kappa}+\frac{y^2}{\beta^2+\kappa}+\frac{z^2}{\gamma^2+\kappa}+\frac{w^2}{\delta^2+\kappa}+\frac{v^2}{\epsilon^2+\kappa}=0, \quad . \quad . \quad . \quad . \quad . \quad (282)$$

has the same principal spheres and the same focal curves as the surface

$$\frac{x^2}{a^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}+\frac{w^2}{\delta^2}+\frac{v^2}{\epsilon^2}=0\,;$$

and subtracting, we have

$$\frac{x^2}{a^2(a^2+\kappa)}+\frac{y^2}{\beta^2(\beta^2+\kappa)}+\frac{z^2}{\gamma^2(\gamma^2+\kappa)}+\frac{w^2}{\delta^2(\delta^2+\kappa)}+\frac{v^2}{\epsilon^2(\epsilon^2+\kappa)}=0,$$

and therefore the spheres whose coordinates are respectively proportional to

$$\frac{x}{a^2},\ \frac{y}{\beta^2},\ \frac{z}{\gamma^2},\ \frac{w}{\delta^2},\ \frac{v}{\epsilon^2}\,;$$

and

$$\frac{x}{a^2+\kappa},\ \frac{y}{\beta^2+\kappa},\ \frac{z}{\gamma^2+\kappa},\ \frac{w}{\delta^2+\kappa},\ \frac{v}{\epsilon^2+\kappa}\,;$$

cut orthogonally; but these spheres evidently touch the surfaces at a common point; hence, confocal cyclides cut orthogonally. And since the above equation (282) is a cubic in κ, it follows that through any point can be drawn three cyclides confocal with a given cyclide, and these surfaces cut orthogonally.

Or, again, let us determine κ so that (282) represents a cubic cyclide; then we see that three cubic cyclides can, in general, be drawn having the same focal curves as a given quartic cyclide, and these cubic cyclides cut orthogonally. Or, if the given cyclide is a cubic cyclide, three quartic cyclides can be drawn with the same focal curves through any point; and two other cubic cyclides can also be drawn with the same focal curves.

260. Let $(\xi\eta\zeta\omega\varpi)$ be any sphere S; this will cut orthogonally one of the bitangent spheres at the point $(x'y'z'w'v')$ of the surface

$$ax^2+by^2+cz^2+dw^2+ev^2=0,$$

if

$$(a-e)x'\xi+(b-e)y'\eta+(c-e)z'\zeta+(d-e)w'\omega=0.$$

Hence, given any sphere S, a series of bitangent spheres belonging to any system can be drawn, cutting S orthogonally, their points of contact lying on the curve in which the cyclide is cut by the sphere

$$(a-e)\xi x+(b-e)\eta y+(c-e)\zeta z+(d-e)\omega w=0,$$

which may be called the *polar* sphere of S with respect to the cyclide.

There are five such polar spheres for any sphere S, each cutting one of the principal spheres orthogonally, and each one clearly intersects S in points lying on the sphere

$$a\xi x+b\eta y+c\zeta z+d\omega w+e\varpi v=0\,;$$

i.e., the five polar spheres of any sphere have with S a common radical plane.

261. If $d=e$, then the equation of the cyclide may be written

$$ax^2+by^2+cz^2=0.$$

It has clearly two *cnic*-nodes: the points common to (x, y, z); and we have seen in § 254 that one of these spheres is imaginary. Also, the surface has three distinct principal spheres; but any pair of orthogonal spheres cutting (x, y, z) orthogonally may clearly be considered as principal spheres. The case corresponds to that of a quadric of revolution.

The coordinates of a bitangent sphere at the point $(x'y'z'w'v')$ must satisfy the equations

$$\frac{\xi}{(a+k)x'}=\frac{\eta}{(b+k)y'}=\frac{\zeta}{(c+k)z'}=\frac{\omega}{kw'}=\frac{\varpi}{kv'}.$$

There are only three systems of bitangent spheres, viz.:—

$$\left.\begin{aligned}
\xi=0,\ &\frac{\eta^2}{a-b}+\frac{\zeta^2}{a-c}+\frac{\omega^2+\varpi^2}{a}=0\\
\eta=0,\ &\frac{\xi^2}{b-a}+\frac{\zeta^2}{b-c}+\frac{\omega^2+\varpi^2}{b}=0\\
\zeta=0,\ &\frac{\xi^2}{c-a}+\frac{\eta^2}{c-b}+\frac{\omega^2+\varpi^2}{c}=0
\end{aligned}\right\}\quad\ldots\ldots\quad(283)$$

The focal curves on the principal spheres will be circles; they are given by

$$\left.\begin{aligned}
x=0,\ &\frac{b}{a-b}y^2+\frac{c}{a-c}z^2=0\\
y=0,\ &\frac{a}{b-a}x^2+\frac{c}{b-c}z^2=0\\
z=0,\ &\frac{a}{c-a}x^2+\frac{b}{c-b}y^2=0
\end{aligned}\right\}\quad\ldots\ldots\quad(284)$$

262. Again, if $b=c$, $d=e$, we have seen that the cyclide has four *cnic*-nodes, and one principal sphere x, which is imaginary. There is one system of bitangent spheres, given by

$$\xi=0,\ \frac{\eta^2+\zeta^2}{a-b}+\frac{\omega^2+\varpi^2}{a-d}=0.\quad\ldots\ldots\quad(285)$$

263. Let us suppose now the radius of one of the principal spheres to be infinite, say r_5; the corresponding focal curve is a plane bicircular quartic, its equation being

$$\frac{x^2}{a-e}+\frac{y^2}{b-e}+\frac{z^2}{c-e}+\frac{w^2}{d-e}=0\quad\ldots\ldots\quad(286)$$

And this plane $v=0$ must clearly pass through the centres of the other spheres. Suppose now this curve to represent a circular cubic, then

$$\frac{1}{r_1^2}\frac{1}{a-e}+\frac{1}{r_2^2}\frac{1}{b-e}+\frac{1}{r_3^2}\frac{1}{c-e}+\frac{1}{r_4^2}\frac{1}{d-e}=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (287)$$

The curve then passes through the centres of the principal spheres; and since these points will then satisfy the condition of being foci, the imaginary circle at infinity must be a cuspidal edge on the cyclide. A cyclide of this nature is called a Cartesian.

264. If the coefficients of the equation

$$ax^2+by^2+cz^2+dw^2+ev^2=0,$$

are connected by the relation

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}+\frac{d}{r_4^2}+\frac{e}{r_5^2}=0,$$

the surface represents a cubic cyclide.

Regarding this as the general cubic cyclide, we see that it passes through the centres of its five principal spheres.

The equation of the tangent plane at the centre of the sphere $x=0$, being

$$\frac{(a-b)}{r_2}y+\frac{(a-c)}{r_3}z+\frac{(a-d)}{r_4}w+\frac{(a-e)}{r_5}v=0\,; \quad . \quad . \quad . \quad . \quad . \quad (288)$$

which is clearly parallel to the plane

$$\frac{ax}{r_1}+\frac{by}{r_2}+\frac{cz}{r_3}+\frac{dw}{r_4}+\frac{ev}{r_5}=0.$$

Thus the tangent planes to the cyclide at the centres of its five principal spheres are all parallel, and they may be regarded as the five tangent planes to the cubic from the line at infinity on the surface.

(B.) *General Nodal Cyclide.*—§§ 265–271.

265. We have seen in § 253, that the equation of a cyclide having one node can be expressed in the form

$$ax^2+by^2+cz^2+dw^2=0.$$

The system of reference being three orthogonal spheres (x, y, z) and their two points of intersection (w, v): so that the equation of the absolute is

$$x^2+y^2+z^2=4wv\,;$$

the coordinates of the plane at infinity being $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{\epsilon}, \frac{1}{\epsilon}$, where r_1, r_2, r_3 are the radii of (x, y, z), and ϵ the distance between (w, v). The point $w=0$ is a *cnic*-node.

266. The coordinates $(\xi, \eta, \zeta, \omega, \varpi)$ of any tangent sphere at the point $(x'y'z'w'v')$ must satisfy

$$\frac{\xi}{(a+k)x'}=\frac{\eta}{(b+k)y'}=\frac{\zeta}{(c+k)z'}=\frac{-2\varpi}{dw'-2kv'}=\frac{-2\omega}{-2kw'}$$

So that the equation of the tangent plane at any point $(x'y'z'w'v')$ is

$$\left(\frac{ax'}{r_1}+\frac{by'}{r_2}+\frac{cz'}{r_3}+\frac{dw'}{\epsilon}\right)(x'x+y'y+z'z-2w'v-2v'w)$$
$$=\left(\frac{x'}{r_1}+\frac{y'}{r_2}+\frac{z'}{r_3}-2\frac{w'+v'}{\epsilon}\right)(ax'x+by'y+cz'z+dw'w); \quad . \quad . \quad . \quad . \quad (289)$$

and the equations of the normal at $(x'y'z'w'v')$ are

$$\begin{Vmatrix} x, & y, & z, & -2v, & -2w \\ x', & y', & z', & -2v', & -2w' \\ ax', & by', & cz', & dw', & 0 \\ \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & -\frac{2}{\epsilon}, & -\frac{2}{\epsilon} \end{Vmatrix} =0. \quad . \quad . \quad . \quad . \quad . \quad (290)$$

267. The three systems of bitangent spheres are given by:

$$\left.\begin{aligned} \xi=0\,; \quad & \frac{\eta^2}{b-a}+\frac{\zeta^2}{c-a}+\frac{4\omega\varpi}{a}-\frac{d}{a^2}\omega^2=0 \\ \eta=0\,; \quad & \frac{\xi^2}{a-b}+\frac{\zeta^2}{c-b}+\frac{4\omega\varpi}{b}-\frac{d}{b^2}\omega^2=0 \\ \zeta=0\,; \quad & \frac{\xi^2}{a-c}+\frac{\eta^2}{b-c}+\frac{4\omega\varpi}{c}-\frac{d}{c^2}\omega^2=0 \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad . \quad (291)$$

the focal curves are given by:

$$\left.\begin{aligned} x=0\,; \quad & \frac{b}{b-a}y^2+\frac{c}{c-a}z^2-\frac{d}{a}w^2=0 \\ & y^2+z^2-4wv=0 \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad . \quad (292)$$

and similar equations. Thus the focal curves are nodal spheri-quadrics on the three principal spheres.

268. From the form of the equations of the focal curves we infer that all surfaces represented by the equation

$$\frac{x^2}{\alpha^2+k}+\frac{y^2}{\beta^2+k}+\frac{z^2}{\gamma^2+k}+\frac{w^2}{\delta^2}=0,$$

are confocal with the surface

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}+\frac{w^2}{\delta^2}=0;$$

and we infer that given any cyclide having one *cnic*-node, through any point three other cyclides can be drawn having the same node and the same focal curves: and these cyclides cut orthogonally.

We also see, by choosing k so that

$$\frac{1}{\alpha^2+k}\frac{1}{r_1^2}+\frac{1}{\beta^2+k}\frac{1}{r_2^2}+\frac{1}{\gamma^2+k}\frac{1}{r_3^2}+\frac{1}{\delta^2}\frac{1}{e^2}=0,$$

that three cubic cyclides can be drawn having the same node and focal curves as a given nodal quartic cyclide.

269. If we have $b=c$, we have seen that the equation

$$ax^2+by^2+bz^2+dw^2=0$$

represents a trinodal cyclide, $w=0$ being one node: its equation may also be written

$$(a+b)x^2+dw^2+4buv=0,$$

or

$$a'x^2+2fwu=0,$$

where u is a sphere passing through the other nodes, but in this case the equation of the absolute will not be so simple. Taking the first form of the equation we see that the system of bitangent spheres is given by

$$\xi=0;\quad \frac{\eta^2+\zeta^2}{b-a}+\frac{4\omega\varpi}{a}-\frac{d}{a^2}\omega^2=0; \qquad (293)$$

and the corresponding focal curve by

$$x=0;\quad \frac{4bv}{b-a}-\frac{d}{a}w=0, \qquad (294)$$

which is a circle on the principal sphere $x=0$.

270. If one of the principal spheres, say z, become a plane, the corresponding focal curve is a plane nodal bicircular quartic, and if it pass through the centres of the other two principal spheres, we have

$$\frac{a}{a-c}\frac{1}{r_1^2}+\frac{b}{b-c}\frac{1}{r_2^2}-\frac{d}{c^2}\frac{1}{e^2}=0, \qquad (295)$$

and if this be satisfied the surface has the circle at infinity for a cuspidal edge; and so may be called a nodal-Cartesian.

271. If the coefficients in the equation

$$ax^3+by^3+cz^3+dw^3=0,$$

are connected by the relation

$$\frac{a}{r_1^3}+\frac{b}{r_2^3}+\frac{c}{r_3^3}+\frac{d}{e^3}=0,$$

the surface represents a nodal circular cubic. It clearly passes through the centres of the three principal spheres, and the tangent planes at these points are parallel to the plane

$$\frac{ax}{r_1}+\frac{by}{r_2}+\frac{cz}{r_3}+\frac{dw}{e}=0;$$

and so the tangent planes at the centres of the principal spheres are the tangent planes drawn from the line at infinity on the surface.

(C.) *Cyclides having a Binode.*—§§ 272–276.

272. The general equation of a cyclide having two principal spheres and a binode can, by § 254, be expressed in the form

$$ax^2+by^2+2hzw=0;$$

the system of reference being the two principal spheres (x, y) and the sphere (z) passing through the node (w), and cutting (x, y) orthogonally in the points (w, v), the equation of the absolute being

$$x^2+y^2+z^2=4wv,$$

and the coordinates of the plane at infinity $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{e}, \frac{1}{e}$.

273. The coordinates $(\xi\eta\zeta\omega\varpi)$ of any tangent sphere at the point $(x'y'z'w'v')$, must satisfy

$$\frac{\xi}{(a+k)x'}=\frac{\eta}{(b+k)y'}=\frac{\zeta}{hw'+kz'}=\frac{-2\varpi}{hz'-2kv'}=\frac{-2\omega}{-2kw'}$$

The equation of the tangent plane at the point $(x'y'z'w'v')$ will be

$$\left(\frac{ax'}{r_1}+\frac{by'}{r_2}+\frac{hw'}{r_3}+\frac{hz'}{e}\right)(x'x+y'y+z'z-2w'v-2v'w)$$
$$=\left(\frac{x'}{r_1}+\frac{y'}{r_2}+\frac{z'}{r_3}-2\frac{w'+v'}{e}\right)(ax'x+by'y+hw'z+hz'w). \quad . \quad . \quad . \quad . \quad (296)$$

The equations of the normals will be

$$\begin{vmatrix} x, & y, & z, & -2v, & -2w \\ x', & y', & z', & -2v', & -2w' \\ ax', & by', & hw' & hz', & 0 \\ \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & -\frac{2}{e}, & -\frac{2}{\epsilon} \end{vmatrix} = 0. \quad . \quad . \quad . \quad . \quad . \quad (297)$$

274. The two systems of bitangent spheres are given by

$$\left.\begin{aligned} \xi=0\,; \quad & \frac{\eta^2}{b-a}+\frac{4\omega\varpi}{a}-\frac{(a\zeta+h\omega)^2}{a^3}=0 \\ \eta=0\,; \quad & \frac{\xi^2}{a-b}+\frac{4\omega\varpi}{b}-\frac{(b\zeta+h\omega)^2}{b^3}=0 \end{aligned}\right\}; \quad . \quad . \quad . \quad . \quad . \quad . \quad (298)$$

and focal curves are given by

$$\left.\begin{aligned} x=0\,; \quad & \left.\begin{aligned} & \frac{b}{b-a}y^2-\frac{2h}{a}zw-\frac{h^2}{a^2}w^2=0 \\ & y^2+z^2=4wv \end{aligned}\right\} \\ \text{and} \\ y=0\,; \quad & \left.\begin{aligned} & \frac{a}{a-b}x^2-\frac{2h}{b}zw-\frac{h^2}{b^2}w^2=0 \\ & y^2+z^2=4wv \end{aligned}\right\} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (299)$$

275. Let us suppose the sphere y to be of infinite radius, then the corresponding focal curve will pass through the centre of the principal sphere x, when

$$\frac{a}{a-b}\frac{1}{r_1^2}-\frac{2h}{b}\frac{1}{er_3}-\frac{h^2}{b^2}\frac{1}{e^2}=0\,; \quad . \quad . \quad . \quad . \quad . \quad . \quad (300)$$

and then the surface will represent a Cartesian having a binode.

276. The surface

$$ax^2+by^2+2hzw=0$$

will represent a cubic cyclide, having a binode when

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{2h}{r_3e}=0.$$

In this case the surface passes through the centres of the two principal spheres (x, y), and the tangents at these points are parallel to the plane

$$\frac{ax}{r_1}+\frac{by}{r_2}+h\frac{z}{e}+h\frac{w}{r_3}=0,$$

and so are the tangent planes drawn through the line at infinity on the surface.

D. *Nodal Cyclides, with One Principal Sphere.*—§§ 277–280.

277. If the cyclide have only one principal sphere, we have seen, in § 255, that its equation may be reduced to the form

$$ax^2+2hyz+dw^2=0,$$

where x is the principal sphere, w the node, and (y, z) two other spheres cutting x and each other orthogonally, and cutting x in the points (w, v).

The equation of the absolute being

$$x^2+y^2+z^2=4wv,$$

and the coordinates of the plane at infinity $\frac{1}{r_1}, \frac{1}{r_2}, \frac{1}{r_3}, \frac{1}{e}, \frac{1}{e}$; we have seen that the node is in general a *cnic*-node; but it is a binode if $a=0$, and a unode if $h=0$.

278. The coordinates $(\xi\eta\zeta\omega\varpi)$ of any tangent sphere at the point $(xyzwv)$ must satisfy

$$\frac{\xi}{(a+k)x'}=\frac{\eta}{hz'+ky'}=\frac{\zeta}{hy'+kz'}=\frac{-2\varpi}{dw'-2kv'}=\frac{-2\omega}{-2kw'}.$$

The equation of the tangent plane at $(x'y'z'w'v')$ will be

$$\left(\frac{ax'}{r_1}+\frac{hz'}{r_2}+\frac{hy'}{r_3}+\frac{dw'}{e}\right)(x'x+y'y+z'z-2w'v-2v'w)$$
$$=\left(\frac{x'}{r_1}+\frac{y'}{r_2}+\frac{z'}{r_3}-2\frac{w'+v'}{e}\right)(ax'x+hy'z+hz'y+dww'). \quad . \quad . \quad . \quad . \quad (301)$$

The equations of the normal at $(x'y'z'w'v')$ are

$$\left\| \begin{array}{ccccc} x, & y, & z, & -2v, & -2w \\ x', & y', & z', & -2v', & -2w' \\ ax', & hz', & hy', & dw', & 0 \\ \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & -\frac{2}{e}, & -\frac{2}{e} \end{array} \right\| =0. \quad . \quad . \quad . \quad . \quad (302)$$

279. The system of bitangent spheres is given by

$$\xi=0, \quad \frac{a(\eta^2+\zeta^2)+2h\eta\zeta}{a^2-h^2}+\frac{d\omega^2}{a^2}-4\frac{\omega\varpi}{a}=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (303)$$

and the corresponding focal curve is given by,

$$\left.\begin{array}{l} x=0, \quad \frac{4h^2wv}{a^2-h^2}+\frac{ah}{a^2-h^2}yz+\frac{d}{a}w^2=0 \\ \qquad\qquad y^2+z^2=4wv \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (304)$$

280. If

$$\frac{a}{r_1^2}+\frac{2h}{r_2r_3}+\frac{d}{s^2}=0,$$

the surface represented by the equation

$$ax^3+2hyz+dw^3=0,$$

will be a cubic surface, which passes through the centre of the principal sphere, the tangent at which point is parallel to the plane

$$\frac{ax}{r_1}+\frac{hz}{r_2}+\frac{hy}{r_3}+\frac{dw}{s}=0,$$

and therefore passes through the line on the surface at infinity.

Chapter VII.—Miscellaneous Theorems.

Equation of a Cyclide referred to Four Spheres Orthogonal to a Principal Sphere.—§§ 281–285.

281. Let the system of spheres $(xyzwv)$ be such that $v=0$ is a principal sphere of a cyclide; then its equation must take the form

$$v^2+\phi(xyzw)=0,$$

and if v be orthogonal to the system $(xyzw)$, then the equation of the absolute must also be of the form

$$v^2+f(xyzw)=0.$$

Hence, by subtraction, we see that the equation of a cyclide can always be written in the form

$$ax^2+by^2+cz^2+dw^2+2fyz+2gzx+2hxy+2lxw+2myw+2nzw=0; \quad (305)$$

the system of reference being four spheres, points or planes orthogonal to a principal sphere.

Now, the equation of any sphere orthogonal to this same principal sphere is of the form

$$\alpha x+\beta y+\gamma z+\delta w=0, \quad \ldots\ldots\ldots \quad (306)$$

where, by § 211, equation (222), α, β, γ, δ must be proportional to the tetrahedral coordinates of the centre of the sphere, referred to the tetrahedron formed by the centres of $(xyzw)$, provided that $(xyzw)$ the coordinates of any point are the powers of the point with respect to the system.

Suppose now that the sphere given by (306) is a bitangent sphere of (305), then clearly we must have

$$\begin{vmatrix} a, & h, & g, & l, & \alpha \\ h, & b, & f, & m, & \beta \\ g, & f, & c, & n, & \gamma \\ l, & m, & n, & d, & \delta \\ \alpha, & \beta, & \gamma, & \delta, & 0 \end{vmatrix} = 0. \quad \ldots \quad (307)$$

Hence the locus of the centres of all bitangent spheres of the system orthogonal to $v=0$ is the quadric surface given by (307). This surface is called by CASEY the "focal quadric." We see that the focal curve on the principal sphere $v=0$ is the curve of intersection of the sphere with the focal quadric.

282. The surface (305) is a cubic cyclide, if it is satisfied by the coordinates (1, 1, 1, 1) of the plane at infinity; it follows that when this is the case the quadric (307) is touched by the plane at infinity, and, therefore, the focal quadrics of cubic cyclides are paraboloids.

283. If we take for our spheres of reference the other four principal spheres, the equation of the cyclide takes the form

$$ax^3+by^3+cz^3+dw^3=0,$$

and then the focal quadric is given by

$$\frac{\alpha^2}{a}+\frac{\beta^2}{b}+\frac{\gamma^2}{c}+\frac{\delta^2}{w}=0;$$

so that the tetrahedron formed by the centres of any four principal spheres of a cyclide is self-conjugate with respect to the focal quadric corresponding to the fifth.

284. In the case of a Cartesian, the focal quadrics must be spheres: hence, if A, B, C, D be the centres of four bitangent spheres which we will take for our spheres of reference, then since the focal quadric must be of the form

$$a^2\beta\gamma+b^2\gamma\alpha+c^2\alpha\beta+a'^2\alpha\delta+b'^2\beta\delta+c'^2\gamma\delta=0,$$

where a, b, c, a', b', c' are the sides of the tetrahedron ABCD, the equation of the cyclide will be

$$\begin{vmatrix} 0, & c^2, & b^2, & a'^2, & x \\ c^2, & 0, & a^2, & b'^2, & y \\ b^2, & a^2, & 0, & c'^2, & z \\ a'^2, & b'^2, & c'^2, & 0, & w \\ x, & y, & z, & w, & 0 \end{vmatrix} = 0; \quad \ldots \quad (308)$$

where $(xyzw)$ are the squares of the tangents from any point P on the surface, to the four bitangent spheres.

285. By § 260 we can draw a series of bitangent spheres belonging to each system orthogonally to any sphere, and the points of contact of each series lie on a sphere which cuts the principal sphere belonging to that series of bitangent spheres orthogonally, and may be called the *polar* sphere of the given sphere with respect to the cyclide. Now let (x, y, z) be any three bitangent spheres cutting a fixed sphere S orthogonally, and let w be the corresponding polar sphere of S, the equation of the cyclide must take the form

$$d^2w^2+a^2x^2+b^2y^2+c^2z^2-2bcyz-2cazx-2abxy=0,$$

and the corresponding focal quadric will be

$$2\frac{\delta^2}{d^2}-\frac{\beta\gamma}{bc}-\frac{\gamma\alpha}{ca}-\frac{\alpha\beta}{ab}=0.$$

Thus the polar of the centre of w with respect to the quadric is the plane passing through the centres of x, y, z.

As a particular case we may suppose S replaced by the centre of the principal sphere, so that (x, y, z) become double tangent planes; the plane $\delta=0$ is now at an infinite distance, and so we see that the centre of the polar of the centre of the principal sphere with respect to the cyclide is also the centre of the corresponding focal quadric. And also the asymptotic cone of the quadric cuts orthogonally the double tangent cone from the centre of the corresponding principal sphere.

Normals to a Cyclide from any Point.—§§ 286, 287.

286. The problem of drawing normals to a cyclide has been extensively discussed by DARBOUX ('Sur une Classe remarquable de Courbes et de Surfaces Algébriques.' Note XI.). To find the number of normals which can be drawn from any point he proceeds thus:—

Let $(\xi\eta\zeta\omega\varpi)$ be any tangent sphere to the cyclide

$$ax^2+by^2+cz^2+dw^2+ev^2=0,$$

then since

$$\frac{\xi}{(a+k)x}=\frac{\eta}{(b+k)y}=\frac{\zeta}{(c+k)z}=\frac{\omega}{(d+k)w}=\frac{\varpi}{(e+k)v};$$

we have the equations

$$\frac{\xi^2}{a+k}+\frac{\eta^2}{b+k}+\frac{\zeta^2}{c+k}+\frac{\omega^2}{d+k}+\frac{\varpi^2}{e+k}=0,$$

and

$$\frac{\xi^2}{(a+k)^2}+\frac{\eta^2}{(b+k)^2}+\frac{\zeta^2}{(c+k)^2}+\frac{\omega^2}{(d+k)^2}+\frac{\varpi^2}{(e+k)^2}=0.$$

Now suppose

$$\frac{\xi}{\alpha+\mu\alpha'}=\frac{\eta}{\beta+\mu\beta'}=\frac{\zeta}{\gamma+\mu\gamma'}=\frac{\omega}{\delta+\mu\delta'}=\frac{\varpi}{\epsilon+\mu\epsilon'},$$

then substituting and eliminating k, we have clearly an equation of the twelfth degree in μ; hence we infer that twelve spheres can be drawn through the circle common to the spheres $(\alpha\beta\gamma\delta\epsilon)$ $(\alpha'\beta'\gamma'\delta'\epsilon')$ to touch the cyclide.

We may deduce from this that through any straight line can be drawn twelve tangent planes to the cyclide, or a cyclide is in general of the twelfth class.

And, again, taking ξ, η, ζ, ω, ϖ proportional to $x+\frac{\mu}{r_1}$, $y+\frac{\mu}{r_2}$, $z+\frac{\mu}{r_3}$, $w+\frac{\mu}{r_4}$, $v+\frac{\mu}{r_5}$ respectively, we see that twelve spheres can be drawn to touch the cyclide, having their centres coincident with the point $(xyzwv)$. Hence twelve normals can be drawn to the cyclide from any point.

287. In § 257, equations (279) give the normal at the point $(x'y'z'w'v')$ of the cyclide

$$ax^2+by^2+cz^2+dw^2+ev^2=0,$$

and we infer that the feet of the normals from the point $(x'y'z'w'v')$ lie on the cubic surfaces:

$$\begin{vmatrix} ax, & by, & cz, & dw, & ev \\ x, & y, & z, & w, & v \\ x', & y', & z', & w', & v' \\ \frac{1}{r_1}, & \frac{1}{r_2}, & \frac{1}{r_3}, & \frac{1}{r_4}, & \frac{1}{r_5} \end{vmatrix}=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (309)$$

Now, $\psi=0$ being the equation of the absolute referred to any system of spheres, ϕ_1, ϕ_2, ϕ_3 any three cyclides; let us form the discriminating quintic of

$$\lambda\phi_1+\mu\phi_2+\nu\phi_3+\rho\psi=0;$$

then we can so choose the ratios $\lambda:\mu:\nu:\rho$ that three of the roots of this quintic shall be zero; and thus the equation will represent two spheres. To determine the ratios $\frac{\lambda}{\rho}, \frac{\mu}{\rho}, \frac{\nu}{\rho}$ we obtain three equations of the fifth, fourth, and third degrees respectively. Hence through the common points of three cyclides a pair of spheres can be drawn in sixty ways.

Let us suppose now that certain of the feet of the twelve normals from the point $(x'y'z'w'v')$ lie on the sphere

$$\xi x+\eta y+\zeta z+\omega w+\varpi v=0,$$

then the rest must lie on another sphere which we may denote by

$$\alpha x+\beta y+\gamma z+\delta w+\epsilon v=0,$$

and then it must be possible so to choose k that the equation

$$k(ax^2+by^2+cz^2+dw^2+ev^2)=(\alpha x+\beta y+\gamma z+\delta w+\epsilon v)(\xi x+\eta y+\zeta z+\omega w+\varpi v), \quad . \quad (310)$$

shall be identical with any one of the cubic cyclides given by (308). We have at once then

$$\frac{\alpha\xi}{a}=\frac{\beta\eta}{b}=\frac{\nu\zeta}{c}=\frac{\delta\omega}{d}=\frac{\epsilon\varpi}{e}=k.$$

Consequently through the feet of the twelve normals from $(x'y'z'w'v')$ can be drawn two spheres, such that $(\xi\eta\zeta\omega\varpi)$ being the coordinates of one of them, $\frac{a}{\xi}, \frac{b}{\eta}, \frac{c}{\zeta}, \frac{d}{\omega}, \frac{e}{\varpi}$ must be proportional to the coordinates of the other.

Also equation (310) must represent a cubic: hence we must have

$$\frac{a}{r_1^2}+\frac{b}{r_2^2}+\frac{c}{r_3^2}+\frac{d}{r_4^2}+\frac{e}{r_5^2}=\left(\frac{\xi}{r_1}+\frac{\eta}{r_2}+\frac{\zeta}{r_3}+\frac{\omega}{r_4}+\frac{\varpi}{r_5}\right)\left(\frac{a}{\xi r_1}+\frac{b}{\eta r_2}+\frac{c}{\zeta r_3}+\frac{d}{\omega r_4}+\frac{e}{\varpi r_5}\right).$$

So that if the given surface is a cubic cyclide, one of these spheres on which the feet of the normals lie must be of infinite radius.

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XVI. *On the Relation between the Thickness and the Surface Tension of Liquid Films.*

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[Plate 33.]

Introduction.

Laplace's theory of capillarity is based upon the assumption that the forces in play between the neighbouring molecules of a liquid are insensible at sensible distances. Each molecule is thus frequently considered as the centre of a sphere which bounds the "insensible" region within which the molecular forces exerted by the central molecule on others are appreciable. The radius of this sphere is called "the radius of molecular action." It is evident from its definition that it is not a determinate physical constant. The point at which a force ceases to be "sensible" depends upon the delicacy of the methods which are employed to detect its effects, and thus also upon the nature of the particular effect which is studied. Many physicists have, however, rightly considered that an approximate measurement of the distance at which the mutual actions of neighbouring molecules become negligible was well worth obtaining. They have, therefore, by different methods sought indications of the magnitude of this distance, and, if successful, have added their result to the other recorded measurements of the radius of molecular action. The nomenclature thus adopted is somewhat vague, but we are not aware that any general misconception exists as to the indefinite character of the magnitude, to which, nevertheless, a definite name is given.

The theory that molecular forces are only sensible at very small distances leads us to regard a liquid as bounded by a very thin external layer, the properties of which are different from those of the mass in the interior. To this the name of the *surface layer* may be given. The definition of its thickness is subject to the same difficulties as the definition of the magnitude of the radius of molecular action. The two, however defined, are closely related and are generally assumed to be equal.

Plateau, arguing that the surface tension would decrease if the thickness of a soap film became less than twice that of the surface layer, first investigated the question as to whether the pressure exerted on the enclosed air depended on the

thickness of the bubble. His method,* as is well known, consisted in forming a bubble at the end of a tube which was bent so as to serve as a water-manometer for the measurement of the internal pressure. He concluded that the radius of molecular action was less than 59×10^{-6} mm., because no regular change was observed in the case of a bubble which thinned till it displayed a colour which, according to NEWTON'S scale, indicated twice that thickness, viz., the pale yellow of the first order.

LÜDTGE† formed a soap film at the end of a tube, and, when it had become thin, closed the other end by a new film. Some air was then forced in, the films assumed the form of spherical segments, and their curvatures were compared. He concluded that the thicker film had the less surface tension, and that almost immediately after its formation the thickness of a film was less than twice the radius of molecular action.

These experiments were carefully repeated by VAN DER MENSBRUGGHE,‡ who kept one of the films moderately thick by dropping liquid upon it, so that it displayed the red and green of the higher orders. He describes the other film as showing bright colours, especially towards the top. He could not detect any change in the curvature of the films, and concluded that the tensions were the same, though the thicknesses were different. Later, VAN DER MENSBRUGGHE'S views appear to have undergone some modification. He based an explanation of a number of curious phenomena on the principle, enunciated by Sir WILLIAM THOMSON, that if a soap film is extended it is cooled. He thought, therefore, that "les expériences de LÜDTGE s'expliquent de la manière la plus nette pourvu que les augmentations de tension admises par le physicien allemand dans certaines parties d'une lame soient attribuées au développement de surfaces fraîches sur les deux faces de cette lame, et à la diminution de température qui en résulte."§

If VAN DER MENSBRUGGHE'S later views are correct it is evident that his earlier test of LÜDTGE'S observations is open to objection, as, if a film be kept thick by dropping liquid upon it, new surfaces must necessarily be produced. Indeed, he seems to think that much less potent causes might produce results which can be easily detected.

We describe one of his experiments, to give an example of his method of applying Sir WILLIAM THOMSON'S law‖:—

A circular hole about 4 cm. in diameter was bored in an iron plate furnished with a handle. The edge of the hole was bevelled, and it was made the common basis of two bubbles forming spherical segments which enclosed a lens-shaped space. If the plate were placed horizontally, if the weights of the films were negligible, and if

* PLATEAU, 'Statique des Liquides,' 1873, vol. 1, p. 210.
† 'Poggendorff, Annalen,' vol. 139, 1870, p. 620.
‡ 'Bruxelles, Acad. Sci. Bull.,' vol. 30, 1870, p. 322.
§ 'Bruxelles, Acad. Sci. Mém.,' vol. 43, 1882 (No. 4, p. 18).
‖ *Loc. cit.*, p. 19.

their surface tensions were identical, the curvature of each would be the same. VAN DER MENSBRUGGHE, however, considers that a zone of liquid in the upper bubble would in descending increase its surface, while in the lower bubble the surface of the descending mass would diminish. Hence the upper bubble would become cooled, while the temperature of the lower one would be raised, and this would be indicated by a difference of curvature, due to the fact that the surface tensions were not the same. Observation showed that such a difference of curvature existed. In one experiment the sagittæ of the spherical segments differed by 1·15 mm., which diminished after 45^{m} to 0·1 mm. When the system had apparently reached a permanent state the plate was reversed, the upper and lower bubbles exchanged places, and the fact that the measurements were to within the error of experiment the same as before was taken as evidence that the phenomenon observed was not due to any inequality produced directly by the weight of the films.

The investigations of these observers have not left the question of the relation between the surface tension and the thickness of a liquid film in a satisfactory state.

We have already pointed out* that the experiment of PLATEAU, who observed a bubble formed of *liquide glycérique* which lasted for three days, and was enclosed in a vessel containing sticks of caustic potash to dry the air, offers no guarantee that the composition of the liquid remained unaltered. As we have found that under circumstances much less favourable to change† a soap film lost in thirty-eight minutes 23 out of the 57·7 volumes of water contained in every 100 volumes of solution, it is practically certain that the constitution of PLATEAU's bubble could not have been constant.

LÜDTGE and VAN DER MENSBRUGGHE were on their guard against this difficulty, and, though they experimented with *liquide glycérique* without any of the elaborate precautions which we have since shown to be necessary if the percentage of water is to remain unaltered, they checked these observations with others made with solutions of Quillaja and of Marseilles soap. Neither, however, appears to have put his conclusions to a certain and satisfactory test. If it be true that the surface tension of a film varies with its thickness it would be a matter of the highest interest to determine the law of change. No such attempt was made by LÜDTGE, who remarked only that the method did not admit of great accuracy, and that one of his experiments indicated a change of tension in the ratio of 2·8 to 2·84 in five minutes. VAN DER MENSBRUGGHE, on the other hand, though he has no doubt suggested a *vera causa* which may completely explain some of the phenomena he describes, made no attempt to determine whether the lowering of temperature to which he ascribes them could be produced by such increments of the surface as could reasonably be supposed to take place. For this purpose the change of tension must be measured, and though he gives the data for the calculation he does not make it. The reason for this probably was

* 'Phil. Trans.,' vol. 172 (Part 2, 1881), p. 448.

† *Loc. cit.*, p. 486.

that he was evidently aware that other causes might affect the result, and that the differences of surface tension given by different experiments varied considerably.

Thus, if we attempt to compare the surface tensions of two spherical films by measuring their sagittæ, the question of the real magnitude of their circular bases becomes important. The fact that a film adheres to its solid support by means of a ring of comparatively thick liquid makes it uncertain whether the dimensions of the film can be directly inferred from those of the solid.

In one of VAN DER MENSBRUGGHE'S experiments the films were formed on a ring 4 cm. in diameter, and their sagittæ were 16·55 and 17·70 mm. respectively. If, therefore, the bases of both segments had the same diameter as the ring, the radii of the spheres, of which they were parts, were 2·036 and 2·015 cm. This might indicate that the surface tensions differed by about 1 per cent. The inequality can also be accounted for by supposing that the films had the same surface tension, but that, owing to some little difference in the size of the thick liquid rings which united them to the solid, the radii of their bases differed by 0·18 mm. This calculation, at all events, proves that small errors of this kind might produce differences of the same order as those ascribed to changes in surface tension.

The experiments of previous observers were all made with comparatively thick films. We have had considerable experience in producing films which display the black of the first order. Their thickness, the measurement of which we have described in a previous communication,* is about one-tenth of that ascribed by PLATEAU to the bubble on which he based his conclusion. It was quite possible that the two surface layers might be separate in his experiment and yet give evidence of mingling in a film ten times thinner.

[† A preliminary objection may be taken to the observations of PLATEAU and LÜDTGE, as it may be urged that, if the tension of a film alters as it becomes thinner, it must break when it has become thin enough for the tensions of the thicker and thinner parts to differ by an appreciable amount. We have elsewhere‡ expressed an opinion that the possibility of the existence of a film in such a state depends on the relative magnitudes of its viscosity, and of the difference of tension in question. As the matter has an important bearing on the subject of this paper, it may be well to discuss it a little more fully.

There can of course be no doubt but that a difference of surface tension in two parts of the same film must produce motion. Is it therefore *a priori* certain that, if large enough to be measured, it must either produce rupture or motion so violent as to prevent the measurements being made?

In considering this question, it must be observed that no calculations had been made previous to our own as to what difference of surface tension is measurable by

* 'Phil. Trans.,' vol. 174 (Part 2, 1883), p. 645.

† The paragraphs within these brackets were added Nov. 10, 1886.

‡ 'Phil. Trans.,' vol. 172 (Part 2, 1881), p. 489.

the various methods which had been employed. Theoretically, these methods can be made infinitely sensitive, and practically, slight variations in the adjustment of the apparatus alter very considerably the value of the experiment as a means of detecting small changes in the tension. VAN DER MENSBRUGGHE had indeed proved that LÜDTGE's experiment is infinitely sensitive when the two films are hemispheres. As we have already stated, he had not, as far as we are aware, calculated the value of the difference of surface tension in play in any particular observation. We have in the following pages discussed the theory of the methods of experiment, instituted a comparison between them, and, in the case of the arrangement which we ourselves adopted, have investigated the various sources of error to which it is subject. For such a purpose calculation alone would not suffice. The practical limit to the sensitiveness is fixed by the impossibility of eliminating small perturbations, and could therefore only be found by trial.

While, therefore, on the one hand, it seemed that experiment alone could decide what could be done by experiment, on the other hand, it was theoretically probable that, if the thickness of a black film was less than twice that of the surface layer, its viscosity might be great. The surface layer of a liquid must consist of elementary layers, the properties of which are different. The mean viscosity of the surface layer being, in the case of soap solution, greater than that of the liquid, the viscosity of some of the elementary layers must be, to an unknown extent, greater than this mean. Thus it is probable that, when the thickness of a film becomes less than twice the radius of molecular action, its viscosity increases. It would not be safe to extend to a black film conclusions drawn from experiments which prove that the liquid of which a moderately thick film is composed moves readily under the influence of a small difference of surface tension.

If, therefore, the general opinion that it was possible that a change in the tension of thin films might be detected by experiment was erroneous, it was not possible to confute it by *a priori* reasoning. It had the support of high authority. Professor CLERK MAXWELL, in his article on "Capillary Action," says, "measurements of the tensions of a film, when drawn out to different degrees of thickness, may possibly lead to an estimate of the range of the molecular forces, or at least of the depth within a liquid mass at which its properties become sensibly uniform. We shall, therefore, indicate a method of investigating the tension of such films," &c.

On the whole, then, though all *a priori* considerations pointed to the conclusion that, if any difference of surface tension existed between the black and coloured parts of a film, it must be small, it appeared to us that the question as to whether it existed could only be settled beyond the possibility of dispute by experiment. We have, therefore, assumed in this paper that such a difference is possible. We have also supposed that the relation between the difference of surface tension and the other properties of the film is such that the only considerable movement to which it gives rise is a change of curvature. Having by experiments, conducted on these favourable

assumptions, decided whether or no evidence of a difference of surface tension exceeding a given amount is forthcoming, we are then at liberty to reconsider the probable constitution of the film in view of the facts which experiment has elicited.]

In studying the properties of the black portion of a soap film, it is also important to remember that it has a well-defined boundary which indicates an apparent discontinuity in the thickness. It may be urged that if the thickness is less than that of two surface layers there must also be an apparent discontinuity in the surface tension at the edge of the black. If this were so, a free globular bubble, consisting of two parts, a coloured portion and a black "polar cap," could not be spherical, but must consist of two spherical segments which meet at a finite angle. This condition is not consistent with equilibrium, and therefore the black and coloured portions of the film must have the same tension.

We think this argument deserves consideration, but we have not thought that it is so conclusive as to obviate the necessity of experiment to determine the facts. In the first place we are not aware that any evidence is on record that a perfectly free spherical bubble can have a black "cap." If the bubble is attached to a tube or other solid support the argument no longer applies, as that part which is in contact with the solid is not constrained to meet the axis of revolution, and may therefore be an unduloid or nodoid which can meet the spherical segment without any discontinuity. In order, therefore, that the argument might have the necessary basis of ascertained fact it would be requisite that bubbles should be formed without any solid supports, and should under these conditions be observed to reach a state in which they were partly black and partly coloured. Spherical bubbles when supported do not ordinarily exhibit more than a very small circular patch of black, and it would probably be very difficult to produce free spherical bubbles which should thin with sufficient rapidity for the purpose in view.

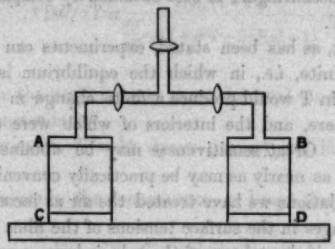
Even supposing that such a result had been obtained, anything like accurate measurement to ascertain the spherical form of the bubble would under the required conditions be difficult or impossible. Unless these measurements were made the experiment would be open to the objection that the bubble might not be truly spherical, and that the same conditions which cause the sudden change in thickness at the edge of the black might also explain a very rapid change in curvature.

It did not therefore seem wise to rely with implicit confidence on *a priori* reasoning as to a region in the film which is the seat of phenomena which are confessedly imperfectly understood. If experiment is necessary, observations on free spherical bubbles did not seem likely to be easy or conclusive. We therefore determined to adopt a modification of LÜDTGE's method, and to balance against each other two films of different degrees of tenuity.

It was essential that our method of experiment should be such as would enable us readily to produce black films. We have found that this end is most easily obtained with cylinders. In most of our experiments, therefore, one of the films has been

cylindrical. The other, though of the same mean curvature, has been of various forms as convenience suggested.

The general idea of the apparatus is represented in the accompanying diagram. Bubbles are blown on the inverted cups A and B, until they adhere to the rings C and D. They can be cut off from or put into communication with each other or with the external air by means of stopcocks. By withdrawing or forcing in air they can be made to assume any required form.



This arrangement, like LÜDTGE's, possesses the great advantage that the indications sought for are not complicated by changes in the temperature or in the pressure of the atmosphere. These may cause the diameter of the films to increase or contract, but will affect them equally. A difference in surface tension would be indicated by a difference in their magnitudes.

The two simplest surfaces which can be formed with such an apparatus are the cylinder and the sphere. The fact that if they change their shape they become unduloids or nodoids has, of course, made the mathematical theory of the experiment more complicated. The justification of the particular method adopted is, however, to be found in the fact that liquid cylinders offer such much greater facilities for obtaining a completely black film than any other geometrical form, that every other consideration has been made to give way to this, which, for reasons to be detailed hereafter, we conceive to be of prime importance.

In the following pages we shall deduce from the assumption that films of different thicknesses have different surface tensions the conclusions to which it leads, and shall then submit to the test of experiment the question as to whether these conclusions are in accord with the observed facts.

On the Sensitiveness of the Methods of measuring Changes in the Surface Tension of a Film.

All the methods which we have described measure the change in the surface tension of the film indirectly by a change in length. In PLATEAU's experiment the movement of the liquid in the manometer, in the others the alterations in the sagittæ or

diameters of the films, furnish the required indication. Let an increment dT in the surface tension T produce an alteration dL in this length. The fraction TdL/dT may then be conveniently taken as a measure of the *sensitiveness* of the experiment. The quantities dL and dT may be finite or infinitely small. In the latter case the expression gives the *limiting sensitiveness* for an infinitely small variation in T.

If the increment dL observed in any experiment is divided by the sensitiveness, the quotient is the fraction of itself by which the tension has altered. The relative values of different methods of measuring dT is best obtained by comparing the sensitivenesses for equal values of dT/T.

It is easy to see that, as has been stated, experiments can be arranged in which the sensitiveness is infinite, *i.e.*, in which the equilibrium is unstable, so that an infinitely small change in T would produce a finite change in L. Two bubbles, each greater than a hemisphere, and the interiors of which were connected, would form such an arrangement. Great sensitiveness may be obtained by approaching the conditions of instability as nearly as may be practically convenient.

In the following calculations we have treated the air as incompressible by the small forces due to slight changes in the surface tensions of the films by which it is enclosed. In LÜDTGE'S experiment, if the volume of the tube is large compared with that of the spherical segments, it might be necessary to allow for variations in the density of the air. If the apparatus be properly designed, the necessity for this complication is avoided. The surface tension of water is about 81 dynes per linear cm. A water bubble 1 cm. in radius would therefore exert upon the enclosed air a pressure of 324 dynes per sq. cm. The changes of surface tension to be studied are but small fractions of the whole, and as the atmospheric pressure is 10^6 dynes per sq. cm. it may safely be said that they could not alter the volume of the enclosed air by more than a few millionths of the total space occupied.

In experiments in which the equilibrium is stable the change of form produced by the change in surface tension is generally such as to reduce the change of pressure which would otherwise be produced. The contraction or expansion of the enclosed air would also affect both films, and its effects would be the same as those of a small change in temperature.

Limiting Sensitiveness of PLATEAU'S *Experiment.*

Let V be the volume of the bubble, r_1 the radius of the manometer tube, and l the distance through which the liquid is depressed. Let r_2 be the radius of the tube from which the bubble hangs, R the radius, and nr_2 the sagitta of the bubble. Then

$$\mathrm{R}=r_2(1+n^2)/2n, \quad \mathrm{V}=\pi r_2^3 n(n^2+3)/6.$$

Also, if $v=\pi r_1^2 l$, we must have

$$\mathrm{V}+v=\text{a constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If p be the pressure due to the film, T the tension, and σ the density of the liquid in the manometer,

$$p=4\mathrm{T}/\mathrm{R}=2\sigma gl.$$

Hence

$$\mathrm{R}=\frac{2\mathrm{T}}{\sigma gl}=\frac{r_2(1+n^2)}{2n} \quad \ldots\ldots\ldots\ldots \quad (2)$$

From these equations we get

$$2\mathrm{T}dl/d\mathrm{T}=\frac{2nr_2^3(1+n^2)^2}{\frac{\sigma g}{4\mathrm{T}}r_2^4(1+n^2)^3-2r_1^2(n^2-1)}.$$

This expression gives the sensitiveness, since the total movement (down and up) of the liquid in the manometer is $2dl$.

In PLATEAU'S experiment $\mathrm{T}=60$ dynes (about) per linear cm., $r_1=0{\cdot}5$ cm., $r_2=0{\cdot}25$ cm., $\sigma=1$.

Substituting these values in the denominator, and equating to zero, we get

$$(1+n^2)^3-31{\cdot}315(n^2-1)=0.$$

This equation in n^2 has one negative and two positive roots. The latter are

$$n^2=1{\cdot}497 \quad \text{and} \quad n^2=2{\cdot}912,$$

which give

$$n=1{\cdot}224 \quad \text{and} \quad n=1{\cdot}707.$$

When n has these values the sensitiveness is infinite. The meaning of this result is as follows. If a bubble be blown at the end of a tube connected with a manometer the bubble will increase, and the liquid in the manometer will become more depressed until the film is a hemisphere. Thereafter the pressure will diminish, and though V will increase v will become less. If the radius of the manometer tube is such that the decrement dv is greater than the increment $d\mathrm{V}$, an increase in the size of the bubble would necessarily be attended with a decrease in the total volume of enclosed air. Under these conditions the equilibrium would be unstable, and the system would transform itself into a form in which the volume of the bubble was such that $d\mathrm{V}+dv$ was positive. If the bubble were originally on the borders of instability a small change in T would produce a finite change in the position of the liquid in the manometer, and thus the sensitiveness would be infinite. In the case of the apparatus used by PLATEAU the whole phenomenon must have occurred within narrow limits and would easily escape attention.

The following table explains itself, and illustrates the fact that as the sagitta increases $\mathrm{V}+v$ has successively a maximum and a minimum value.

TABLE I.

R	n	$2Tdl/dT$	l	V/π	v/π	(V+v)/π
cm.			cm.			
2	$8+\sqrt{63}$	0·12	0·06116	10·6660	0·01529	10·6813
1	$4+\sqrt{15}$	0·25	0·12232	1·3323	0·03058	1·3629
0·5	$2+\sqrt{3}$	0·56	0·24464	0·16452	0·06116	0·22568
0·286	1·707	$+\infty-$	0·42771	0·02629	0·10693	0·13322
0·271	1·500		0·45128	0·02051	0·11282	0·13333
0·256	1·224	$-\infty+$	0·47783	0·01433	0·11946	0·13379
0·250	1·000	0·98	0·48930	0·01042	0·12232	0·13274
0·5	$2-\sqrt{3}$	0·02	0·24464	0·00799	0·06116	0·06916
∞	0	0	0	0	0	0

The importance of this table for our present purpose depends, however, chiefly on the figures in the first three rows. The radii are chosen to correspond with those of bubbles which PLATEAU used. As he does not mention the radius of the bubble on which he based his estimate of the superior limit of the radius of molecular attraction, it is impossible to estimate exactly the sensitiveness of his experiment. Probably it did not exceed 0·56.

Limiting Sensitiveness of LÜDTGE'S *Experiment.*

The expression for this quantity has been calculated by VAN DER MENSBRUGGHE.* Let the radius of the tube at the extremities of which the films are formed be r, and let the sagittæ of the two spherical segments be nr and $n'r$ respectively. Then from equation (2)

$$p=\frac{8Tn}{r(1+n^2)}=\frac{8T'n'}{r(1+n'^2)}.$$

If the tensions alter, the change in the pressure exerted on the internal air must be the same for both films, so that

$$\frac{n\,dT}{1+n^2}+\frac{T(1-n^2)}{(1+n^2)^2}\,dn=\frac{n'\,dT'}{1+n'^2}+\frac{T'(1-n'^2)}{(1+n'^2)^2}\,dn'.$$

If both films had at first the same surface tension, $T=T'$ and $n=n'$. Also, since the total volume is constant, $dn=-dn'$. Hence

$$2Tr\,dn/(dT-dT')=-nr(1+n^2)/(1-n^2).$$

In the following table we have taken $r=1$, so that the sensitiveness for a tube of any given radius is obtained by multiplying the figures in the second column by the radius expressed in centimetres.

* 'Bruxelles, Acad. Sci. Bull.,' vol. 30, 1870, pp. 322–332.

If one of the films be maintained in a constant state the increment of T for that film is zero.

TABLE II. ($n=1.$)

n	$2Tds/dT$
0	0
0·25	0·2833
0·50	0·8333
0·75	2·6786
1·00	∞

If $n>1$ the arrangement is unstable.

VAN DER MENSBRUGGHE'S experiments described above are, from the mathematical point of view, particular cases of LUDTGE'S method, and do not require separate discussion.

On the Limiting Sensitiveness of the Experiments described in this Paper.

In most of the experiments hereafter described we have balanced two cylinders against each other. Such figures deform into unduloids, the equations to which could be expressed by the approximate relation

$$y=a+c\sin\frac{x}{a}.$$

We have also balanced spheres against spheres, and a sphere against a cylinder, and it therefore seemed better to deduce general expressions from which the sensitiveness could in each case be calculated.

BEER* has shown that, if the axis of x be the axis of revolution, the equation to the curve which generates a surface of revolution of constant curvature is

$$dx=\pm\frac{y^2\pm\alpha\beta}{\sqrt{(\alpha^2-y^2)(y^2-\beta^2)}}\,dy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

In this expression α and β are both taken as positive quantities, and the curve is an unduloid or nodoid according as the positive or negative sign is taken before the product $\alpha\beta$. We shall omit the alternative sign and, taking α positive and always numerically $>\beta$, shall assume that β may be either positive or negative. We shall also assume that normally the origin lies on a maximum ordinate. In that case dx/dy is negative for positive values of y which correspond to values of x less than that for which the first minimum value of y occurs if β is positive, and less than that for which $y^2+\alpha\beta=0$ if β is negative. In most of the cases with which we have had to deal

* 'Tractatus de theoriâ mathematicâ phænomenorum in liquidis actioni gravitatis detractis observatorum,' Bonn, 1857. See also PLATEAU, 'Statique des Liquides,' vol. 1, p. 139.

there has only been one maximum or minimum value of y in the part of the curves under consideration.

Again, when $y^2 < -\alpha\beta$, which can only be the case when β is negative, the curve is convex towards the axis of revolution, and the pressure is exerted outwards. This is also a case we have not practically realized, so that it will be convenient to take the square root as affected with the negative sign.

If then, with BEER, we assume

$$y^2 = \beta^2 \sin^2\phi + \alpha^2 \cos^2\phi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$k^2 = (\alpha^2 - \beta^2)/\alpha^2 \text{ and } \Delta = \sqrt{1 - k^2 \sin^2\phi},$$

we get from the differential equation

$$x = \alpha E + \beta F,$$

where F and E are elliptic integrals of the first and second kinds respectively.

As the lower limit of the integrals is zero, the origin is assumed to lie on a maximum ordinate, for $y = \pm a$ when $\phi = 0$.

When it is convenient to transfer the origin to a minimum ordinate, ϕ is $> \pi/2$ for positive values of x; and, taking this as the upper limit, we may write

$$x = \alpha(E - E_1) + \beta(F - F_1),$$

where F_1 and E_1 are the complete integrals taken between the limits $\pi/2$ and 0.

Again, if V is the volume included between the plane through the origin perpendicular to the axis of x and any other parallel plane,

$$V/\pi = \left(\tfrac{2}{3}\alpha^3 + \beta\alpha^2 + \tfrac{2}{3}\alpha\beta^2\right)E - \tfrac{1}{3}\alpha\beta^2 F + \tfrac{1}{6}\alpha(\alpha^2 - \beta^2)\Delta \sin 2\phi \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This expression has been given by BEER. If the origin lies on a minimum ordinate we must, as before, write $E - E_1$ and $F - F_1$ for E and F respectively, remembering that ϕ is $> \pi/2$.

Let us suppose that two rings of radius Y are arranged with their planes perpendicular to the straight line on which their centres lie, the distance between them being 2X.

Let the rings be joined by a liquid film, the maximum or minimum ordinate of which lies half-way between them. It is convenient to speak of this as the *principal ordinate*. Let ϕ_1 be the value of ϕ which corresponds to Y, and Δ_1 the value of Δ when $\phi = \phi_1$.

If then the form of the film be altered, but so that it remains a surface of revolution of constant curvature, the quantities α, β, and ϕ_1 must vary subject to the conditions that X and Y are constant.

Hence, differentiating the expressions for X and Y given above, we have

$$d\alpha\left\{\mathrm{E}-\frac{\beta^2}{\alpha^2}\int\frac{\sin^2\phi}{\Delta}d\phi+\frac{\beta^2}{\alpha^2}\int\frac{\sin^2\phi}{\Delta^3}d\phi\right\}+d\beta\left\{\mathrm{F}+\frac{\beta}{\alpha}\int\frac{\sin^2\phi}{\Delta}d\phi-\frac{\beta^2}{\alpha^2}\int\frac{\sin^2\phi}{\Delta^3}d\phi\right\}$$

$$+d\phi_1\left\{\alpha\Delta_1+\frac{\beta}{\Delta_1}\right\}=0, \quad \ldots\ldots\ldots\ldots \quad (6)$$

and

$$\beta\sin^2\phi_1 d\beta+\alpha\cos^2\phi_1 d\alpha-(\alpha^2-\beta^2)\sin\phi_1\cos\phi_1 d\phi_1=0 \ldots\ldots \quad (7)$$

The limits of the integrals are ϕ_1 and 0, or ϕ_1 and $\pi/2$, according as the origin lies on a maximum or minimum ordinate. In the latter case $\mathrm{E}-\mathrm{E}_1$ and $\mathrm{F}-\mathrm{F}_1$ must be written for E and F.

But

$$\int\frac{\sin^2\phi}{\Delta^3}d\phi=\frac{1}{k^2}\int\frac{1-(1-k^2\sin^2\phi)}{\Delta^3}d\phi=\frac{1}{k^2}\int\frac{d\phi}{\Delta^3}-\frac{\mathrm{F}}{k^2},$$

$$\int\frac{\sin^2\phi}{\Delta}d\phi=\frac{\mathrm{F}-\mathrm{E}}{k^2}, \text{ and } \int\frac{d\phi}{\Delta^3}=\frac{\alpha^2}{\beta^2}\left\{\mathrm{E}-\frac{k^2\sin\phi_1\cos\phi_1}{\Delta_1}\right\}.$$

Hence, substituting from these expressions, and eliminating $d\phi_1$ between (6) and (7),

$$d\alpha\{\beta^2\mathrm{F}-\alpha^2\mathrm{E}-\alpha^2\Delta_1\cot\phi_1\}+\alpha^2 d\beta(\mathrm{E}-\mathrm{F}-\Delta_1\tan\phi_1)=0 \ldots\ldots \quad (8)$$

Again, differentiating the expression for V given in equation (5), we get in the general case, when $d\alpha$, $d\beta$, and $d\phi_1$ are independent,

$$dV/\pi=d\alpha\left\{(2\alpha^2+2\alpha\beta+\frac{2}{3}\beta^2)\mathrm{E}-\frac{1}{3}\beta^2\mathrm{F}\right.$$

$$+\frac{1}{6}(3\alpha^2-\beta^2)\Delta_1\sin 2\phi_1-\frac{\beta^2}{\alpha^2}\left(\frac{2}{3}\alpha^2+\alpha\beta+\frac{2}{3}\beta^2\right)\int\frac{\sin^2\phi}{\Delta}d\phi$$

$$\left.-\frac{1}{3}\frac{\beta^4}{\alpha^2}\int\frac{\sin^2\phi}{\Delta^3}d\phi-\frac{1}{6}\frac{\beta^2(\alpha^2-\beta^2)}{\alpha^2\Delta_1}\sin^2\phi_1\sin 2\phi_1\right\}$$

$$+d\beta\left\{(\alpha^2+\frac{4}{3}\alpha\beta)\mathrm{E}-\frac{2}{3}\alpha\beta\mathrm{F}-\frac{\alpha\beta}{3}\Delta_1\sin 2\phi_1\right.$$

$$+\left(\frac{2}{3}\alpha^2+\alpha\beta+\frac{2}{3}\beta^2\right)\frac{\beta}{\alpha}\int\frac{\sin^2\phi}{\Delta}d\phi+\frac{1}{3}\frac{\beta^3}{\alpha}\int\frac{\sin^2\phi}{\Delta^3}d\phi$$

$$\left.+\frac{1}{6}\frac{\beta(\alpha^2-\beta^2)}{\alpha\Delta_1}\sin^2\phi_1\sin 2\phi_1\right\}$$

$$+d\phi_1\left\{\left(\frac{2}{3}\alpha^3+\alpha^2\beta+\frac{2}{3}\alpha\beta^2\right)\Delta_1-\frac{1}{3}\frac{\alpha\beta^2}{\Delta_1}\right.$$

$$\left.+\frac{1}{6}\alpha(\alpha^2-\beta^2)\left(2\Delta_1\cos 2\phi_1-\frac{\alpha^2-\beta^2}{\alpha^2}\frac{\sin^2 2\phi_1}{2\Delta_1}\right)\right\}.$$

Whence, substituting as above for the integrals, we get

$$dV/\pi = d\alpha\left[\frac{\alpha}{\alpha-\beta}\{(2\alpha^2-\beta^2)E-F\beta^2\}+\alpha^2\Delta_1\sin\phi_1\cos\phi_1\right]$$
$$+d\beta\left[\frac{\alpha}{\alpha-\beta}\{(\alpha^2-2\beta^2)E+F\beta^2\}-\alpha\beta\Delta_1\sin\phi_1\cos\phi_1\right]$$
$$+d\phi_1\Delta_1\alpha^2(\alpha+\beta)\left(1-\frac{\alpha-\beta}{\alpha}\sin^2\phi_1\right).$$

If Y is constant this becomes from (7)

$$dV/\pi = \frac{\alpha}{\alpha-\beta}[d\alpha\{(2\alpha^2-\beta^2)E-F\beta^2+\Delta_1\alpha^2\cot\phi_1\}$$
$$+d\beta\{(\alpha^2-2\beta^2)E+F\beta^2+\Delta_1\beta^2\tan\phi_1\}]. \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and, if both X and Y are constant, we get from (8)

$$dV/\pi = d\alpha\times\frac{\alpha+\beta}{\alpha}\{(2E\alpha^2-F\beta^2+\Delta_1\alpha^2\cot\phi_1)(2E-F-\Delta_1\tan\phi_1)$$
$$-E^2\alpha^2\}\div(E-F-\Delta_1\tan\phi_1)$$
$$=-d\beta\times\alpha(\alpha+\beta)\{(2E\alpha^2-F\beta^2+\Delta_1\alpha^2\cot\phi_1)(2E-F-\Delta_1\tan\phi_1)$$
$$-E^2\alpha^2\}\div(F\beta^2-E\alpha^2-\alpha^2\Delta_1\cot\phi_1). \quad . \quad . \quad . \quad . \quad . \quad (10)$$

These equations applied to the problem under discussion enable us in the first place to determine what arrangements of two films in communication will be stable.

Maxwell * has shown that a slight bulge in a cylinder will increase or decrease the pressure according as the cylinder is shorter or longer than half the circumference of the rings to which it is attached. Hence two cylinders of equal length and radius will be unstable if their lengths exceed the limit thus laid down. For, if one cylinder expands and the other contracts a little, the pressure in the first will fall off, and that in the second will increase. Hence, the tendency will be to depart still further from the position of equilibrium. We found (as was to be expected) that it was impossible to keep the cylinders steady if we approached too near to the limit of stability. The theoretical limit is $X=1{\cdot}57Y$, but in practice X should not be $>1{\cdot}25Y$. Hence, in general, the length of the cylinders has been one-and-a-quarter diameters.

It is easy to apply the formulæ to the case of any two similar surfaces of constant curvature.

The unduloid and nodoid are the roulettes of the foci of an ellipse and hyperbola respectively. Lindelöf has shown that the sum of the principal curvatures at any point in the surfaces of revolution described by these curves about their axes is the

* 'Encyclop. Brit.,' Art. "Capillary Action."

same as the curvature of the circle the diameter of which is the major axis of the generating ellipse in the one case, and the transverse axis of the hyperbola in the other.

Now, since α and β are the maximum and minimum ordinates, it is evident that in the ordinary notation

$$\alpha = a(1+e), \quad \beta = a(1-e).$$

Hence

$$\frac{1}{R}+\frac{1}{R'}=\frac{1}{a}=\frac{2}{\alpha+\beta}.$$

If then α and β alter, the increment in the curvature is

$$-2(d\alpha+d\beta)/(\alpha+\beta)^2.$$

If the variations take place subject to the conditions that X and Y are constant, and if (8) be written in the form

$$M d\alpha + \alpha^2 N d\beta = 0,$$

we have

$$\frac{(\alpha+\beta)^2}{2} d\left(\frac{1}{R}+\frac{1}{R'}\right) = -\frac{\alpha^2 N - M}{\alpha^2 N} d\alpha = \frac{\alpha^2 N - M}{M} d\beta. \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In the case of a spherical film

$$\beta = 0, \ k^2 = 1, \ \Delta_1 = \cos\phi_1, \ E = \sin\phi_1, \ F = \log_e \tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right).$$

Hence

$$M = -\alpha^2/\sin\phi_1, \ N = -\log_e \tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right),$$

and from (11)

$$d\left(\frac{1}{R}+\frac{1}{R'}\right) = \frac{2}{a^3} \frac{\operatorname{cosec}\phi_1 - \log_e \tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right)}{\log_e \tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right)} . d\alpha.$$

The denominator of this expression is positive, and possible for positive values of $\phi_1 < \pi/2$. The numerator may be shown by trial to be positive for values of $\phi_1 < 56^\circ\ 28'$. When this value is reached it becomes zero and changes sign.

Now in the case of a sphere

$$X = \alpha \sin\phi_1; \quad \text{and, since } \alpha^2 = X^2 + Y^2, \quad Y = X \cot\phi_1.$$

Therefore, if $\phi_1 = 56^\circ\ 28'$,

$$2X = 3{\cdot}0179\,Y = 0{\cdot}4803 \times 2\pi Y,$$

and

$$\alpha = 1{\cdot}8102\,Y.$$

Thus a slight increase in the principal ordinate of a spherical film converts it into a nodoid the mean curvature of which is greater or less than that of the sphere according as the distance between the rings is less or greater than the circumference of a ring $\times 0{\cdot}4803$.

If this distance between the rings is exceeded, an arrangement consisting of two spherical films, the interiors of which are in communication, will be unstable.

We have next to investigate the sensitiveness of our experiment.

If two films, which are in communication, are in equilibrium, the pressure exerted on the internal air must be the same. Hence

$$p=\frac{4T}{\alpha+\beta}=\frac{4T'}{\alpha'+\beta'}.$$

If the equilibrium is maintained after a change in any or all of these quantities,

$$\frac{dT}{\alpha+\beta}-\frac{T(d\alpha+d\beta)}{(\alpha+\beta)^2}=\frac{dT'}{\alpha'+\beta'}-\frac{T'(d\alpha'+d\beta')}{(\alpha'+\beta')^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The six quantities $d\alpha$, $d\beta$, $d\phi_1$, and $d\alpha'$, $d\beta'$, $d\phi_1'$ can be determined from the six conditions that the values of X and Y shall in each case remain unaltered, that the changes of volume shall be equal in magnitude and opposite in sign, and that the change of pressure shall be the same for each film.

If the two films are similar the condition $dV=-dV'$ leads in general to the result $d\alpha=-d\alpha'$ and $d\beta=-d\beta'$.

In the case of two cylinders an expansion of the one is accompanied by a contraction of the other. If both were initially similar in all respects it is easy to show that $d\alpha=-d\beta'$, where the latter symbol refers to the film which has undergone a contraction.

For, since the lengths of the cylinders were the same, and in the case of the cylinder

$$\alpha=\beta=Y \quad \text{and} \quad E=F=\phi,$$

we have

$$X=\alpha E+\beta F=2Y\phi_1=2Y\left(\phi_1'-\frac{\pi}{2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and from (10), since $dV=-dV'$,

$$d\alpha\{\phi_1(\cot\phi_1-\tan\phi_1)-1\}/\tan\phi_1=d\beta'\left\{\left(\phi_1'-\frac{\pi}{2}\right)(\cot\phi_1'-\tan\phi_1')-1\right\}/\cot\phi_1',$$

which, by (13), gives $d\alpha=-d\beta'$, and in like manner $d\alpha'=-d\beta$.

Hence, from (12), if we put $dT'=0$,

$$2(d\alpha+d\beta)T=(\alpha+\beta)dT=2Y\,dT,$$

and from (8)

$$\frac{d\alpha}{\tan\phi_1}=\frac{d\beta}{-\cot\phi_1}=\frac{d\alpha+d\beta}{-2\cot 2\phi_1},$$

so that

$$4d\alpha T/dT=-4Y\sin^2\phi_1/\cos 2\phi_1.$$

This expression gives the limiting sensitiveness, since the total alteration in the lengths of the two diameters is $2(d\alpha-d\beta')=4d\alpha$.

If the films were initially spheres and the arrangements were in all respects the same for both we have $d\alpha=-d\alpha'$, $d\beta=-d\beta'$. Also, substituting the proper values of E and F, given above in equation (8),

$$\frac{d\alpha}{\log\tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right)}=-\frac{d\beta}{\operatorname{cosec}\phi_1}=\frac{d\alpha+d\beta}{\log\tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right)-\operatorname{cosec}\phi_1},$$

whence, and from (12), if we put $dT'=0$,

$$4d\alpha T/dT=\frac{2\alpha\log\tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right)}{\log\tan\left(\frac{\pi}{4}+\frac{\phi_1}{2}\right)-\operatorname{cosec}\phi_1}.$$

By the aid of these expressions we have calculated the following table. The radius of the rings is taken as the unit of length. The value of X is therefore numerically equal to the distance between the rings in terms of their diameters. The values of ϕ_1 have been so chosen as to give approximately equal values of X in the parts of the table which refer to the cylinders and spheres respectively.

TABLE III.

Two cylinders (Y = 1).				Two spheres (Y = 1).			
ϕ_1 expressed in		$X=2\phi_1$.	$4d\alpha T/dT$.	ϕ_1 expressed in degrees.	$X=\tan\phi_1$.	$\alpha=\sec\phi_1$.	$4d\alpha T/dT$.
Circular measure.	Degrees.						
1/8	7°16	0·25	0·064	14° 0′	0·249	1·0306	0·13
1/4	14·32	0·50	0·279	27 0	0·509	1·1223	0·64
3/8	21·49	0·75	0·734	37 0	0·754	1·2521	1·80
1/2	28·65	1·00	1·700	45 0	1·000	1·4142	4·68
5/8	35·81	1·25	4·343	51 21	1·250	1·6011	14·24
π/4	45·00	π/2	∞	56 28	0·961 × π/2	1·8102	∞

If a sphere and cylinder are in communication, the radius of the sphere must be twice that of the cylinder if the films are to be in equilibrium when they have the same surface tension. The following results apply to a case in which the radii of the rings which support the sphere are 1·6 times greater than that of those to which the cylinder is attached. The symbols with dashes affixed refer to the sphere.

$$Y=1, \quad X=1{\cdot}25, \quad \phi_1=0{\cdot}625=35^\circ\ 48',$$
$$\alpha=\beta=1, \quad E=F=\phi_1, \quad \Delta_1=1,$$
$$Y'=1{\cdot}6, \quad X'=1{\cdot}25, \quad \alpha'=2, \quad \beta'=0,$$
$$\operatorname{Sec}\phi_1'=\alpha'/Y'=1{\cdot}25, \quad \therefore \phi_1'=36^\circ\ 52',$$
$$E=\sin\phi_1'=0{\cdot}59995, \quad F=\log_e\tan\left(\frac{\pi}{4}+\frac{\phi_1'}{2}\right)=0{\cdot}69307,$$
$$\Delta_1'=\cos\phi_1'.$$

Since $dV=-dV'$, we get from equation (10)

$$d\alpha\{1-\phi_1(\cot\phi_1-\tan\phi_1)\}/\tan\phi_1$$
$$=2d\alpha'\left\{1-(\sin\phi_1'+\operatorname{cosec}\phi_1')\log_e\tan\left(\frac{\pi}{4}+\frac{\phi_1'}{2}\right)\right\}/\log_e\tan\left(\frac{\pi}{4}+\frac{\phi_1'}{2}\right),$$

or

$$d\alpha=-2{\cdot}0343d\alpha'.$$

In like manner, from equation (8)

$$d\alpha+d\beta=-0{\cdot}9223d\alpha,$$
$$d\alpha'+d\beta'=-1{\cdot}405d\alpha'.$$

If, as before, $dT'=0$, we get from (12)

$$dT=\frac{T}{\alpha+\beta}\times 3{\cdot}2812d\alpha'=-\frac{T}{\alpha+\beta}\times 1{\cdot}6129d\alpha,$$

or, since $\alpha+\beta=2$,

$$2(d\alpha-d\alpha')T/dT=-3{\cdot}699, \quad \ldots\ldots\ldots \quad (14)$$

which gives the sensitiveness in this case.

The general result of this investigation is to prove that our apparatus was capable of being used so as to be far more sensitive than PLATEAU's, and probably than LÜDTGE's, as used by him. The latter does not mention the diameter of the tube he employed; but if we assume that PLATEAU's bubble had a radius of 0·5 cm. (which seems to have been the smallest he used), and that LÜDTGE's tube had a radius of 1 cm., and that the sagittæ of the spherical segments were half the radius, we find

from Tables I. and II. that the limiting sensitiveness of the experiments was 0·56 in the one case and 0·83 in the other. MENSBRUGGHE purposely used segments which differed but little from hemispheres, and may therefore have attained very great sensitiveness. The highest value to be deduced from the experiments he describes is 11·5.

We used two sets of rings the diameters of which were 2·08 and 3·25 cm. respectively. When the two cylinders were employed the sensitivenesses (see Table III.) were

$$4{\cdot}34 \times 1{\cdot}04 = 4{\cdot}51 \text{ and } 4{\cdot}34 \times 1{\cdot}62 = 7{\cdot}03,$$

the distance between the rings being in each case 1·25 times the diameter.

In like manner when two spheres were used the values were 14·81 with the small rings and 23·06 with the large ones. Such high sensitiveness was found in practice inconvenient, and this arrangement was not often employed.

We have for simplicity calculated the sensitiveness of the sphere and cylinder on the assumption that the ratio of the diameter of the rings used was 1·6. It was really 3·25/2·08 = 1·56. We may, however, without serious error, assume the value 3·7 to be correct.

If, then, we credit PLATEAU and VAN DER MENSBRUGGHE with the most favourable arrangements described by them, and make the above probable assumptions for LÜDTGE, we get the following Table of sensitiveness :—

PLATEAU	0·56	MENSBRUGGHE	11·5
LÜDTGE	0·83 (?)	R. & R. Two spheres—	
R. & R. Sphere and cylinder	3·7	Small rings	14·81
" Two cylinders—		Large rings	23·06
Small rings . .	4·51		
Large rings . .	7·03		

These numbers express the relative merits of the different experiments regarded as null methods. They are inversely proportional to the differences of surface tension which could in each case exist without detection. It is, however, proper to point out that the more sensitive methods are also more liable to accidental disturbances of various kinds, and do not therefore present so great an advantage as the theory would indicate.

This remark does not apply to a method which we employed, and by which the sensitiveness is doubled, viz., keeping each film thick in turn. By this means the indications of a difference of surface tension were twice as great as they would otherwise have been, and the figures given above may be multiplied by two.

Description of the Apparatus.

The film box was rectangular in shape, 20 cm. long, 10 cm. broad, and 17 cm. high. It was made entirely of thick plate glass, with the exception of the cover, which was of ebonite and about half an inch thick. The cover fitted accurately into its place, and to it were attached the thermometers, hygrometer, supports for the film, &c., so that all the apparatus connected with the films could be removed from the box by lifting the cover.

The supports for the films are shown at A, B, A′, B′ (Plate 33, fig. 2). A and A′ are cylindrical platinum cups, 20·8 mm. in diameter, with carefully turned edges. They are screwed to brass tubes C and C′, the upper portions of which carry rackwork and are moved up and down the larger tubes D, D′, by the pinions seen at E, E′. The supports for the pinions, as well as the tubes in which the racks slide, are attached to brass discs F, F′ (figs. 1 and 2), the lower faces of which are ground plane, and rest, when they are in their places, in good contact with the top of the ebonite cover. The holes in the cover, closed by these discs, are large enough to allow the upper cups to pass freely through, so that the latter may be introduced and adjusted to their proper positions after the cover has been put in its place.

The cylindrical rings B, B′, are the lower supports of the films. They are about 5 mm. deep, of the same diameter as the upper cups, and are screwed to platinum wires bent at right angles and soldered to brass rods H, H′.

These rods pass through tubes K, K′, fixed to the ebonite cover. By the ebonite buttons a, a', the platinum rings can be turned on one side in such a way as to allow the upper cups to be brought into contact with the soap solution, with which the floor of the box is covered to a depth of 2·5 mm.

Both the upper and lower supports for the films can be unscrewed, and others of different size or material substituted for them. We have used (as already stated) platinum rings of 20·8 mm. and 32·5 mm. in diameter respectively, and occasionally glass rings 32·5 mm. in diameter.

In order to maintain one or other of the two films thick, or to thicken them after they had become thin, two different methods were adopted. For carrying out the first method, strips of linen were cut about 16 mm. broad, and long enough to pass one and a half times round one of the upper cups. Each strip was folded so as to have half its original width, and was wrapped round the cup in such a way that its edge projected above it, and formed a cavity into which liquid could be poured. It was secured in its place by cotton thread. Liquid was supplied to the cavity formed in this way by the bent glass tubes L, L′, which were open at the top, and were drawn out to conical terminations below. They were supported by corks placed in holes cut in the cover, and could be turned round or moved up and down, so as to bring their points just over the cavity formed by the linen wrappers. By pouring a little of the soap solution down the tube, the linen could be made to supply liquid at a greater or less

rate to the film to prevent its thinning. It is convenient to describe this operation, to which we shall have frequently to refer, as *flooding* the film. As it was important that the total quantity of liquid on the floor of the box should not increase, as this would have involved a slow change in the forms of the films, the liquid used in flooding was always taken from the box itself. Two holes R, R′ (fig. 1), were provided at the back of the cover for the insertion of a small pipette, and liquid was drawn up to a fixed mark on the tube so as to supply on each occasion the same amount of liquid to the film. By this means also we ensured, as far as possible, that the temperature of the liquid used in flooding should be the same as that of the film itself.

The second method which was used to prevent or retard thinning consisted in passing an electric current up the film. The binding screws for attaching the leads are seen at *b*, *b′*, *c*, *c′* (fig. 2). The electromotive force generally used was about 45 volts, and a reflecting galvanometer in the circuit indicated the strength of the current. An account of the effect of an electric current in retarding or accelerating the thinning of a liquid film has been already published by us.*

The wires M, M′, pass through perforations in two discs of sheet india-rubber, and can be moved in any direction. If the pointed ends are wetted with the solution, they may be used to detach and carry away small bubbles which sometimes cling to the supports, and which might, if allowed to remain, cause irregular thinning.

A glass partition N, about 3 inches broad, is attached to the cover of the box, and extends to within half an inch of the liquid at the bottom. It was designed to prevent the spray caused by the bursting of one of the films from injuring the other.

Two thermometers T, T′, were fitted by corks into holes in the lid near the back of the box, one at each side, and a hair hygrometer P served to indicate any change which might occur in the hygrometric state of the air.

To ensure constancy of temperature, the film box was enclosed in a glass tank filled with water, the dimensions of which were such that there were 3 inches of water on every side of the box except at the top and bottom.

The experiments described below were all made under such conditions that the variations of temperature and hygrometric state were within the limits which previous experiment had proved to be necessary to maintain the films in a constant state. We have not therefore thought it necessary to give the thermometer and hygrometer readings.

The lengths of the films were measured by a cathetometer placed at a distance of 11 feet from the box. Their diameters were measured by a small telescope moving along an optical bench of Prof. CLIFTON'S pattern. The position of the telescope could be read by a vernier to one or two hundredths of a millimetre. The optical bench rested on a solid stone table, and was about 8 feet from the film box. Usually a narrow strip of millboard was placed behind each film, and the edge of the film was thus viewed by reflected light. When the film was thick enough to show colours, the readings

* 'Phys. Soc. Proc.,' vol. 6, p. 357.

could be made with considerable precision. The difficulty was greater when it became black, but even in this case the accuracy attainable was greater than might have been expected considering the small amount of light reflected.

The liquid used was, unless the contrary is stated, made with Potash Soap (Brit. Pharm.). Its composition has been described by us in the paper on the influence of the electric current on the rate of thinning of films previously referred to. Its refractive index was 1·3366 for the sodium line D.

As nothing in this paper depends on the very accurate measurement of the thickness of the films, we estimated it approximately by noting the colour near the edge, so that the light was nearly at grazing incidence. The thickness corresponding to any colour may therefore be taken as one and a half times greater than if the incidence had been normal. The error made in this assumption is probably not more than 2 per cent. Inasmuch also as we had to deal with much thinner films than those used in our experiments on the electrical conductivity of the films, the difficulty of estimating the colour accurately was much greater.

The thickness of the film indicated by a certain colour is thus 1·5/1·3366=9/8, very nearly, of that of a plate of air showing the same colour by means of light falling upon it at normal incidence.

In the following Table, Column I. contains the name of the colour, Column II. the symbol of the middle of the colour on the system previously adopted by us. In these symbols the letter is the first letter of the name of the colour, the first figure in the brackets indicates the order. The colour is divided into ten parts, numbered 0, 1, 2, &c., in the order of increasing thickness, and the second number indicates the part of the colour displayed. In the third column are the thicknesses expressed in millionths of a millimetre, corresponding to the colours when displayed in a thin plate of air at normal incidence. In the first order these numbers are (with the exception of the red and black) taken from NEWTON'S table. They are probably only very roughly accurate. The other orders are taken from our own table.* They are only approximate below the blue of the second order. Above that colour they are probably correct to 1 per cent. In the fourth column are the same numbers as in Column III., each increased by one-eighth. They give the thickness of the film corresponding to a given colour when mentioned in the present paper.

* 'Phil. Trans.,' vol. 172 (Part 2, 1881), p. 456.

TABLE IV.

I.	II.	III.	IV.
First order—			
Black	*b*	..	12
Grey	G (1, 5)	(?)	(?)
White	W (1, 5)	131	147
Yellow	Y (1, 5)	178	200
Orange	O (1, 5)	200	225
Red	R (1, 5)	284	320
Second order—			
Violet	V (2, 5)	305	343
Blue	B (2, 5)	353	397
Green	G (2, 5)	409	460
Yellow	Y (2, 5)	454	511
Orange	O (2, 5)	491	552
Red	R (2, 5)	522	587
Third order—			
Purple	P (3, 5)	559	629
Blue	B (3, 5)	603	678
Green	G (3, 5)	656	738
Yellow	Y (3, 5)	710	799
Red	R (3, 5)	765	861
Bluish-red	BR (3, 5)	815	917
Fourth order—			
Green	G (4, 5)	893	1005
Yellow-green	YG (4, 5)	964	1085
Red	R (4, 5)	1052	1184
Fifth order—			
Green	G (5, 5)	1188	1337
Red	R (5, 5)	1335	1502
Sixth order—			
Green	G (6, 5)	1479	1664
Red	R (6, 5)	1627	1830
Seventh order—			
Green	G (7, 5)	1787	2010
Red	R (7, 5)	1936	2178

In describing the experiments it is convenient to have a symbol which enables us to express the colour shortly. The two films are distinguished as the right and left respectively, and C_r and C_l are used to indicate their colours.

Thus, if the film was uniform in tint, C_r=G (2, 5) means—the colour of the right film was the middle of the green of the second order.

If two colours are indicated between brackets they are those of the top and bottom of the film respectively. Thus

$$C_l = \{B\ (2, 5),\ G\ (3, 5)\}$$

may be read—the colour of the left film varied between the middle of the blue of the second order at the top and the middle of the green of the third order at the bottom.

If the film displayed any black, the length in millimetres is written first; thus

$$C_r = \{5b,\ W\ (1, 5),\ O\ (1, 0)\}$$

means—the right film displayed five millimetres of black at the top, and the tint of the remainder varied between the middle of the white of the first order, which was in contact with the black, and the beginning of the orange of the first order at the bottom.

Results of Preliminary Experiments.

The first experiments which we performed were designed to test the observations of Lüdtge and Van der Mensbrugghe as to the difference of tension between two films, of which one had been formed more recently than the other.

The symbols ρ and λ indicate the principal diameters of the right and left films respectively, and $\delta=\rho-\lambda$.

The diameter of the rings is 2Y, the distance between them 2X.

All lengths are given in millimetres except when otherwise stated. In calculations in which the sensitiveness is used they must be expressed in centimetres.

The following experiment was made as a test of the rapidity with which the films would return to their position of equilibrium when disturbed. This gave an idea of the readiness with which the arrangement would respond to a change of tension.

Experiment I.

$2Y=20{\cdot}8$, $2X=2{\cdot}5Y=26{\cdot}0$.

Two films were made, flooded, and put into communication. Five minutes and eight minutes afterwards the diameters were measured. The time elapsed since the formation of the films is indicated by t. The results were as follows:—

t	C_r	C_l	ρ	λ	δ
m.					
5	Colourless		20·90	20·83	0·07
8	G (4, 5)	R (4, 5)	20·94	20·89	0·05

The two films were now separated, and the right hand one was blown out a little. The diameters were then measured.

t	C_r	C_l	ρ	λ	δ
m.					
11	G (3, 5)	G (3, 5)	22·18	20·91	1·27
13	"	"	22·20	20·88	1·32

At 17m the stop-cock was opened, so that the interiors of the films were again in communication, and the following observations taken. The colour of both films was still the green of the third order.

t	ρ	λ	δ
m.			
19	21·58	21·43	0·15
21	21·74	21·52	0·22

The result of this experiment was satisfactory. The mean difference of the diameters, having been increased from 0·06 mm. to 1·30 mm., was reduced to 0·18 mm. two minutes after communication between the films was re-established. There appears to have been a slight alteration of the zero reading, but the new position of equilibrium was rapidly taken up. When the large rings were used the movements of the films were slower, but, as all the principal experiments described in this paper were made with the small rings, the above observation is the only one which need be given in full to illustrate the point.

Experiment II.

The following is a typical example of a large number of experiments which proved that in general the thicker film behaved as though it had the greater surface tension. The left film was flooded at intervals of ten minutes, while the right was allowed to thin. The diameters were measured every ten minutes. When the first traces of black appeared in the right film it was flooded, and the left was allowed to thin till traces of black appeared. The interval between the two extreme measurements, which are alone given below, was $1^h\ 20^m$. During this time the temperature of the film box was steady at 19°·2 C.

The glass rings were used. $2Y=32·5$, $2X=2·5Y$.

C_r	C_l	$\delta=\rho-\lambda$
{R (2, 3), R (2, 9)} Colourless	Colourless {R (2, 1), P (2, 5)}	3·18 −2·25

We have already remarked that the alternate flooding of the two films doubles the sensitiveness of the apparatus as a means of detecting small changes in surface tension. If, therefore, the values given on p. 645 may be applied to such considerable differences of the diameters, we have

$$dT/T=0·543/2\times 7·03\equiv 3·8 \text{ per cent.}$$

This calculation is given chiefly as an illustration of the sensitiveness of our method. A number of possible objections have to be considered before it can be accepted as a trustworthy determination of an actual difference of surface tension between the films.

In the first place, we may state that we were most scrupulously careful about the cleanliness of the apparatus. It was always taken to pieces as soon as the experiments were over, and the glass box, rings, tubes, linen bands, &c., were all washed in distilled water. The bands were left to soak all night, and when required were wrung out and dried with blotting paper. The films were always flooded before an experiment began, so that the cups and bands were well washed down with the solution. We found also that successive films, when thus treated, behaved in a much more uniform way than if the flooding was omitted. We have at times replaced the metal rings by the glass ones, and been most careful that the liquid should touch nothing but glass and linen, but the phenomenon above described was as marked as before. Lastly, if one of the films was thickened by the electric current it contracted just as it would do if thickened by flooding. The fact that the thicker film displayed the greater surface tension cannot, therefore, be due to any peculiarity of the apparatus or mode of thickening adopted.

Experiment III.

As a crucial test as to whether the weight of the thicker film had anything to do with the phenomenon, we placed the larger rings on the upper and the smaller rings on the lower supports. We thus obtained two unduloidal films, which may be described as approximately cones with the smaller ends below. It is evident that if one of these films were heavier than the other the pressure exerted on the enclosed air would thereby be diminished, and the heavier film would bulge. Experiment proved that, on the contrary, it contracted, thus indicating that the pressure was increased.

Experiment IV.

The question as to whether the phenomenon was due to the fact that in the thicker film the liquid is falling more or less rapidly was answered by closing the cock until the down-rush was practically over. When communication between the films was restored the thicker one contracted as usual.

The fact that the film contracted if it was slowly thickened from below by the electric current pointed to the same conclusion.

Experiment V.

It was, in the next place, important to determine whether the gradual disappearance of the liquid rings by which the thinning film is attached to the solid supports could produce changes in the value of Y which would account wholly or partially for the phenomenon observed.

If an unduloid differs but little from a cylinder of radius Y, we may write for its maximum and minimum ordinates $Y+d\alpha$ and $Y+d\beta$. Any other ordinate may be expressed by $Y+dy$.

Hence, if we neglect the squares of small quantities,

$$k^2=(\alpha^2-\beta^2)/\alpha^2=2(d\alpha-d\beta)/\mathrm{Y} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

If x be the difference of the abscissæ of two points for which ϕ_1 and ϕ_2 are the values of ϕ, then, to the same approximation,

$$x=\alpha\int_{\phi_2}^{\phi_1}\Delta\, d\phi+\beta\int_{\phi_2}^{\phi_1}\frac{d\phi}{\Delta}=2\alpha\left(1-\frac{k^2}{4}\right)(\phi_1-\phi_2)=2\mathrm{Y}\left\{1+\frac{d\alpha+d\beta}{2\mathrm{Y}}\right\}(\phi_1-\phi_2). \quad . \quad (16)$$

Let

$$x/2\mathrm{Y}=\xi.$$

Then

$$\phi_1-\phi_2=\xi\left\{1-\frac{d\alpha+d\beta}{2\mathrm{Y}}\right\}.$$

Now from equation (4)

$$(\mathrm{Y}+dy_1)^2=(\mathrm{Y}+d\alpha)^2\cos^2\phi_1+(\mathrm{Y}+d\beta)^2\sin^2\phi_1,$$

or, neglecting the squares of small quantities,

$$dy_1=\frac{d\alpha+d\beta}{2}+\frac{d\alpha-d\beta}{2}\cos 2\phi_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

If now we take three points on the unduloid, such that the distances between consecutive points measured parallel to the axis are equal, then to the first approximation

$$\phi_1-\phi_2=\phi_2-\phi_3=\xi.$$

But, from two equations similar to (17),

$$dy_1+dy_3=d\alpha+d\beta+(d\alpha-d\beta)\cos 2\phi_2\cos 2\xi.$$

Also

$$dy_2=\frac{d\alpha+d\beta}{2}+\frac{d\alpha-d\beta}{2}\cos 2\phi_2.$$

Multiplying the last equation by $2\cos 2\xi$, and subtracting,

$$d\alpha+d\beta=\frac{dy_1+dy_3-2dy_2\cos 2\xi}{2\sin^2\xi}.$$

Since $d\alpha$ and $d\beta$ are independent, we may suppose the unduloid to have been derived from any convenient cylinder, and the calculations are simplified if we select that the radius of which ($\overline{\mathrm{Y}}$) is the mean of the three quantities $\mathrm{Y}+dy_1$, &c.

In this case,

$$dy_1+dy_2+dy_3=0,$$

and

$$d\alpha+d\beta=-dy_2\frac{1+2\cos 2\xi}{2\sin^2\xi}.$$

This formula affords a means of deciding whether any slipping of the liquid ring takes place. We have carried out the above calculations to the second degree of approximation, but, as the numerical work is troublesome and the correction introduced is small, we here neglect the squares of small quantities.

A vertical millimetre scale was placed by the film box, and the reading on this was determined, which corresponded to the middle of the distance between the cups to which the films were attached. The diameters were then measured not only in this horizontal, but also in planes 1 cm. above and below it.

The values of $\overline{Y}$, dy_1, dy_2, and dy_3 were thus found, and $d\alpha+d\beta$ was determined without any assumption as to the nature of the attachments of the films to the cups.

Since the pressures exerted by the films on the internal air were the same, we have

$$\frac{T+dT}{2\overline{Y}+d\alpha+d\beta}=\frac{T}{2\overline{Y}'+d\alpha'+d\beta'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

whence dT/T is found.

It can also be calculated from the measurements of the principal ordinates and from the sensitiveness. If the results of the two methods agree the assumption made in the earlier part of this paper that the generating curve of the film passes through the edges of the upper and lower cups is justified.

In making an experiment with a thick film as the standard of comparison, it is necessary that the same amount of liquid should always be used in flooding, and that the measurements should always be made at the same interval after that operation. In the experiment now to be described the flooding took place between each pair of readings near the top, middle, and bottom of the films. One minute elapsed between the flooding and the beginning of the measurements, and three minutes between consecutive floodings. The readings were taken in the order: top, bottom, middle, bottom, top, &c., so that the measurement of the principal diameter corresponded to the mean of the two for the top and bottom, from which it was in point of time equidistant. Any error due to slow changes in the position of equilibrium was thus as far as possible obviated.

The following results were obtained when the left film was colourless, and when $C_r=\{1b, Y(2,9)\}$:—

	Right.	Left.
Diameter	23·05	18·80
$2\overline{Y}$	22·213	19·643
$2dy_3$	+0·837	−0·843
ξ	25° 48′	29° 10′

Using equation (18), we find that the surface tension of one film exceeded that of the other by 8·8 per cent.

Also, since $\delta=0{\cdot}425$ cm. and the limiting sensitiveness for the two cylinders is 4·51,

$$dT/T=0{\cdot}425/4{\cdot}51=0{\cdot}094.$$

So that by this formula the difference of surface tension is given as 9·4 per cent.

Considering that both equation (18) and the limiting sensitiveness are here applied to films which differ considerably from cylinders, these results are in satisfactory agreement and prove that the greater part of the difference of form of the two films cannot be ascribed to any slipping of the liquid attachments to the solid supports.

To show that this point needed careful investigation, we may remark that it seemed *prima facie* probable that the bulging film might be forced out and the contracting film drawn in a little so that they were virtually attached to rings a little larger and a little smaller than those actually employed. If this had affected the radii only to the extent of 0·1 mm. the effect on the calculated difference of surface tension would have been very great. Thus, assuming the principal diameters to have remained unaltered, but the others to have been increased in the right and diminished in the left film by 0·2 mm., the values of $2\overline{Y}$ in the above table would have been 22·346 and 19·510. The difference of surface tension given by these numbers is 3·8 per cent., while that calculated from the sensitiveness remains, as before, 9·4 per cent. It is evident, therefore, that it was only by careful measurement that we could assure ourselves that the source of error we are now investigating was not affecting our results to a very serious amount.

We have made a considerable number of similar experiments which prove that the method of measuring three ordinates generally gives results from 0·5 per cent. to 1 per cent. lower than those obtained from the sensitiveness. The following Table is a sample of our results when the films had settled down to a constant state. It shows the numbers obtained by a series of consecutive measurements on the same film. The value of dT is given in percentages. The left film was flooded at three minute intervals.

TABLE V.

C_r	δ	dT found by measurement of	
		(1) Three ordinates.	(2) One ordinate.
		Per cent.	Per cent.
{2·5b, G (3, 5)}	2·35	4·1	5·2
{3b, Y (2, 6)}	2·69	5·5	6·0
{8b, B (2, 5)}	2·42	4·5	5·4
{10b, O (1, 9), B (2, 5)} . .	2·53	5·3	5·6
{18b, O (1, 9), B (2, 5)} . .	2·37	4·3	5·3

When both films are flooded, and are therefore presumably in the same state, there is, notwithstanding, in general a more or less marked difference between their diameters. To this we generally refer as the zero error.

The following results were obtained under these conditions with six different films:—

TABLE VI.

δ	dT found by measurement of	
	(1) Three ordinates.	(2) One ordinate.
	Per cent.	Per cent.
0·76	0·7	1·7
0·36	1·0	0·8
0·19	0·8	0·4
0·12	1·1	0·3
0·43	−0·4	1·0
0·01	0·0	0·0
Mean . .	0·53	0·70

These results do not show such a constant difference between the two methods as is exhibited in the previous table, and it is, therefore, probable that when the bulging and contraction of the films becomes considerable there is a little slipping.

Taking, however, the lowest figures, these experiments are, we think, fatal to VAN DER MENSBRUGGHE'S explanation. He explains similar phenomena by means of the general principle: "Si la couche superficielle d'un liquide augmente, ou qu'elle devienne le siége d'une énergie potentielle qu'elle ne possédait pas d'abord, il y a refroidissement, et la tension est plus grande que primitivement, &c."*

From this point of view the surface of the liquid of which the thick film is formed is continually being renewed, it is therefore cooled, and the result is due to the increase of surface tension which follows.

* 'Bruxelles, Acad. Sci. Mém.,' vol. 43 (No. 4, p. 7).

It is, however, impossible to account for the smallest of the differences of surface tension shown in Table V. in this way.

If a gramme of water were drawn out into a film of 100 sq. cm. the cooling effect would be 1/4803 degree C.* But the thickness of the thinnest soap films has been proved by us to be about 10^{-6} cm. Hence, in thinning to this degree of tenuity, the film would be cooled only a little more than 2° C. But since, if t is the temperature Centigrade,

$$\frac{dT}{dt} = -\frac{T}{550},$$

the increase of surface tension due to the cooling would be only about 0·4 per cent.

Now a colourless film, which has been recently flooded, is certainly 250 times thicker than a black one, so that if the heat were distributed uniformly throughout the whole mass of the film the increase of surface tension would be only 0·0016 per cent.

The enormous discrepancy between this number and that actually given by our experiments (9 per cent. in Experiment III.) cannot be explained away by the small corrections which would no doubt have to be made if data were used which apply accurately to the soap solution instead of those derived from water. One of these corrections would indeed strengthen the argument, for, as the surface tension of the solution is less than that of water, the cooling effect of the stretching would be decreased.

Calculations based on VAN DER MENSBRUGGHE'S experiments indicate differences of surface tension as great as those on which our argument depends. We have, however, proved that the various possible sources of error discussed above, and which have not been previously investigated, do not seriously affect the result. We may add that, although we have only cited a few experiments, we have performed many. Thus when we balanced a sphere against a sphere the sensitiveness was so great that the films broke before the limit to the movement was approached. The change of form was so great that the limiting sensitiveness could not be considered to apply even approximately.

It is not, however, easy to decide what is the particular cause, if it can be referred to one cause only, to which the change in the surface tension of one of the films is due.

Experiment VI.

We tried the effect of increasing largely the quantity of carbonic acid and of oxygen in the air within and without the films, and to this end passed gentle currents of these gases through the film box, and formed the films by means of them instead of air.

As was to be expected, the liquid became turbid under the action of the carbonic

* Baynes' 'Thermodynamics,' p. 136.

acid, and if fresh liquid which had not been exposed to the action of the gas was used to flood the thick film it bulged instead of contracting.

Oxygen, on the other hand, appeared to increase the phenomenon ordinarily observed, or, at all events, to accelerate it.

Thus, when the film box was filled with air, the difference of the diameters increased by 2·07 mm. in 24^m, when the left film was flooded at 3 minute intervals, the right film being allowed to thin.

The films were then blown with purified oxygen, and a current was passed through the box.

In two successive experiments which were then made under the same conditions as before, except that oxygen was used, the change in the diameter was 2·96 mm. and 2·80 mm. in 9 minutes.

These experiments cannot be considered conclusive, but they indicate that the phenomenon under discussion is affected by the nature of the gas with which the films are in contact. To have put the matter to a thoroughly satisfactory test would have required a complete remodelling of our apparatus. We did not carry the inquiry further, partly on this account, and partly because we had collected evidence which proved that the greater tension of the thicker film was not due merely to its thickness. The cause of the difference was therefore outside the immediate scope of our investigation.

One reason for this conclusion is that the results obtained with the same liquid are very irregular. In three consecutive experiments, performed on the same day, in each of which the thin film displayed the colours of the third order, the values of δ were 3·37, 2·50, and 1·73. On the previous day, when a longer interval elapsed between the floodings, two consecutive experiments gave for similar thicknesses $\delta=1{\cdot}05$ and 1·56. Again, different solutions behaved very differently. The above results were obtained with a solution of potash soap which had been recently made. An exactly similar solution a year old gave $\delta=0{\cdot}48$ when displaying the green of the third order.

In the next place, so much of the change takes place before the thinner film is thin enough to display any colours that if it is really due to the thinness of the film the radius of molecular attraction must be at least $=10^{-3}$ mm. Many of our experiments have shown that between one half and one quarter of the total change takes place while the thin film is still colourless.

On the whole, then, we are inclined to the view that this phenomenon is merely a striking instance of the difficulty which many observers have found in preserving a liquid surface pure. Oberbeck found that the superficial viscosity of water exposed for 24 hours to the air became immeasurably large, but regained its former value when the surface was cleansed by a clean strip of platinum. He concludes, "Wir müssen daher schliessen, entweder, dass der freien Wasseroberfläche ein recht bedeutender Oberflächenwiderstand zukommt, oder dass eine reine Wasseroberfläche in

Berührung mit Luft überhaupt nicht existirt."* It seems probable that a liquid drawn out into a film would be especially liable to such changes, and that the constant renewal of the surface of the thick film would tend to maintain its purity, and therewith its surface tension.

Adopting, then, the conclusion that the rapid fall in the surface tension of a newly formed film is due, not to its thinning, but to a disturbing cause, it remains to be seen whether this can be reduced or eliminated so as to permit a comparison of films of very different thicknesses. We have found that this is possible, but, though the methods which we have adopted are such as would naturally suggest themselves, the determination of the best conditions of experiment has cost a large expenditure of time and trouble.

In the first place, we have employed the method of measuring the principal ordinates only, and using the sensitiveness to calculate the difference of tension (if any) between the films. In what follows these differences will be very small, and we have shown that, in this case, both methods lead to very similar mean results. The method of measuring three ordinates is also too laborious to be used if another will serve equally well. It necessitates continual measurements being made, and, though very useful for the elucidation of the particular point to which it was directed, is very troublesome and fatiguing, as twenty-three readings are necessary to obtain a single comparison. Again, we were not satisfied with the flooded film as a standard with which to compare that which was allowed to thin. The difference of the surface tensions, and, therefore, of the forms of the films, was too great for accuracy. It was thought possible that the comparatively small disturbance produced by retarding the thinning by the electric current might make a film, up which a current was either constantly or at intervals passed, a better standard than one which was flooded between consecutive measurements.

The following experiment, which was made to see if a film could be kept thick and in an approximately constant state by short intermittent applications of the current, will perhaps serve to show the difficulty of eliminating the disturbing cause by any such means.

Experiment VII.

Two cylinders were formed, flooded, and put into communication.

Five minutes afterwards $\delta = -0{\cdot}20$ mm. Both films were colourless for the most part, but displayed thin rings of colour at the top.

The current was then sent up the right cylinder for 10^s. The coloured rings disappeared.

At 9^m from the beginning of the experiment, and about 3^m after the passage of the current, the right film was colourless.

* 'Wiedemann, Annalen,' vol. 11, 1880, p. 650.

C_l=G (6, 5) with rings of higher orders at the top.

$\delta = -0{\cdot}38$.

At 14^m, C_l=G (5, 5); the right was colourless, and both had rings at the top.

$\delta = -0{\cdot}27$.

Now, remembering that $\delta = \rho - \lambda$, and that a negative value of δ therefore means that the left film had the larger diameter, this experiment indicates that the passage of the current for 10^s only produced a difference in the diameter of the rings which 3^m afterwards amounted to 0·18 mm., and 8^m afterwards to 0·07 mm., and that the film disturbed by the current had the greater surface tension.

At 15^m the current was sent up the right film for 12^s. The coloured rings disappeared.

At 19^m the right film was faintly coloured, and

$C_l = \{0{\cdot}5b,\ \mathrm{YG}\ (4,5)\}$,

$\delta = -0{\cdot}28$.

The disturbance produced by the current was less than before, which is accounted for by the fact that the film, though colourless, was really thinner than when it was previously applied, and thus, the resistance in circuit being greater, the current was weaker. That the film was thinner is proved by the fact that, whereas after the first application of the current the right film remained for the most part colourless for eight minutes, on the second occasion it showed faint colours after four minutes. This observation shows that in a film which is not very thick the current may be passed for a few seconds without leaving any trace of its action three minutes afterwards, but unfortunately it also proves that such treatment is not sufficient to prevent the film from thinning.

At 20^m the current was sent up the right film for 20^s, and the following measurements were made:—

At 24^m—

C_r=R (5, 0), $C_l = \{1{\cdot}5b,\ \mathrm{G}\ (4, 0)\}$.

$\delta = -0{\cdot}90$.

At 29^m—

C_r=G (5, 7), $C_l = \{3{\cdot}0b,\ \mathrm{Y}\ (3, 9)\}$.

$\delta = -0{\cdot}44$.

In this case the longer passage of the current caused a considerable disturbance, the effects of which were very marked 9^m afterwards. The experiment was continued in the same way, the measurement showing that a current which produced an appreciable retardation in the thinning also produced an appreciable difference in the diameters.

Seventy-one minutes after the beginning the state of things was as follows:—

$C_r = \{1{\cdot}0b,\ \mathrm{B}\ (3, 5)\}$, $C_l = \{11{\cdot}0b,\ \mathrm{Y}\ (2, 7)\}$.

$\delta = -0{\cdot}77$.

The films were then cut off from communication, and the current was passed up the right film for 90^s: the black disappeared and the lower part became colourless. The stop-cock was opened again eight minutes after the passage of the current, and a measurement taken when it had been open two minutes.

$$C_r = \{B\ (3,\ 0),\ \text{colourless}\},\quad C_l = \{14{\cdot}0b,\ R\ (2,\ 0)\}.$$
$$\delta = -1{\cdot}18.$$

Shortly afterwards the thin film broke.

The foregoing observations proved that the disturbance produced by thickening one of the films died out some minutes after the disturbing cause was removed. If, then, the surface tension assumed its final value while a film was still moderately thick it might be possible, by balancing films of very different thicknesses, to obtain some evidence as to whether any further difference of tension due only to the difference in thickness could be detected. We, therefore, gave one of the films a start by thickening the others some time after their first formation, and then compared them when another considerable interval of time had elapsed.

This method is attended with several difficulties. In the first place, the operation takes a long time, during which slow changes in the zero error take place. Thus, if the two films are first compared when both have been recently flooded, and afterwards when one is black and the other coloured, a slight difference in the position of equilibrium may be observed, due to changes in the liquid rings, &c. On some occasions, when one of the films has been entirely black, the lower liquid ring has disappeared. This phenomenon is accompanied by a slight shift in the position of equilibrium. Again, if the start is too short the films are too similar to make the comparison of much use; if it is too long the thin film is apt to break before the thick one is ready for observation. These difficulties are increased by the fact that it is not safe to leave the films in communication while one of them is being thickened. A black film is almost certain to break if subject to a disturbance which may alter its diameter by one or two millimetres. It is, therefore, necessary to separate the films while the thickening takes place, and not to put them again into communication until after the interval which experience has proved to be sufficient to prevent any great change in diameter. We have attempted as far as possible to overcome these difficulties, and have carried out our observations in two different ways.

First Method of Experiment.

The fact that the method of measuring three ordinates confirmed that based upon the measurement of the principal diameters only, in indicating a small difference of surface tension between the films, even when both had been recently flooded, proves either that such differences really exist or that some slight outstanding cause of error affects both methods equally. In either case it seemed best to argue in each experi-

ment, not from the theoretical position of equilibrium, but from that assumed by two films which were as similar as we could make them.

Two cylindrical films were therefore flooded and put into communication. Their diameters were measured several times, and the difference, if constant, gave the zero reading. After an interval, which depended on the rate of thinning, the films were separated, and one was thickened by means of the current, generally until the colours of the third order reached the top. If no black appeared on the thicker film communication was re-established after 17^m, and the diameters were measured about 3^m afterwards. If the black began to appear at the top of the thick film before 17^m after thickening, the stop-cock was opened sooner. By this means we generally succeeded in comparing two films, one of which was nearly all black, while the other displayed a little black and the colours of the first or second orders. If the position of equilibrium was the same as when both films were thick the experiment gave no evidence of any difference of surface tension between a black film and one many times thicker than itself. If it was not the same the apparent difference of surface tension was calculated from the sensitiveness.

Both films were then allowed to thin, and, assuming that that which was already black remained in a constant state, we were able to observe whether any change of diameter took place as the coloured film became black. If it did it furnished us with another measurement of the difference of surface tension under investigation. It is evident that the mean of two such measures obtained from the same films would probably be free from error caused by any slight change in the position of equilibrium due to causes other than the increasing tenuity of the films. Accidental errors would of course disappear on the mean of a number of experiments.

This method is, however, open to an objection which requires discussion. We were seeking for evidence of a difference of surface tension (if any such existed) between a black and a coloured film. Most of the films observed were, however, partly black and partly coloured. If any difference of tension existed between their different parts the films would not be cylinders or simple unduloids, but the black and coloured portions would be of different curvatures. Measurements based on the sensitiveness would not therefore be trustworthy if, upon the comparatively simple changes of form assumed in the calculation of that quantity, other and more complex variations were superposed. We have therefore investigated the form of a film consisting of two parts of different surface tensions, assuming that it does not differ much from a cylinder.

If the surface tensions are T and $T+dT$, the generating curve of the film may be regarded as consisting of two parts, PR and RQ, which produce unduloidal surfaces of different curvatures.

The conditions which these curves must satisfy are as follows:—

(1.) If X_1 and X_2 are the differences of the abscissæ of R and P, and Q and R,

respectively, then X_1 and X_2 may be considered as constants which are given by the conditions of the problem.

(2.) The values of the ordinates which correspond to P and Q must both be Y.

(3.) The ordinate of R($=y$) must be the same for both curves.

(4.) The tangent at R must be the same for both curves.

(5.) The sum of the volumes of the two unduloids contained between the rings may be taken as given.

(6.) The pressures exerted by the two parts of the film on the enclosed air must be the same.

These eight conditions may be used to determine the eight unknowns, viz., $d\alpha_1$, $d\beta_1$, $d\alpha_2$, $d\beta_2$, ϕ_1 and ϕ_2, the values of ϕ which correspond to P and Q, and ψ_1 and ψ_2, the values of ψ which correspond to R, considered as belonging in the first case to PR, and in the second to RQ.

The discontinuity at R is dependent on the fact that in the cylinder the value of ϕ which corresponds to any given point is indeterminate. We shall carry out the calculation to the first order of small quantities only.

Using the same notation as before, we have, as in equations (15) and (16),

$$k_1^2=\frac{2(d\alpha_1-d\beta_1)}{Y}, \qquad k_2^2=\frac{2(d\alpha_2-d\beta_2)}{Y}, \quad \ldots\ldots (19)$$

$$\left.\begin{aligned}\psi_1-\phi_1&=\xi_1\left\{1-\frac{d\alpha_1+d\beta_1}{2Y}\right\}\\ \phi_2-\psi_2&=\xi_2\left\{1-\frac{d\alpha_2+d\beta_2}{2Y}\right\}\end{aligned}\right\}. \quad \ldots\ldots (20)$$

In like manner we get from (5)

$$V_1/\pi=2\alpha_1^3(1-\tfrac{3}{4}k_1^2)(\psi_1-\phi_1)+\frac{\alpha_1^2k_1^2}{2}(\sin 2\psi_1-\sin 2\phi_1),$$

whence, substituting from (19) and (20),

$$V_1/\pi=2Y^3\xi_1\left(1+\frac{d\alpha_1+d\beta_1}{Y}\right)+Y^2(d\alpha_1-d\beta_1)(\sin 2\psi_1-\sin 2\phi_1).$$

But $2\pi Y^3\xi_1=\pi Y^2X_1$, so that, if dV be the difference between the sum of the volumes of the figures generated by PR and RQ and the volume of the cylinder,

$$\begin{aligned}dV/\pi Y^2=&2\xi_1(d\alpha_1+d\beta_1)+(d\alpha_1-d\beta_1)(\sin 2\psi_1-\sin 2\phi_1)\\&+2\xi_2(d\alpha_2+d\beta_2)+(d\alpha_2-d\beta_2)(\sin 2\phi_2-\sin 2\psi_2). \quad \ldots\ldots (21)\end{aligned}$$

Again, since

$$Y^2=\beta_1^2\sin^2\phi_1+\alpha_1^2\cos^2\phi_1,$$

$$\left.\begin{array}{c} d\beta_1\sin^2\phi_1+d\alpha_1\cos^2\phi_1=0 \\ \text{and similarly} \\ d\beta_2\sin^2\phi_2+d\alpha_2\cos^2\phi_2=0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Equating the two expressions for the square of the common ordinate at R, we get

$$d\beta_1\sin^2\psi_1+d\alpha_1\cos^2\psi_1=d\beta_2\sin^2\psi_2+d\alpha_2\cos^2\psi_2. \quad . \quad . \quad . \quad . \quad (23)$$

Since, by BEER's equation,

$$\frac{dy}{dx}=-\frac{\sqrt{(\alpha^2-y^2)(y^2-\beta^2)}}{y^2+\alpha\beta},$$

we have, by equating the expressions obtained by substituting the values of y in terms of α_1, β_1, and ψ_1, and α_2, β_2, and ψ_2,

$$(d\alpha_1-d\beta_1)\sin 2\psi_1=(d\alpha_2-d\beta_2)\sin 2\psi_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

Substituting from (20) in (23) and (24),

$$d\beta_1\sin^2(\phi_1+\xi_1)+d\alpha_1\cos^2(\phi_1+\xi_1)=d\beta_2\sin^2(\phi_2-\xi_2)+d\alpha_2\cos^2(\phi_2-\xi_2),$$

and

$$(d\alpha_1-d\beta_1)\sin 2(\phi_1+\xi_1)=(d\alpha_2-d\beta_2)\sin 2(\phi_2-\xi_1).$$

Again, substituting in these equations from (22),

$$\frac{d\alpha_1}{d\alpha_2}=\frac{\sin^2\phi_2\cos^2(\phi_2-\xi_2)-\cos^2\phi_2\sin^2(\phi_2-\xi_2)}{\sin^2\phi_1\cos^2(\phi_1+\xi_1)-\cos^2\phi_1\sin^2(\phi_1+\xi_1)}\times\frac{\sin^2\phi_1}{\sin^2\phi_2}, \quad . \quad . \quad . \quad . \quad (25)$$

and also

$$=\frac{\sin 2(\phi_2-\xi_2)}{\sin 2(\phi_1+\xi_1)}\times\frac{\sin^2\phi_1}{\sin^2\phi_2},$$

therefore

$$\frac{\sin(2\phi_2-\xi_2)\sin\xi_2}{-\sin(2\phi_1+\xi_1)\sin\xi_1}=\frac{\sin 2(\phi_2-\xi_2)}{\sin 2(\phi_1+\xi_1)}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Also from (21) and (24)

$$d\nabla/\pi Y^2=2\xi_1(d\alpha_1+d\beta_1)+2\xi_2(d\alpha_2+d\beta_2)-(d\alpha_1-d\beta_1)\sin 2\phi_1+(d\alpha_2-d\beta_2)\sin 2\phi_2$$

$$=\frac{d\alpha_2}{\sin^2\phi_2}(\sin 2\phi_2-2\xi_2\cos 2\phi_2)-\frac{d\alpha_1}{\sin^2\phi_1}(\sin 2\phi_1+2\xi_1\cos 2\phi_1). \quad . \quad . \quad . \quad (27)$$

Now, if we are dealing with one film only which is not connected with another, and if it would assume the cylindrical form when of uniform surface tension, $d\mathrm{V}=0$. In this case we get from equations (26) and (27)

$$\frac{\tan 2\phi_2 \sin 2\xi_2 - 1 + \cos 2\xi_2}{-\tan 2\phi_1 \sin 2\xi_1 - 1 + \cos 2\xi_1} = \frac{\tan 2\phi_2 \cos 2\xi_2 - \sin 2\xi_2}{\tan 2\phi_1 \cos 2\xi_1 + \sin 2\xi_1} = \frac{\tan 2\phi_2 - 2\xi_2}{\tan 2\phi_1 + 2\xi_1},$$

and, solving these equations, we get

$$\left.\begin{aligned} \tan 2\phi_1 &= \frac{\xi_1 \sin^2 \xi_2 - \xi_2 \sin \xi_1 \sin (\xi_1 + 2\xi_2) + \sin \xi_1 \sin \xi_2 \sin (\xi_1 + \xi_2)}{\sin (\xi_1 + \xi_2)\{\xi_2 \cos (\xi_1 + \xi_2) - \sin \xi_2 \cos \xi_1\}} \\ \tan 2\phi_2 &= -\frac{\xi_2 \sin^2 \xi_1 - \xi_1 \sin \xi_2 \sin (\xi_2 + 2\xi_1) + \sin \xi_1 \sin \xi_2 \sin (\xi_1 + \xi_2)}{\sin (\xi_1 + \xi_2)\{\xi_1 \cos (\xi_1 + \xi_2) - \sin \xi_1 \cos \xi_2\}} \end{aligned}\right\} \quad . \quad . \quad (28)$$

If dp is the difference between the internal pressure and that exerted by a cylindrical film of radius Y,

$$dp = -\frac{\mathrm{T}}{\mathrm{Y}^2}(d\alpha_1 + d\beta_1) = \frac{2d\mathrm{T}}{\mathrm{Y}} - \frac{\mathrm{T}}{\mathrm{Y}^2}(d\alpha_2 + d\beta_2),$$

or, from (22),

$$dp = \frac{\mathrm{T}}{\mathrm{Y}^2}\frac{\cos 2\phi_1}{\sin^2 \phi_1} d\alpha_1 = \frac{2d\mathrm{T}}{\mathrm{Y}} + \frac{\mathrm{T}}{\mathrm{Y}^2}\frac{\cos 2\phi_2}{\sin^2 \phi_2} d\alpha_2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The angles ϕ_1 and ϕ_2 can now be found from (28). By means of (25) and (29) $d\alpha_1$ and $d\alpha_2$ are given in terms of $d\mathrm{T}$, and thus $d\beta_1$ and $d\beta_2$ are known from (22).

We shall suppose that the suffix 2 refers to the black part of the film. The general nature of the changes of form which the film would undergo as ξ_2 increased can easily be deduced from the equations. When $\xi_2=0$, $\tan 2\phi_2=0$, $\tan 2\phi_1=0/0$.

From the first of these conditions we deduce $\phi_2=0$ or $\pi/2$. The first of these values corresponds to the case where $d\mathrm{T}$ is negative, when the black part of the surface would bulge; the second to the case where $d\mathrm{T}$ is positive. Since in each case the tangent to the unduloid is parallel to the axis, an infinitely thin ring, of different surface tension to the rest of the film, would not cause any finite change of form.

Evaluating $\tan 2\phi_1$, we get

$$\tan 2\phi_1 = (\sin 2\xi_1 - 2\xi_1)/2 \sin^2 \xi_1,$$

which gives the initial value of ϕ_1.

The distance from the point halfway between the rings of the point at which the initial unduloid would cut the generating line of the cylinder is $\mathrm{Y}(2\phi_1 - \xi_1)$, where ϕ_1 and ξ_1 have, of course, their initial values.

If ξ_1' and ξ_2' be such that $\xi_1=\xi_2'$ and $\xi_2=\xi_1'$, it is evident from (28) that $\tan 2\phi_1 = -\tan 2\phi_2'$ and $\tan 2\phi_2 = -\tan 2\phi_1'$, so that $\phi_1+\phi_2'=\phi_2+\phi_1'=\pi/2$.

But from (29) and (25)

$$\frac{d\alpha_1}{\sin^2\phi_1}\left\{\frac{\cos 2\phi_1 \sin 2(\phi_2-\xi_2)-\cos 2\phi_2 \sin 2(\phi_1+\xi_1)}{\sin 2(\phi_2-\xi_2)}\right\}=\frac{2Y dT}{T},$$

and from the same equations and (22)

$$-\frac{d\beta_2'}{\cos^2\phi_2'}\left\{\frac{\cos 2\phi_1' \sin 2(\phi_2'-\xi_2')-\cos 2\phi_2' \sin 2(\phi_1'+\xi_1')}{\sin 2(\phi_1'+\xi_1')}\right\}=\frac{2Y dT}{T},$$

whence, substituting for ϕ_1', ϕ_2', ξ_1', and ξ_2' in terms of ϕ_1, ϕ_2, ξ_1, and ξ_2, $d\alpha_1=-d\beta'_2$, and in like manner $d\alpha_2=-d\beta_1'$. It follows that the maximum distances of the two parts of the film from the cylinder are the same whether the black covers $1/n$th or $1-1/n$th of the entire film, but that the maximum contraction in the one case is equal to the maximum bulge in the other, and *vice versâ*.

The following Table contains in the first column the fraction of the entire length of the film over which the change in surface tension is supposed to have extended. In the second and third columns are the corresponding values of ϕ_1 and ϕ_2, calculated for the case in which $X=1{\cdot}25/Y$. The next four columns contain quantities from which $d\alpha_1$, &c., can be determined. In the last column is the ratio of the change of pressure caused by a partial change of surface tension to that produced by the extension of the change over the entire film. Throughout dT is considered positive.

TABLE VII.

$\xi_2/(\xi_1+\xi_2)$	ϕ_1	ϕ_2	$Td\alpha_1/YdT$	$Td\beta_1/YdT$	$Td\alpha_2/YdT$	$Td\beta_2/YdT$	Ydp/dT	dp/dp'
0	−23° 16′	90° 0′	0	0	2	0	0	0
0·2	−21 40	100 30	0·037	−0·234	1·867	−0·064	0·197	0·0985
0·3	−20 35	103 40	0·069	−0·489	1·679	−0·099	0·420	0·2100
0·5	−17 54	107 54	0·116	−1·116	1·116	−0·116	1·000	0·5000
0·7	−13 40	110 35	0·099	−1·679	0·489	−0·069	1·580	0·7900
0·8	−10 30	111 40	0·064	−1·867	0·234	−0·037	1·803	0·9015
1·0	0	113 16	0	−2	0	0	2	1·0000

Since, when dT is positive, $d\beta_1$ and $d\alpha_2$ refer to parts of the two unduloids which do not lie between the rings, it follows from this Table that the greatest divergence of any part of the generating curves from the generating line of the initial cylinder is $0{\cdot}116Y dT/T$.

If, therefore, as in most of our experiments, $Y=10$ mm. (nearly), and if we take $dT/T=0{\cdot}01$, it follows that the greatest change in the diameter of the film caused by the gradual spread of a change of surface tension of 1 per cent. would be only a little greater than 0·02 mm., a quantity which certainly lies within the limits of the error of experiment. If, therefore, a film was half black and half coloured it would differ

inappreciably from a cylinder if the difference of surface tension was only 1 per cent. The pressure, however, would have altered by 0·5 per cent., and if the film were now put into communication with another which was coloured throughout an appreciable change in the principal diameters would take place. This is evident from the fact that, if the change in the pressure exerted by the first film had been due to its surface tension having altered throughout by 0·5 per cent., the difference in the principal diameters would have been $0{\cdot}451/2=0{\cdot}225$ mm. It follows, therefore, that if two films are in communication, and if the surface tension of a part of one differs slightly from that of the remainder, the asymmetry of this film is small compared with the change produced in the principal diameters of the two films. We may, therefore, as a first approximation to the truth, regard the sensitiveness of an experiment in which part only of the film is black as proportional to dp/dp'.

The numbers given in the last column of Table VII. may be plotted in a curve, the abscissæ being the fractions of the whole length of the cylinder covered by the black, and the ordinates the values of dp/dp'.

If in any experiment the length of the black part of the film increased from (say) 0·2 to 0·8 of the whole length of the cylinder, the effective length would be 0·6; and the alteration of pressure is shown by the curve to be from 0·1 to 0·9, *i.e.*, 0·8 of the whole alteration which would be produced if a coloured film became completely black.

If the thicker film displays some black, the effective length of the black in the thinner film is the difference of the lengths in the two films. It will be observed that different changes in the pressure may be produced by the same change in the effective length. Thus dp/dp' is 0·66 or 0·8 according as the length of the black increases from 0 to 0·6, or from 0·2 to 0·8 of the whole length of the cylinder.

We now proceed to describe the experiments. It is necessary to do this in considerable detail, as no two were alike, and the results do not admit of being stated shortly so as to include all facts which might have affected the results.

Experiment VIII.

Three measurements were made when the films were in similar states.

t	C_r	C_l	δ	Mean.
m.				
24	{2·5*b*, G (2, 2)}	{0·5*b*, B (3, 0)}	−0·83	
32	{4*b*, B (2, 3)}	{2*b*, G (2, 9)}	−0·86	
37	{5*b*, R (1, 0)}	{3*b*, B (2, 6)}	−0·84	−0·84

The films were then separated, and the current was sent up the left film till the colours of the third order reached the top. This operation was repeated at 70^m. The current was taken off at 74^m; at 88^m the films were again connected, and the following observations were taken :—

t	C_r	C_l	δ	Mean.
h. m.				
1 30	All black	{1*b*, G (2, 0)}	−0·52	
35	" "	{2·5*b*, B (2, 2)}	−0·78	
40	" "	{4*b*, O (1, 8)}	−0·69	−0·66

In this case the final measurements gave a comparison between two films, of which one was black and the other displayed on the average 2·5 mm. of black only. The difference between the principal diameters was, however, only 0·18 mm. different from what it was when the films were in very similar states, the change being such as to prove that the left film had contracted a little, *i.e.*, that the surface tension of the right film when black was a little less than when it was coloured.

If this were so, however, since 2·5 mm. of the left film were black, the effective difference between the tensions of the two films was 0·98 of what it would have been, had the left film been coloured throughout. The sensitiveness is therefore $4{\cdot}51 \times 0{\cdot}98 = 4{\cdot}42$, and thus

$$dT/T = -0{\cdot}018/4{\cdot}42 = -0{\cdot}0041.$$

Hence according to this experiment the surface tension of a black film is less than that of a coloured one by 0·41 per cent.

Experiment IX.

In this case one of the films which had become partly black in a previous experiment was used. Hence no initial zero was obtained. The thicker film was again thickened by the current 14^m after the films were first put into communication. The

circuit was broken at 23m, and the stop-cock was opened at 32m. The following observations were taken :—

t	C_r	C_l	δ	Mean.
m.				
41	All black	{3b, B (3, 0)}	+0·51	
45	,,	{4b, O (2, 0)}	+0·23	
49	,,	{5b, G (2, 5)}	+0·39	+0·38

The value of δ fell as the left film thinned. The last two observations were as follows :—

t	C_r	C_l	δ	Mean.
h. m.				
1 39	All black	{21b, W (1, 5)}	−0·05	
0 44	,,	{23b, W (1, 5)}	−0·16	−0·10

The right film may be presumed to have remained in the same state throughout the experiment. The length of the black part of the left film increased from 4 mm. to 22 mm. The difference of pressure must therefore have increased from 0·05 dp' to 0·95 dp', *i.e.*, by 0·9 of the total difference between a coloured and a black film. During the same time δ changed by 0·48 mm., the movement being such as to indicate an increase in the diameter, *i.e.*, a falling-off in the tension of the film which was becoming black.

Hence

$$dT/T = -0{\cdot}048/4{\cdot}51 \times 0{\cdot}9 \equiv -1{\cdot}18 \text{ per cent.}$$

Experiment X.

The films were made as usual, and the following preliminary observations were taken :—

t	C_r	C_l	δ	Mean.
m.				
5	R (5, 5)	R (5, 0)	−0·34	
7	R (4, 5)	R (4, 5)	−0·42	
12	R (3, 5)	R (3, 5)	−0·38	−0·38

The right film was then thickened by means of the current. The circuit was broken at 17m, the stop-cock opened at 35m. The first reading gave $\delta = -0{\cdot}16$, and the value of δ increased (algebraically). The following observations were taken at the times indicated :—

t	C_r	C_l	δ	Mean.
h. m. 0 54 0 58	{7b, B (2, 3)} {8b, B (2, 0)}	{23b, W (1, 3)} All black	+0·21 +0·11	+0·16
1 31 1 36	{17b, G (2, 0)} {20b, G (2, 2)}	All black „	+0·21 +0·20	+0·20

The first comparison showed that δ had altered by 0·54 mm., the movement being such as to indicate an increase in the surface tension of the film which displayed the most black. Since the thicker film had 7·5 mm. of black, while the thinner had an average amount of 24·5 mm., the difference of pressure could only have been 0·8 of its full amount. Hence

$$dT/T = 0{\cdot}054/4{\cdot}51 \times 0{\cdot}8 \equiv 1{\cdot}5 \text{ per cent.}$$

During the second part of the experiment the movement of the zero continued in the same direction, which must be interpreted as a falling-off in the surface tension of the right film, which was now becoming black.

As the black extended only from 7·5 to 18·5 mm.,

$$dT/T = -0{\cdot}004/4{\cdot}51 \times 0{\cdot}6 \equiv -0{\cdot}15 \text{ per cent.}$$

This experiment illustrates the importance of obtaining, if possible, two measurements with the same pair of films. The fact that the change in the magnitudes of the principal diameters went on without reference to the thinning of the films is strong evidence that that change was not caused by the alterations in their relative thicknesses.

Experiment XI.

The same films were used as in Experiment I. The final observations in that experiment gave 0·18 as the initial difference of the diameters. At the end both films were first thickened by the current and then flooded. The value of δ found was 0·44. The mean of the two values is 0·32.

During the greater part of the time that the experiment lasted the current was passing. The circuit was broken at $2^h\ 11^m$ after the films were first made, and the following observation was taken :—

t	C_r	C_l	δ
h. m. 2 16	{17b, W (1, 0)}	{0·5b, W (1, 5), Y (1, 5)}	0·41

The change in the value of δ from the mean of the two values obtained when the films were in the same state was 0·09 mm.

Hence

$$dT/T = -0{\cdot}009/4{\cdot}51 \times 0{\cdot}72 \equiv -0{\cdot}28 \text{ per cent.}$$

Experiment XII.

The following observations were made to determine the zero :—

t	C_r	C_l	δ	Mean.
m.				
17	Colourless	Colourless	1·16	
22	{1*b*, B (2, 5)}	R (2, 5)	1·12	1·14

The current was then passed up the left film, but in spite of this it continued to become thinner. The circuit was broken at 31^m, the stop-cock was opened at 37^m, and the following observations taken :—

t	C_r	C_l	δ
m.			
39	All black	{5*b*, Y (1, 5), B (2, 5)}	1·12
53	„ „	All black	0·92

The first comparison gives

$$dT/T = +0{\cdot}002/4{\cdot}51 \times 0{\cdot}89 \equiv +0{\cdot}05 \text{ per cent.}$$

The second gives

$$dT/T = -0{\cdot}020/4{\cdot}51 \times 0{\cdot}90 \equiv -0{\cdot}49 \text{ per cent.}$$

Experiment XIII.

On this occasion the films were allowed to thin for some time before one of them was thickened. The value of δ changed considerably. Thus at 13^m $\delta = 0{\cdot}09$, at 16^m $\delta = 0{\cdot}42$, at 19^m $\delta = 0{\cdot}60$. The last three readings gave the following results :—

t	C_r	C_l	δ	Mean.
m.				
24	{0·75*b*, Y (3, 5)}	{G (4, 0)}	0·58	
29	{0·75*b*, G (3, 0)}	{G (3, 9)}	0·61	
33	{0·75*b*, R (2, 9)}	{1*b*, R (2, 6)}	0·66	0·62

The current was used to accelerate the thinning of the one film and to retard that of the other. The circuit was broken at 53^m, and afterwards the following measurements were made:—

t	C_r	C_l	δ	Mean.
h. m.				
1 18	{6b, O (1, 5)}	{22b, W (1, 5)}	0·10	
21	{8b, Y (1, 5)}	{25b, W (1, 5)}	0·02	
24	{10b, Y (1, 2)}	All black	0·10	0·07
59	{24b, Y (1, 5)}	All black	−0·14	−0·14

The effective change in the length of the black in the first comparison was from 8 mm. to 24 mm., the alteration in δ was 0·55 mm. in the direction indicating a falling-off in the surface tension of the black film. Hence

$$dT/T = -0{\cdot}055/4{\cdot}51 \times 0{\cdot}77 \equiv -1{\cdot}59 \text{ per cent.}$$

From the second part of the experiment we get

$$dT/T = 0{\cdot}021/4{\cdot}51 \times 0{\cdot}77 \equiv 0{\cdot}60 \text{ per cent.}$$

Second Method of Experiment.

Our second group of experiments was intended to utilize a fact which we had observed, viz., that a spherical film thins more slowly than a cylinder. A cylinder of the usual dimensions (length = 1·25 × diameter) was formed on one of the smaller pairs of rings. The larger rings were used to form the sphere, and, since the ratio of their diameter to that of the smaller pair was very approximately 1·6, it is evident, if the distance between them was made = 1·2 × 2Y, that a sphere would be formed the radius of which was 2Y, and which would therefore be in equilibrium with the cylinder if the surface tensions were identical. In consequence of the slow thinning of the sphere, a longer interval could generally be allowed after flooding with this arrangement than when two cylinders were employed.

It was found that, even when the films were newly formed, the ratio of the two diameters always differed appreciably from its theoretical value of 2 : 1. This was probably due to slight errors of adjustment and to the effect of the liquid rings. The experiment consisted in testing whether the difference between the diameters was the same when the two films were thick as it was when the cylinder was much thinner than the sphere.

The sensitiveness was 3·7.

The spherical film rarely showed more than one or two millimetres of black, the

effect of which may be neglected. When the cylindrical film was only partially black the theoretical sensitiveness can be calculated as in the last group of experiments.

We give, as in the last group, *all* the experiments which were made in which a sufficient difference of thickness was established to make the comparison useful.

Experiment XIV.

The stop-cock was opened five minutes after the films had been made and flooded. No further flooding took place. At 9^m after communication had been established readings were begun.

The colours of the sphere and cylinder are indicated by C_s and C_c, the difference between their diameters by δ.

t	C_s	C_c	δ	Mean.
m.				
9	R (4, 5)	Y (3, 5)	17·60	
12	G (4, 5)	B (3, 7)	17·82	17·71

We now omit a number of readings, the differences given by which were

$$\delta = 17{\cdot}77,\ 17{\cdot}67,\ 17{\cdot}83, \text{ and } 17{\cdot}79.$$

Afterwards the following observations were taken :—

t	C_s	C_c	δ
m.			
21·5	B (3, 0)	{ 5*b*, W (1, 9)}	17·90
24·5	R (2, 5)	{23*b*, (?) }	17·89

The sudden formation of the black was in this experiment accompanied by no change in the value of δ, hence $d\mathrm{T}=0$.

Experiment XV.

At the beginning of the experiment the following measurements were made :—

t	C_s	C_c	δ	Mean.
m.				
6	Colourless	Colourless	20·36	
11	,,	G (5, 2)	20·36	20·36

The films were then separated. At 27^m the sphere was flooded. The stop-cock was opened at 46^m, and the following values of δ were obtained, beginning at 48^m :—

$$\delta = 19{\cdot}87,\ 20{\cdot}03,\ 20{\cdot}15,\ 20{\cdot}04,\ 20{\cdot}11.$$

The last three readings were as follows :—

t	C_s	C_c	δ	Mean.
h. m.				
1 2	{1b, G (3, 9)}	{23b, Y (1, 3)}	20·09	
5	{1b, G (3, 0)}	{24b, Y (1, 3)}	20·12	
9	{1b, R (2, 6)}	All black	19·96	20·06

We noticed that in this case, when the black reached the bottom, the liquid ring disappeared, and we are, therefore, inclined to attribute the last reading to some slight change in Y.

The decrease in the value of δ indicates a bulging of the cylinder, *i.e.*, a falling-off in its surface tension.

$$dT/T = -0{\cdot}030/3{\cdot}7 \times 0{\cdot}98 \equiv -0{\cdot}83 \text{ per cent.}$$

Experiment XVI.

The first two readings were as follows :—

t	C_s	C_c	δ	Mean.
m.				
10	Colourless	R (6, 0)	19·85	
15	,,	R (6, 0)	19·78	19·81

The films were separated. At 27^m the sphere was flooded. The stop-cock was opened at 49^m, and the following values of δ were obtained, beginning at 51^m :—

$$\delta = 19{\cdot}51,\ 19{\cdot}54,\ 19{\cdot}40,\ 19{\cdot}46.$$

The last four readings were as follows :—

t	C_s	C_c	δ	Mean.
h. m.				
1 7	{1b, BR (3, 5)}	{24b, O (1, 5)}	19·55	
12	{1b, Y (3, 5)}	All black	19·65	
17	{1·5b, G (3, 2)}	,,	19·49	
22	{2b, B (3, 7)}	,,	19·55	19·56

In this case the liquid ring did not disappear when the black reached the bottom.

$$dT/T = -0{\cdot}025/3{\cdot}7 \equiv -0{\cdot}68 \text{ per cent.}$$

Experiment XVII.

Two measurements were made, when the sphere was colourless, and the cylinder also, with the exception of about 2 mm. at the top, which showed rings of colour.

t	δ	Mean.
m.		
5	20·67	20·68
8	20·69	

The films were then separated, and the sphere was flooded. At 40^m the stop-cock was again opened, and the following readings, beginning at 44^m, gave

$$\delta = 20{\cdot}58,\ 20{\cdot}69,\ 20{\cdot}71,\ 20{\cdot}66,\ 20{\cdot}66.$$

The last two readings were as follows :—

t	C_s	C_e	δ	Mean.
m.				
58·5	{2*b*, Y (2, 0)}	{17*b*, Y (1, 5), R (1, 5)}	20·76	20·77
60·0	{2·5*b*, G (2, 5)}	{18*b*, Y (1, 5)}	20·79	

$$dT/T = 0{\cdot}009/3{\cdot}7 \times 0{\cdot}76 \equiv 0{\cdot}32 \text{ per cent.}$$

Experiment XVIII.

Three measurements were made when the films were colourless, or very nearly so.

t	δ	Mean.
m.		
7	20·69	
10	20·88	20·78
12	20·78	

The following readings gave $\delta = 20{\cdot}61,\ 20{\cdot}85,\ 20{\cdot}76,\ 20{\cdot}80,\ 20{\cdot}71,\ 20{\cdot}84,\ 20{\cdot}96,\ 20{\cdot}75,\ 20{\cdot}97$.

The last two observations were as follows :—

t	C_s	C_e	δ	Mean.
h. m.				
1 11	{0·75*b*, G (4, 5)}	{18*b*, G (2, 5)}	21·12	21·15
1 18	{0·75*b*, BR (3, 8)}	{22*b*, W (1, 5)}	21·18	

$$dT/T = 0{\cdot}037/3{\cdot}7 \times 0{\cdot}87 \equiv 1{\cdot}15 \text{ per cent.}$$

Experiment XIX.

When both films were colourless the following observations were made :—

t	δ	Mean.
7	19·08	
8	19·25	19·16

Successive readings gave afterwards

$$\delta = 19{\cdot}08,\ 19{\cdot}03,\ 19{\cdot}12,\ 19{\cdot}19,\ 18{\cdot}94,\ 19{\cdot}26, \text{ and } 19{\cdot}14.$$

The last two observations were as follows :—

t	C_r	C_r	δ	Mean.
47	{1b, G (5, 6)}	{23b, Y (1, 5)}	19·29	
51	{1b, G (5, 6)}	{25b, W (1, 5)}	19·08	19·18

Hence

$$dT/T = 0{\cdot}002/3{\cdot}7 \times 0{\cdot}98 \equiv 0{\cdot}06 \text{ per cent.}$$

Results of the Experiments.

The above experiments give no indication of a continuous change in the surface tension as the film becomes thinner. The differences observed are irregular and such as would arise from accidental causes.

Thus, in Experiment VII., the black film showed a deficiency of surface tension of 0·41 per cent. when balanced against a film which displayed the blue and green of the second order. In Experiment IX. the surface tension of the black film, also balanced against another, which showed the blue of the second order, was in excess by 1·5 per cent.

In all cases where the same experiment has furnished two results they make dT of opposite signs, thereby indicating that the difference between the two films is not due to the fact that one is thicker than the other.

The results are summed up in the following table :—

TABLE VIII.

No. of Experiment.	Average effective length of black in m.m.	$\frac{dp}{dp'}$	$\frac{dT}{T}$ (in percentages).
VIII.	23·5	0·98	−0·41
IX.	18·0	0·90	−1·18
X.	17·0	0·80	1·50
	11·0	0·60	−0·15
XI.	16·5	0·72	−0·28
XII.	20·0	0·89	0·05
	21·0	0·90	−0·49
XIII.	16·0	0·77	−1·59
	16·0	0·77	0·60
XIV.	18·0	0·0	0·0
XV.	24·0	0·98	−0·83
XVI.	26·0	1·00	−0·68
XVII.	17·5	0·76	0·31
XVIII.	20·0	0·87	1·15
XIX.	24·0	0·98	0·06
Mean	..	..	−0·13

The experiments are not all of the same value. Those in which dp/dp' is most nearly unity are better than those in which it is small. So many other considerations, however, affect the weight of the observation, *e.g.*, the time during which it lasts, &c., &c., that we have made no attempt to discriminate between them.

The range of the above observations is from the green of the fifth order to the black of the first. In Experiment XIX. a film almost entirely black was balanced against another, the colour of which was G(5, 6). This tint corresponds to a thickness of about 1350×10^{-6} mm. The value of dT/T was in this case unusually small.

In all these experiments the black extended over the film quietly and regularly. At times the black is formed with something like a convulsion. Not only does it spread with extraordinary rapidity, but the edge is violently disturbed and large patches rise through the coloured part of the film. Whenever this occurs the film breaks before long, but in four cases we were able to obtain measurements before rupture. We are not able to produce this phenomenon at will, but the few observations we have been able to make on it are in agreement among themselves, and we therefore describe them, as we think they make it necessary to generalise our results with very great caution.

In each case we give the colours and value of δ from observations taken immediately before and after the sudden increase of the black surface. A rapid increase was observed in Experiment XIV., but the general character of the mode of formation was normal. The increase tabulated in the last column is the change in δ taken positive when the thinner film bulged.

Experiment XX.

t	C_r	C_l	δ	Increase.
m.				
21	{5*b*, O (1, 3)}	{7*b*, W (1, 7)}	−0·25	
24	{10*b*, W (1, 7)}	{20*b*, W (1, 5)}	−0·60	+0·35

Experiment XXI.

t	C_r	C_l	δ	Increase.
m.				
16	{2·5*b*, R (1, 0)}	{2*b*, R (1, 0)}	+0·12	
20	All black	{13*b*, Y (1, 0)}	+0·65	+0·53

Experiment XXII.

t	C_r	C_l	δ	Increase.
m.				
20	{15*b*, Y (1, 3)}	{4*b*, B (2, 0)}	−0·27	
24	All black	{5*b*, Y (1, 8)}	+0·48	+0·75

Experiment XXIII.

t	C_r	C_l	δ	Increase.
m.				
22	{3*b*, W (1, 5)}	{0·5*b*, O (1, 5)}	+0·40	
25	{23*b*, O (1, 5)}	{3*b*, W (1, 5)}	+1·01	+0·61

In all these cases there was a sudden increase in the diameter of the film which thinned most rapidly in the peculiar way just described. We do not wish to lay too much stress on experiments which can only be produced by accidental circumstances, but they certainly suggest the inquiry whether one of those circumstances may not be something abnormal in the thickness of the black. If its surface tension were really lower than usual, the rapid formation of the black surface and the convulsions by which it is accompanied would be intelligible.

We therefore think it best to state the general result of our inquiry as follows:—

When the black part of a soap film forms in a normal way, spreading slowly over the surface, no evidence of any change in surface tension dependent on the thickness of the film is furnished by a direct comparison of the tensions of thin and thick films over a range of thickness extending from 1350 *to* 12 *millionths of a millimetre.*

Our observations also justify the statement that—

This conclusion is based upon a method of experiment by which a change of one-half per cent. in the value of the tension must have been detected, had it existed.

The further question remains, can any inference be drawn from these observations as to the magnitude of the radius of molecular action?

On this point we venture to think that, if the magnitude of that length is to be settled merely by reference to the equality of the surface tensions of thick and thin films, we have found evidence in favour of its being less than 6×10^{-6} mm. (half the thickness of an ordinary black film), far stronger than that on which the frequently quoted statement of PLATEAU that it is less than 50×10^{-6} mm. is based.

Our observations have been more numerous and more sensitive than his, and on the other hand we have used films much thinner than those employed by him, or by VAN DER MENSBRUGGHE in his confirmatory observations.

We must admit, however, that we are unwilling to draw any such conclusion until an explanation of the discontinuity in the thickness at the edge of the black is forthcoming, which is not incompatible with it. The regular recurrence of that phenomenon suggests that it must in some way be connected with the extreme tenuity of the film on one side of it. We thought it well that PLATEAU'S method should be pushed (as we venture to think we have pushed it) beyond the point where he and VAN DER MENSBRUGGHE left it, but the fact that it has led to a negative result does not diminish the significance of the sharp edge of the black.

As far as we are aware, no suggestion has yet been made as to the cause of the discontinuity. We have necessarily had our attention directed to it, and, though our views are at present little more than speculations, yet it may perhaps conduce to a useful discussion if we suggest a cause which might, we think, furnish a satisfactory solution of the difficulty.

Suggestion as to the Cause of the Discontinuity at the Edge of the Black Part of a Soap Film.

It may be convenient if we collect together the various facts which constitute all that is certainly known as to the behaviour and constitution of a black film.

1.* The black part of a soap film in equilibrium, and not subjected to the action of an external force, is always separated from the rest of the film by a clearly defined boundary.

A gradual transition from the black to the thicker parts of the film is only observed either in a film which thins rapidly and breaks soon (when it is a very transient phenomenon) or when an electric current is passed through the film.

2. The discontinuity in the thickness indicated at the boundary varies very much

* 'Roy. Soc. Proc.,' vol. 26, p. 334.

on different occasions. We have seen black at the top of a film of which the rest was colourless, and therefore probably 250 times thicker than the black part.

3.* The thickness of the black in different films varies between narrow limits only, viz., from 7 to 14 millionths of a millimetre.†

4.* In the same film the thickness of the black remains constant, at all events from a short time after its first formation.

5. No certain difference of surface tension can be detected by direct experiment between a persistent black film and a film more than one hundred times thicker.

6.‡ If an electric current of sufficient intensity is passed through a film which is partly black, the boundary becomes ill-defined, and there is a gradual transition from the thickness of the coloured to that of the black part of the film.

7. When the current is broken the grey colour which has bridged over the gap between the black and the rest of the film disappears, and the definite boundary, indicating a discontinuity in thickness, is re-established. This operation occupies only a few seconds.

The existence of the discontinuity and statements (6) and (7) can best be explained if we suppose that for certain thicknesses intermediate between the black and the white of the first order the film is in a state of instability. It may be maintained in this state by the application of an external force, but spontaneously abandons it as soon as the constraint is removed.

The observations recorded in this paper prove that, under ordinary circumstances, the black and coloured films have the same surface tension to within 0·5 per cent. [The re-establishment of the discontinuity between the black and coloured parts of the films, referred to in statement (7) above, is not instantaneous, but takes place in from ten to fifteen seconds. If, then, the force which produces this result is a difference of surface tension, it must be very small. We are not, therefore, compelled to suppose that the viscosity of the black is great, as would have been necessary, had a measurable difference of surface tension been detected.—Nov. 16, 1886.] On the other hand, there is evidence that a difference of surface tension, so small that it would probably be impossible to measure it, may produce movements in the surface of a moderately thick bubble.

Thus Félix Plateau§ found that the slight elevation of temperature caused by bringing the finger near a film produced movement, and every one is acquainted with the violent currents seen in a newly-formed bubble.

If this be admitted, the discontinuity at the edge of the black could be explained if we suppose that the surface tension has a critical value in the grey part of the film,

* 'Phil. Trans.,' vol. 174 (Pt. 2, 1883), p. 645.

† In fifteen out of eighteen measurements made on single films the results lay between 10·7 and 13·4 millionths of a millimetre.

‡ 'Phil. Mag.,' vol. 19, 1885, p. 94.

§ 'Statique des Liquides,' vol. 1, p. 294.

which is always missing if the film is in equilibrium (*i.e.*, not thinning very rapidly) and under the action of its own molecular forces only.

In discussing such a theory, it is of primary importance to determine whether a critical value of the surface tension is physically possible.

It is usual to assume that if the thickness of the film is less than that of two surface layers the tension must be less than that of the liquid in mass.

Thus PLATEAU* argues that in such a film the two layers "doivent nécessairement exercer des actions moins fortes, et conséquemment la somme de celles-ci, c'est-à-dire la pression sur l'air intérieur, doit être plus petite que ne l'indique la formule."

Such arguments are based upon the assumption that the force in play between neighbouring molecules is attractive at all distances. The force exerted on a molecule by its neighbours must no doubt be, on the whole, attractive, and if the film is thin enough the surface tension must diminish. It is, however, usual, in dealing with other problems in molecular mechanics, to assume the existence of repulsive forces acting at all events between certain limits as to distance. We need only refer to the well-known instance of some forms of the dynamical theory of gases, and to the fact that the most recent theory of the nature of matter lately put forward by Professor OSBORNE REYNOLDS† leads to the conclusion that the force between two molecules which approach each other is alternately attractive and repulsive.

If such an alternation really takes place the surface tension of a thinning film will alternately increase and diminish. This is clearly pointed out by MAXWELL in his article on "Capillary Action" ('Encycl. Brit.'), in which he says:—"The force between the particle and the liquid is certainly on the whole attractive, but if between any two small values of c [the distance of a particle from the surface] it should be repulsive then for films whose thickness lies between these values the tension will increase as the thickness diminishes, but for all other cases the tension will diminish as the thickness diminishes."

It appears, therefore, that a critical value of the surface tension is not inconsistent with received views as to the nature of the forces in play between the particles of a liquid, and if it existed it would produce phenomena similar to those observed at the edge of a black film. In all cases we have supposed that the surface tension finally diminishes when the film is very thin, and is constant when it is very thick. If it has a critical value between these two states, the two simplest suppositions which we can make are (1) that it has a maximum and a minimum value, (2) that it has a maximum value only.

(1.) The first of these, which is the more complex supposition, affords in some respects the better explanation of the behaviour of the black part of the film. Let us suppose that the change in the surface tension as the film becomes thinner is represented by the annexed curve (fig. 1).

* 'Statique des Liquides,' vol. 1, p. 205.

† 'Phil. Mag.,' vol. 20, 1885, p. 479.

When the film became a little thinner than the degree of tenuity represented by P a state of unstable equilibrium would be reached. The thicker parts of the film would tear the thinner parts asunder, but rupture would be prevented by the fact that when the point of minimum surface tension was passed the resistance would increase until, when the thickness corresponding to Q was attained, the tensions of the thin and thick parts of the film would be the same and equilibrium would again be possible. The equilibrium would be stable, because if the film became thinner its surface tension would increase, and it would tend to contract and thus to thicken.

Such a theory would explain the approximately constant thickness of the black, on the ground that it would be impossible for it to exist except at a thickness such that its surface tension was nearly equal to that of the liquid in mass. When the film is very thin its tension would probably alter rapidly with the thickness, and thus the latter would be confined within narrow limits.

These considerations would not apply to the liquid on the thicker side of the boundary. Its surface tension being independent of the thickness, the magnitude of the latter would be indifferent.

If, as is possible by means of the electric current, the discontinuity was filled up, by liquid being forced into the black space, the equilibrium would be unstable. As soon as the external force was removed the "grey," intermediate in thickness to the black and coloured films, would be absorbed or stretched until it became black.

(2.) If the simpler supposition is made that there is only one alternation of repulsion and attraction to be considered, the law of the change of the surface tension might be represented as in fig. 2.

In this case there would be a preliminary difficulty in explaining the formation of the black, inasmuch as when the point P′ was reached any further decrease in thickness would be resisted by an increase in tension. As a matter of fact, however, the black is generally first formed in small specks, and if, owing to a sudden disturbance, the film were thinned to below the maximum of surface tension, equilibrium would be possible if the thickness were that corresponding to Q′. The parts of the film of thickness intermediate to P′ and Q′ would shrink, and an apparent discontinuity would be established.

This supposition is also less satisfactory than the last, inasmuch as the equilibrium would be unstable. An accidental thinning below Q′ would produce rupture. The fact that this does not occur at once might be explained by supposing that the surface viscosity would play a more important part as the film became thinner.

We need hardly point out that the best way of attempting to test such theories would be to measure the surface tension of a grey film. Unfortunately, however, such a film can only be obtained when the electric current is passing, and the very fact that the grey displays itself is a proof that changes in the surface are going on which may complicate the conclusions to be drawn from the experiment.

We have frequently tried to secure satisfactory observations of this kind, but have not succeeded in obtaining any in which we could with certainty discriminate between the possible effects of the renewal of the surface and a real change in surface tension due to a change in the thickness. Experiments XX. to XXIII. are the only observations we have made in which an apparent change of surface tension could not be accounted for by the change of surface. In these observations, however, the latter cause would have produced a contraction instead of a bulging of the black film, and they cannot therefore be set down as particular cases of the phenomenon studied in the earlier part of the paper.

On one occasion also, when we thickened a black film by passing a current through it till it became grey, no appreciable change of the diameters took place. This is, perhaps, on the whole in favour of the first of the above hypotheses, as the renewal of the surface and the decrease of the "specific" tension would act in opposite directions.

Our observations do not, therefore, throw much light on this intricate question. Experiments which might do so would have to be made with far more complicated apparatus, and with very uncertain prospect of success. Nothing that we have noticed, however, negatives the view that the discontinuity at the edge of the black is connected with a critical value of the surface tension, caused by an alternation from attraction to repulsion in the inter-molecular forces, if the variations in the magnitude of the surface tension for thicknesses $> 12 \times 10^{-6}$ mm. are small. On the other hand,

we are not aware of any other suggested explanation. That which we offer is based upon a view of the nature of the inter-molecular forces in a liquid which is frequently neglected, but which was recognised by MAXWELL as tenable. The first hypothesis offers, at all events, a plausible explanation of the discontinuity, of the persistence of the film in spite of the sudden decrease in thickness, and of the constant thickness of the black portion of the film.

POSTSCRIPT.

(Added June 18, 1886.)

Sir WILLIAM THOMSON has informed us in the course of conversation that he suggested a minimum of surface tension as the cause of the sharp edge of the black in a lecture delivered at the Royal Institution on Friday, January 29, 1886. The discourse has not yet been published, but we are, by his kindness, enabled to give the passage in question from a proof:—

"Well-known phenomena of bubbles, and of watery films wetting solids, make it quite certain that the molecular attraction does not become sensible until the distance is much less than 250 micro-millimetres. From the consideration of such phenomena, QUINCKE ('Poggendorff, Annalen,' 1869) came to the conclusion that the molecular attraction does become sensible at distances of about 50 micro-millimetres. His conclusion is strikingly confirmed by the very important discovery of REINOLD and RÜCKER that the black film, always formed before an undisturbed soap bubble breaks, has a uniform or nearly uniform thickness of about 11 or 12 micro-millimetres. The abrupt commencement and the permanent stability of the black film demonstrate a proposition of fundamental importance in the molecular theory. The tension of the film, which is sensibly constant when the thickness exceeds 50 micro-millimetres, diminishes to a minimum, and begins to increase again when the thickness is diminished to 10 micro-millimetres. It seems not possible to explain this fact by any imaginable law of force between the different portions of the film supposed homogeneous, and we are forced to the conclusion that it depends upon molecular heterogeneousness."

It is, perhaps, unnecessary to say that we were unaware of Sir W. THOMSON's views till within the last few days, or that we are glad to find that our opinion has the support of so high an authority.

XVII. *On the Blood-Vessels of Mustelus Antarcticus: a Contribution to the Morphology of the Vascular System in the Vertebrata.*

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[Plates 34—37.]

Contents.

	page.
Introductory	685
I. The Arteries.	
1. The ventral aorta and afferent branchial arteries	686
2. The efferent branchial arteries, epibranchial arteries, and dorsal aorta	687
3. The peripheral arteries	692
A. Arising from the efferent branchial arteries	693
B. Arising from the dorsal aorta	696
II. The Veins	702
A. System of the sub-intestinal vein	703
i. Renal portal section	703
ii. Hepatic portal section	705
B. System of the cardinal veins	710
C. „ „ lateral „	716
D. „ „ coronary „	720
E. „ „ cutaneous „	720
General considerations	721
Methods of investigation	722
Bibliography	725
Description of plates	726

The present inquiry was undertaken with a view of settling, if possible, one or two doubtful points in our knowledge of the vascular system of Fishes, and of giving, in an accessible form, a fairly complete account of the blood-vessels of a typical Selachian, since, as far as I am aware, this has not yet been done. The arteries and veins of the Skate are figured, for the most part very accurately, by Monro (16)*; the arteries of

* The figures between brackets refer to the bibliographical list at the end of the paper (p. 725).

Raja and *Torpedo* are described and figured in detail by Hyrtl (11), and there are good general accounts of the vascular system in both orders of *Plagiostomi* in the works of Müller (17), Stannius (25), and Milne Edwards (14). By all these authors, however, several points of considerable importance are either missed or but slightly referred to, while others are more or less inaccurately described.

In all the more modern text-books of comparative anatomy to which I have had access the vascular system of Fishes is very meagrely treated, the manuals of Owen (19), Huxley (10), Claus (4), Gegenbaur (7), Rolleston (24), Macalister (13), Günther (8), and Wiedersheim (26), adding little or nothing to the excellent though brief account in Stannius's handbook just referred to. Indeed, the only general work I have seen which gives any important information not to be found in Stannius is Milne Edwards's 'Leçons,' in which the description of the vascular system, and especially of the arteries of Fishes, is full, and, like everything else in that invaluable book, admirably clear.

Great attention has been directed of late years to the Selachians, since it is evident that they are the nearest existing representatives of the ancestral stock of the higher or gnathostomatous Vertebrata. Their skeleton, muscles, nervous system, urinogenital organs, and general embryology have all been made the subjects of special study within the last few years, but, as far as I am aware, the only important contributions to our knowledge of their vascular system, since the publication of Milne Edwards's third volume, are Jourdain's discovery of their renal-portal system (12) and Balfour's researches on the development of their veins (1 and 2).

I. The Arteries.

1. *The Ventral Aorta and Afferent Branchial Arteries.*

   a. Ventral aorta.

   Branchial artery, Monro (16), Owen (19), Gegenbaur (7).
   Arteria branchialis, Müller (17).
   Artère branchiale, Milne Edwards (14).
   Kiemenarterienstamm, Stannius (25).
   Cardiac aorta, Huxley (10).
   Aorta, aorta ascendens, cardiac aorta, Claus (4).
   Ventral aorta, branchial artery, Balfour (2).
   Hauptstamm der Kiemenarterie, Wiedersheim (26).

   b. Afferent branchial arteries.

   Branches of branchial artery, Monro (16).
   Kiemenarterien, Hyrtl (11).
   Artères branchiales propres, Milne Edwards (14).
   Branchial arteries, Claus (4).

The ventral aorta (Plate 34, figs. 1 and 2; Plate 37, fig. 18; *V. Ao.*) of *Mustelus antarcticus* presents no points of special interest. Immediately after its origin from the short conus arteriosus (*C. Art.*) it gives off on each side a short outwardly-directed trunk which almost immediately divides into the afferent arteries of the fourth and fifth gills (*Af. br. A.* 4 and *Af. br. A.* 5), that is, those borne by the third and fourth branchial arches respectively. Passing forwards, the ventral aorta gives off a single trunk on either side to the third gill (*Af. br. A.* 3), and, at its anterior termination, divides into two vessels, each of which passes directly outwards, soon dividing into the first and second afferent branchial arteries (*Af. br. A.* 1 and *Af. br. A.* 2). Of these the former (*Af. br. A.* 1) is distributed to the hyoidean hemibranch,* the latter (*Af. br. A.* 2) to the gill of the first branchial arch or first complete gill.

The precise position of the afferent artery with regard to the various constituents of the gill is best seen in transverse section and will be described immediately (p. 688).

2. *The Efferent Branchial Arteries, Epibranchial Arteries, and Dorsal Aorta.*

a. Efferent Branchial Arteries.

Branchial Veins, Monro (16), Owen (19), Huxley (10), Gegenbaur (7), Rolleston (24), Macalister (13), Balfour (2).

Kiemenvenen, Hyrtl (11), Stannius (25).

Venæ branchiales, Müller (17).

Artères epibranchiales (in part), Milne Edwards (14).

Epibranchial arteries, branchial veins, Claus (4).

b. Epibranchial Arteries.

Aortenwurzeln, Hyrtl (11), Dohrn (5).

Radices aortæ, Wiedersheim (26).

Venæ branchiales communes, Müller (17).

Artères epibranchiales (in part), Milne Edwards (14).

c. Dorsal Aorta.

Trunk of descending aorta, Monro (16).

Aorta descendens, Müller (17).

Aorta, Hyrtl (11), Stannius (25), Owen (19).

Aorta abdominalis, Wiedersheim (26).

Aorta dorsale, Milne Edwards (14).

Dorsal aorta, Huxley (10).

Dorsal aorta, aorta descendens, Claus (4).

As usual in Plagiostomes, which in this, as in many other points of structure, differ from all other amphirhinous fishes, there is an efferent branchial artery for each

* I have adopted (in 23) this very convenient word as the equivalent of Milne Edwards's "demi-branchie." It means the whole set of gill-filaments on one side of a gill pouch in a Cyclostome or an Elasmobranch. A complete gill, composed of two hemibranchs, may be appropriately termed a *holobranch*.

hemibranch or half-gill, and consequently nine such arteries in all (Plates 34 and 35, figs. 1, 2, and 6, *Ef. br. A.* 1–9), one for the hyoidean hemibranch (*Ef. br. A.* 1) and two for each of the holobranchs or complete gills borne by the first four branchial arches. The anterior artery of each holobranch (*Ef. br. A.* 2, 4, 6, and 8) is markedly larger than the posterior.

These vessels are usually, but very incorrectly, called *branchial veins*. It would be quite as justifiable to speak of the *portal artery* as to call these obviously arterial vessels veins; a capillary system may be interposed in the course either of an artery or of a vein, but this does not make the efferent trunk in the one case a vein, nor the afferent trunk in the other case an artery.

The relations of the afferent and efferent branchial arteries to one another and to the other structures of the gill are well seen in transverse section. Fig. 5 represents a section of the second holobranch taken through the dorsal end of the cerato-branchial. It shows the crescentic transverse section of the cerato-branchial (*C. br.*) and the reniform section of the extra-branchial (*Ex. br.*); one of the cartilaginous branchial rays (*Br. r.*) is cut longitudinally, as also are two branchial filaments (*Br. fil.*) belonging to the anterior and posterior hemibranchs respectively. A muscle which flexes the epi- upon the cerato-branchial, and may be called the M. epi-cerato-branchialis (*M. ep. c. br.*), is shown in transverse section, and the great radiating muscle of the gill (*M. rad.*) in longitudinal section. It is seen that the afferent artery (*Af. br. A.* 3) lies between the radiating muscle and the branchial ray, and immediately external to the branchial arch; while of the two afferent vessels, one (*Ef. br. A.* 4) is situated a short distance anterior to the radiating muscle, the other (*Ef. br. A.* 5) immediately posterior to the branchial ray. The two branches of the vagus (nv^1, nv^2) are situated immediately cephalad of the radiating muscle and caudad of the branchial ray respectively.

The radiating muscle may be taken as indicating the position of the head cavity in the embryonic gill-arch, so that, as DOHRN has shown (5), the afferent and the posterior efferent arteries lie caudad, the anterior efferent artery cephalad of the head cavity.

The efferent branchial arteries unite with one another in a characteristic manner (Plate 35, fig. 6), the arteries of the anterior and posterior hemibranchs of each gill-pouch—not of each gill—anastomosing both dorsally and ventrally so as to form a complete loop. Thus the artery of the hyoidean hemibranch (*Ef. br. A.* 1) unites with that of the anterior hemibranch of the first branchial arch (*Ef. br. A.* 2); the artery of the posterior hemibranch of the first branchial arch (*Ef. br. A.* 3) with that of the anterior hemibranch of the second arch (*Ef. br. A.* 4), and so on. Four arterial loops are thus formed on each side, the ventral ends of which are united by a longitudinal commissure (*com.* 1).

Further, the two arteries of each holobranch are placed in communication at about the middle of their length by two short transverse anastomoses (*com.* 2):

it is through these that the ninth or last efferent artery (*Ef. br. A.* 9) pours its blood into the eighth; it forms, of course, no loop, since the fifth branchial arch is gill-less.

From the dorsal end of each arterial loop an *epibranchial artery** (*Epbr. A.* 1–4) is continued backwards and inwards: by uniting with one another successively in pairs these four trunks form the *dorsal aorta* (*D. Ao.*). As seen in fig. 6 the first pair (*Epbr. A.* 1) unite in the middle line and form a short median trunk; this is joined by the second pair, a thicker trunk being the result; then the third pair join in like manner, and finally the fourth (*Epbr. A.* 4).

The regular manner in which the epibranchials of *Mustelus* unite is worthy of notice as a clearly primitive arrangement. In the Rays (23, fig. 20), *Holocephali* (Plate 36, fig. 17), and *Teleostei* (23, fig. 32), two or more of the epibranchials of the same side unite with one another before joining with those of the opposite side.

In the embryo the aortic arches are continued directly from the ventral to the dorsal aorta. In *Holocephali* (Plate 36, fig. 17) and *Teleostei* there is only one efferent artery to each gill, corresponding to the anterior of the two efferent arteries in the plagiostome holobranch. This is very evident in *Callorhynchus*, in which the single efferent artery of each gill is always cephalad of the corresponding afferent trunk. These facts tend to confirm the opinion to which one is led by the simple inspection of the parts in the adult *Mustelus* (compare figs. 6 and 17); namely that the anterior efferent artery of each holobranch (*Ef. br. A.* 2, 4, 6, and 8) is to be looked upon as its primary revehent trunk and as strictly continuous with the corresponding epibranchial artery, the posterior efferent artery being a secondary vessel which debouches not into the primary trunk of its own, but into that of the next following gill.

The foregoing paragraph was originally written fully a year ago, before I had seen Dohrn's paper on the "Development of the Gill-Arches" referred to above (5). It is satisfactory to find that the views just enunciated are largely confirmed by embryology. Dohrn shows that the aortic arch in *Pristiurus* is at first single, lying caudad of the head cavity. When the gill filaments are developed, accessory blood-currents (Nebenströme) are set up, and give rise first to the posterior and then to the anterior efferent artery; both of these open at first into the dorsal region of the aortic arch, which afterwards becomes the epibranchial artery. Gradually the afferent trunk, formed from the ventral region of the original aortic arch, loses its connexion with the dorsal portion, so that afferent and efferent blood-streams, communicating only by capillaries, are now definitely established. Then, as development goes on, the anterior efferent artery increases at the expense of the posterior; much of the blood from the

* Milne Edwards uses this word so as to include the corresponding efferent branchial arteries, but I think there can hardly be two opinions as to the convenience of restricting it to that portion of the efferent series of vessels lying dorsad of the gill.

latter is poured into the anterior trunk by the transverse commissures (*com.* 2), and, finally, the posterior efferent artery loses altogether its connexion with the aortic arch of its own branchiomere, and acquires a secondary one with the succeeding arch.

From the above considerations one is led to look upon the connexion of the first (hyoidean) efferent artery (*Ef. br. A.* 1) with the first epibranchial artery (*Epbr. A.* 1) as a secondary one, and it then becomes a matter of considerable interest to find in *Mustelus antarcticus* distinct remains of the dorsal portion of the hyoidean aortic arch, and of its connexion with the dorsal aorta.

From the dorsal end of the first efferent branchial artery arises a large vessel, the *posterior carotid artery* (*Post. car. A.*). This trunk passes forwards and inwards, ventrad of the proximal end of the hyomandibular, to the ventral surface of the auditory capsule, and then through a foramen in the skull floor to the orbit. Its further course will be described hereafter (p. 695), the point of interest for the present purpose is that shortly before entering the foramen just mentioned—at the point x in fig. 6—it gives off a very slender vessel (y), which passes backwards and inwards along the ventral aspect of the skull and vertebral column, and joins with its fellow of the opposite side to form a delicate longitudinal median trunk (z), which is continued backwards to the junction of the first pair of epibranchial arteries.

I think there can be no doubt that the posterior carotid artery, from its origin to the point x, together with its backward continuation y, represents the dorsal portion of the hyoidean aortic arch, or *hyoidean epibranchial artery*, the altered direction of the vessels being accounted for by the changed position of the hyoid arch.

The median trunk z is, as obviously, the actual anterior portion, or what may be called the inter-hyoidean section, of the dorsal aorta. It has clearly nothing whatever to do with the *arteria vertebralis impar* of Myxinoids (17) which it resembles at first sight, since the latter is a secondary forward prolongation of the aorta altogether cephalad of the gills. As this anterior portion of the dorsal aorta undergoes complete atrophy—if indeed it ever exists—in the Rays as well as in *Holocephali*, it is a matter of some interest to find it persisting in a typical Selachian, and one is led to inquire whether it is actually absent in those forms the arteries of which have been described, or whether it has hitherto been overlooked. I can only say that I have failed to find any mention of it.

The course of the mandibular aortic arch is by no means so clear and can only be decided by a detailed study of development. From about the middle of the first efferent branchial artery is given off the *pseudobranchial artery* (fig. 6, *Psbr. A.*), which passes forwards external to the articulation of the hyomandibular and epihyal, and then extends forwards, upwards, and inwards to the spiracular pseudobranch (*Psbr.*), the rudiment of the mandibular gill of the embryo. The blood is collected from this organ by the *anterior carotid artery* (*Ant. car. A.*) which is continued inwards and forwards to the orbit, passes through a foramen in the side wall of the skull, and becomes the cerebral artery (*Cereb. A, vide infra*, p. 694): while

imbedded in the perichondrial lining of the cranial cavity, it sends off a branch (*w*) which passes inwards and backwards, crossing its fellow of the opposite side in the thickened perichondrium of the pituitary space, and finally anastomosing with the posterior carotid of the opposite side.

From the ventral end of the first efferent branchial artery springs the *mandibular artery* (figs. 2, 6, and 10, *Mandib. A.*), which extends upwards along the outer face of the hyoid arch and just internal to the mandible, thus running parallel to the efferent branchial arteries. Distally it breaks up into several branches at about the level of the pseudobranchial artery (fig. 10). In *Callorhynchus* the evidently homologous vessel (Plate 36, fig. 17, *Mandib. A.*) anastomoses with the proximal end of the anterior carotid—which, in the absence of a pseudobranchia, springs directly from the first efferent branchial artery—and appears from its position and relations to represent the mandibular aortic arch.

From the analogy of *Callorhynchus* it would seem, therefore, that the mandibular artery is to be looked upon as a part of the original mandibular aortic arch, the rest of the ventral portion of which, represented by the pseudobranchial artery, has acquired a secondary connexion, comparable to the transverse commissures (*com.* 2), with the first efferent branchial artery. Whether the anterior carotid, together with the anastomotic trunk *w*, has anything to do with the original connexion of the mandibular aortic arch with the dorsal aorta (*mandibular epibranchial artery*) it is of course impossible to decide from anatomical evidence alone.

The following table shows what I conceive to be the relation of the arteries of the adult to the embryonic aortic arches :—

In the Embryo.	In the Adult.	
	Dorsal Section.	Ventral Section.
1*st Aortic arch* (mandibular) . .	Commissure *w* (?), anterior carotid artery.	Pseudobranchial artery. Mandibular artery.
2*nd Aortic arch* (hyoidean) . . .	Vessel *y*; posterior carotid from origin to *x*.	1st afferent branchial artery. 1st efferent branchial artery.
3*rd Aortic arch* (1st branchial) . .	1st epibranchial artery . . .	2nd efferent branchial artery. 2nd afferent branchial artery. 3rd efferent branchial artery.
4*th Aortic arch* (2nd branchial) . .	2nd epibranchial artery . .	4th efferent branchial artery. 3rd afferent branchial artery. 5th efferent branchial artery.
5*th Aortic arch* (3rd branchial) .	3rd epibranchial artery . . .	6th efferent branchial artery. 4th afferent branchial artery. 7th efferent branchial artery.
6*th Aortic arch* (4th branchial) . .	4th epibranchial artery . .	8th efferent branchial artery. 5th afferent branchial artery. 9th efferent branchial artery.

3. *The Peripheral Arteries.*

The peripheral arteries of *Mustelus* may be classified as follows :—

A. Arising from the efferent branchial arteries.
   1. The pseudobranchial and anterior carotid arteries.
      a. The ophthalmic arteries.
      b. The cerebral arteries.
         α. The anterior cerebral arteries.
         β. The posterior cerebral arteries.
         γ. The myelonal artery.
   2. The posterior carotid arteries.
      a. The orbital arteries.
      b. The buccal arteries.
      c. The maxillo-nasal arteries.
      d. The rostral arteries.
   3. The mandibular arteries.
      a. The sub-mental arteries.
      b. The nutrient hyoidean arteries.

B. Arising from the dorsal aorta.
   4. The subclavian arteries.
      a. The hypobranchial arteries.
         α. The anterior lateral arteries.
         β. The coronary arteries.
         γ. The nutrient branchial arteries.
         δ. The pericardial artery.
      b. The brachial arteries.
   5. The cœliaco-mesenteric artery.
      a. The cœliac artery.
         α. The anterior gastric artery.
         β. The right hepatic artery.
         γ. The left hepatic and pyloric arteries.
         δ. The ventral gastric artery.
      b. The anterior mesenteric artery.
         α. The ventral intestinal artery.
         β. The intra-intestinal artery.
   6. The anterior spermatico-mesenteric artery.
      a. The anterior spermatic artery.
      b. The dorsal intestinal artery.
   7. The lieno-gastric artery.
      a. The pancreatic arteries.
      b. The dorsal gastric arteries.
      c. The splenic artery.

8. The oviducal arteries (in the female).
9. The posterior spermatico-mesenteric artery.
   a. The posterior spermatic arteries.
   b. The posterior mesenteric artery.
10. The renal arteries.
11. The dorso-lumbar (segmental) arteries.
12. The iliac arteries.
   a. The femoral arteries.
   b. The posterior lateral arteries.
13. The caudal artery.

A. *Arteries arising from the Efferent Branchial Arteries.*
   (1.) *The pseudobranchial and anterior carotid arteries.*
      a. The pseudobranchial artery.
         Arterienstamm der Spritzloch-Nebenkieme, Müller (17).
         Abführendes Gefäss der Spritzlochkieme, Hyrtl (11).
      b. The anterior carotid artery.
         Carotis anterior v. *Carotis interna anterior*, Müller (17).
         Carotis anterior, Stannius (25).
         Carotide interne, Milne Edwards (14).
         Zuführendes Gefäss der Spritzlochkieme, Hyrtl (11).

The course of these vessels has already been briefly described (p. 690). They are considered together because, although one is the afferent and the other the efferent trunk of the pseudobranch, they are practically continuous even in forms like *Mustelus* and *Raja*, in which the pseudobranch is present, while in those species—*e.g.*, *Callorhynchus* (fig. 17)—in which the mandibular gill completely disappears, they form a single vessel to which the name *anterior carotid* is applied.

The application of the name "carotid" to the cephalic arteries of fishes must of course be taken to imply nothing more than a general correspondence with the similarly-named vessels in the higher Vertebrata. In Sauropsida and Mammalia the external carotid is formed from the ventral ends, the internal carotid from the dorsal ends of the aortic arches, the former supplying the head generally, the latter the brain and eye. In Plagiostomes the artery here called posterior carotid is, from its mode of origin, better entitled to the name "internal" carotid than any other vessel, and the mandibular artery to the name "external" carotid. As regards their distribution, however, it will be seen that these two arteries jointly perform the function of external carotid, while that of internal carotid is assigned to the artery now under consideration, the "anterior" carotid of Müller.

These considerations make it desirable to drop the names "internal" and "external"

carotids, for the head arteries of fishes, altogether, and to follow MÜLLER in the use of the terms "anterior" and "posterior" carotid; at least until the embryology of the vessels is thoroughly made out in a large number of forms. At present any attempt to introduce a systematic nomenclature applicable to the whole of the Vertebrata could, it appears to me, only result in failure.

The pseudobranchial artery gives off no branches until it ramifies in the pseudobranch (Plate 35, figs. 6 and 10, *Psbr. A.*), from which the blood is collected by the anterior carotid artery (*Ant. car. A.*). This vessel passes inwards and forwards along the floor of the orbit, crossing ventrad of the posterior carotid (*Post. car. A.*). It sends off a small *ophthalmic artery* (*Ophth. A.*) to the eye, and then almost immediately enters the carotid foramen, an aperture in the skull wall about 5 mm. caudad of the optic foramen. Having entered the carotid foramen it passes obliquely inwards and forwards (fig. 6) through the thick perichondrial lining of the skull, is joined by the anastomotic branch *w* from the posterior carotid of the opposite side, and emerges into the cranial cavity as the *cerebral artery*.

The main *cerebral artery* (Plate 35, figs. 6 and 8, *Cereb. A.*) is a short trunk; it divides almost immediately, opposite the mesencephalon, into an anterior and a posterior cerebral.

The *anterior cerebral artery* (*Ant. cereb. A.*) passes forwards along the outer and ventral side of the prosencephalon, to which it sends a considerable branch, and finally breaks up into a brush-work of small arteries on the ventral surface of the olfactory lobe.

The *posterior cerebral artery* (*arteria profunda cerebri*, HYRTL, 11) gives branches to the diencephalon, mesencephalon, and cerebellum (*Post. cereb. A.*), and, passing to the ventral face of the medulla oblongata, unites with its fellow in the middle line, forming the median *myelonal artery* (*arteria spinalis inferior*, HYRTL, 11), which (*Myel. A*) is continued along the ventral face of the spinal cord.

According to HYRTL (11) the blood, in *Raja* and *Torpedo*, reaches the cerebral artery, not, as described above, by the anterior carotid, but by the anastomotic branch *w*, which he calls the internal carotid (*vide infra*, p. 695). Part of the blood from this trunk is then taken by the anterior carotid (zuführendes Gefäss der Spritzlochkieme) to the pseudobranch, and thence by the pseudobranchial artery (abführendes Gefäss der Spritzlochkieme) to the first efferent branchial artery. So that, according to HYRTL, the anterior carotid is the afferent, the pseudobranchial artery the efferent trunk of the pseudobranch.

It seems to me that the relative sizes of the vessels sufficiently disprove this theory as far as *Mustelus* is concerned, the anterior carotid being a comparatively large vessel, many times larger than the anastomotic branch *w*, which is supposed to supply both it and the cerebral artery. In *Callorhynchus*, again, in which the anastomotic branch *w* is absent, the anterior carotid (Plate 36, fig. 17, *Ant. car. A*) obviously becomes the cerebral (or more accurately cerebro-rostral) artery. So that I can see no reason for

supposing the embryonic blood-current to undergo the reversal which is pre-supposed by HYRTL's view of the circulation through the pseudobranch.

(2.) *The posterior carotid artery.*
Carotis posterior v. *Carotis interna posterior*, MÜLLER (17).
Carotis posterior, STANNIUS (25).
Carotide interne postérieure, MILNE EDWARDS (14).
Carotis communis plus *Carotis externa*, HYRTL (11).

The course of the posterior carotid has already been partly described (p. 690). From its origin from the first efferent branchial artery it passes forwards and inwards (Plate 35, fig. 6, *Post. car. A.*) along the ventral face of the hyomandibular and auditory capsule, and enters the orbit through a foramen in its cartilaginous floor. The anastomotic trunk *w*, already referred to (*Carotis interna*, HYRTL), is given off just before entering this foramen; it passes inwards and slightly forwards along a groove on the skull floor. The groove soon becomes a canal through which the vessel in question reaches the pituitary space, where, imbedded in the thickened perichondrium, it crosses its fellow of the opposite side and unites with the anterior carotid of the opposite side, as already described (p. 694). The anastomotic trunk from the left posterior carotid passes ventrad of that from the right: HYRTL found exactly the opposite arrangement in *Raja clavata*.

In HYRTL's paper (11) the posterior carotid, from its origin to the point of junction of the trunk *w*, is called the "common" carotid: the remainder of the artery is the "external" carotid, and the anastomotic branch *w*, the "internal" carotid. I have adopted these names in my "Zootomy" (23), but have now come to the conclusion, as stated above (p. 693), that it is advisable to discontinue their use. That they were ever employed is a striking instance of the danger of basing a nomenclature on the study of specialized forms.

Immediately after entering the orbit the posterior carotid gives off the *orbital artery* (Plate 35, figs. 6, 10, and 11, *Orbit. A.*). This sends off twigs to the posterior wall of the orbit, and then divides into two branches, a dorsal which supplies the superior rectus, superior oblique and inferior oblique muscles, and a ventral which supplies the internal and external recti. The inferior rectus apparently receives its blood supply from a special branch of the posterior carotid.

The posterior carotid then passes obliquely forwards and outwards, emerging from the orbit at its antero-ventral corner (fig. 10); it then gives off a large *buccal artery* to the M. adductor mandibulæ, and almost immediately divides into two trunks, the *maxillo-nasal artery* which passes forwards and downwards along the anterior edge of the upper jaw (pterygo-quadrate cartilage), and a *rostral artery* (*Rost. A.*) which goes at first forwards, upwards, and inwards, then forwards and inwards along the outer border of the lateral (paired) rostral cartilage.

3. *The mandibular artery.*

This (Plate 34, figs. 1 and 2, and Plate 35, figs. 6 and 10, *Mandib. A.*) is a considerable vessel springing from the ventral end of the first afferent branchial artery. It passes at first forwards, then outwards, upwards, and slightly backwards along the outer face of the hyoid arch, and just within the ventral border of the mandible. Close to its origin it sends off several small vessels (fig. 2, *Cor. mand.*) to the coraco-mandibularis muscle, and then a large branch, the *sub-mental artery* (*S. ment. A.*), which goes along the inner face of the mandible to the symphysis, also supplying the M. coraco-mandibularis. Distad of the origin of this latter vessel the mandibular artery gives off small branches at the angle of the mouth (*Add. mand.*) to the M. adductor mandibulæ, and nutrient arteries to the hyoidean hemibranch (*Nu. hy. A.*), and finally breaks up into a number of small twigs, which pass, for the most part, external to the pseudobranchial artery and supply the anterior part of the M. constrictor superficialis.*

B. *Arteries arising from the dorsal aorta.*

4. *The subclavian artery.*

Arteria subclavia, Hyrtl (11).

Artère claviculaire, Milne Edwards (14).

The subclavian arteries (Plate 34, figs. 1 and 2, and Plate 35, fig. 6, *Subcl. A.*) are comparatively small vessels, arising one on each side from the dorsal aorta immediately cephalad of the junction with it of the fourth pair of epibranchial arteries. Each subclavian passes outwards with a slight backward inclination, and, at about the junction of the cardinal with the precaval sinus (*vide infra*, pp. 728 and 729, figs. 1 and 2, *Card. S.*, *Pr. cav. S.*) turns forwards and downwards, and divides into two arteries, the brachial and the hypobranchial.

The origin of the subclavians from the dorsal aorta is worthy of notice, since in the higher Vertebrata the corresponding vessels always arise from an aortic arch.

The brachial artery

(*Brach. A.*) passes at first slightly forwards and downwards, then outwards through a foramen in the shoulder girdle, situated at about the level of the mesopterygium, to the pectoral fin, which it supplies.

* In the nomenclature of the muscles I follow Vetter, "Untersuchungen zur vergleichenden Anatomie der Kiemen- und Kiefermusculatur der Fische," 'Jenaische Zeitschrift,' vol. 8, 1874, p. 405.

The hypobranchial artery.

Figured, but not named, by Monro (16).

Ventrale Verlängerungen der Venen des zweiten Kiemensackes (in *Raja*), Hyrtl (11).

Grosse branche anastomotique, &c., Milne Edwards (14).

Hypobranchial arteries, Parker (23).

This remarkable system of arteries, forming an anastomosis between the efferent branchial and the subclavian arteries, was, as far as I am aware, first described by Monro in the skate, and was afterwards more accurately figured in the same type by Hyrtl. I have seen no account of any such vessels in Selachians; both Müller and Milne Edwards write as if they were confined to the Batoidei, and no mention of them is made in any of the text-books I have had the opportunity of consulting.

After separating from the brachial, the hypobranchial artery (Plate 34, fig. 2, *Hypbr. A.*) gives off a branch to the lateral body muscles, and then passes forwards and inwards through the ventral wall of the pericardial cavity, finally uniting with its fellow of the opposite side at about the level of the junction of the ventricle with the conus arteriosus. About midway between this junction and its origin from the subclavian each hypobranchial sends off a small *anterior lateral* artery (figs. 1 and 2, *Ant. lat. A.*), which accompanies the lateral vein and helps to supply the anterior body muscles.

The median trunk formed by the union of the paired hypobranchial arteries passes forwards and upwards amongst the muscles of the throat until it reaches the ventral aorta about 2 cm. cephalad of its origin from the conus arteriosus; it then breaks up (fig. 2) into a simple plexus, which is united by two pairs of commissural trunks with the longitudinal vessels (*com.* 1) uniting the ventral ends of the efferent branchial arteries. The anterior of these (*com.* 3) arises between the fourth and fifth efferent arteries and joins the median hypobranchial immediately ventrad of the origin of the third afferent branchial artery from the ventral aorta. The posterior commissure (*com.* 4) arises between the sixth and seventh efferent arteries and joins the hypobranchial at the level of the origin of the fourth and fifth afferent arteries. Cephalad of the anterior of these commissural trunks the median hypobranchial artery is continued forwards as a slender vessel, which, at the level of the anterior end of the ventral aorta, breaks up into a number of branches distributed to the M. coracomandibularis.

The coronary arteries

(Plate 34, fig. 2, *Cor. A.*) are two in number, and arise from the azygos hypobranchial artery close to the posterior pair of commissures (*com.* 4) from the efferent branchial arteries. The two coronaries pass backwards along the ventral face of the conus arteriosus, the right soon dividing into two, and branch out over the ventricle.

I have found no trace of a posterior coronary artery such as exists in the skate (11 and 23).

The nutrient branchial arteries.
Bronchialarterien, MÜLLER (17).

The hyoidean hemibranch, as stated above (p. 696), is supplied by several twigs of the mandibular artery. The first holobranch, or complete gill, receives its nutrient artery (*Nu. br. A.* 1) from the longitudinal commissure between the second and third efferent branchial arteries. The nutrient artery of the second holobranch (*Nu. br. A.* 2) arises from the azygos hypobranchial close to its junction with the anterior pair of commissures (*com.* 3) from the efferent branchial arteries; that of the third complete gill (*Nu. br. A.* 3) springs from the azygos hypobranchial near its junction with the posterior pair of commissures (*com.* 4).

The nutrient artery of each holobranch passes along the anterior border of the corresponding extra-branchial cartilage. There is a good deal of variation in the precise mode of origin of these vessels.

The pericardial artery

(Plate 34, fig. 2, *Pcard. A.*) is given off from the left posterior anastomotic trunk (*com.* 4) between the median hypobranchial and the efferent branchial arteries. It passes backwards and at the same time medianwards in the dorsal wall of the pericardial cavity, soon dividing into right and left branches.

5. *The cœliaco-mesenteric artery.*
Arteria cœliaco-mesenterica (in *Torpedo*), HYRTL (11).

There is an immense amount of variation in the mode of origin and in the distribution of the splanchnic arteries in Elasmobranchs. In the skate (11, 23) there are large cœliac and anterior mesenteric arteries and a small posterior mesenteric, or, more correctly, spermatico-mesenteric. In *Scymnus* (22), the large cœliac artery is supplemented by three small mesenteric arteries. In *Callorhynchus* there are two splanchnic arteries, a cœliaco-mesenteric and a posterior mesenteric, both of considerable size. In *Mustelus* and in *Torpedo* (11) the main arteries for the alimentary canal and its glands are branches of a cœliaco-mesenteric; in *Mustelus* there are, however, in addition three comparatively small vessels springing separately from the dorsal aorta; these I call, from their distribution, the lieno-gastric, and the anterior and posterior spermatico-mesenteric. Thus the names given to the more important splanchnic arteries of one Elasmobranch are often quite inapplicable to those of another, and I have found it impossible in many cases to use HYRTL's names, since their employment would indicate homologies which do not exist.

The large and important cœliaco-mesenteric artery (Plate 34, fig. 1; Plate 35, figs. 6, 7, and 10; Plate 36, fig. 12, *Cœl. Mes. A.*) arises from the dorsal aorta immediately caudad of the junction with the latter of the fourth pair of epibranchial arteries. It is about 3 cm. long in the adult, dividing into sub-equal cœliac and mesenteric arteries.

The cœliac artery.

"An artery like to our cœliac," Monro (16).

Arteria cœliaca, Hyrtl (11).

The anterior of the two divisions of the cœliaco-mesenteric artery (Plate 35, fig. 7; Plate 36, fig. 12, *Cœl. A.*) passes backwards along the inner or right side of the cardiac division of the stomach. It first gives off the *anterior gastric artery* (*Ant. gast. A.*) which supplies the proximal third of the dorsal wall of the cardiac division of the stomach; then a comparatively small *right hepatic artery* (*R. hep. A.*) to the right lobe of the liver; then a vessel which divides almost immediately into the *left hepatic artery* (*L. hep. A.*) supplying the left lobe of the liver, and the small pyloric artery (*Pyl. A.*) going to the distal end of the pyloric division of the stomach. The cœliac then becomes the main or *ventral gastric artery* (*Vent. gast. A.*), which branches out in the fold of mesentery between the cardiac and pyloric divisions of the stomach, sending numerous branches to both. Of these the branches to the cardiac division are distributed exclusively to its ventral wall.

The anterior mesenteric artery.

"An artery resembling our superior mesenteric artery," Monro (16).

Arteria mesenterica anterior, Hyrtl (11).

This, the posterior division of the cœliaco-mesenteric artery (Plate 35, fig. 7; Plate 36, fig. 12, *Ant. Mes. A.*), passes backwards alongside the portal vein (fig. 12, *Hep. port. V.*) and divides, immediately cephalad of the pylorus, into a ventral branch, the *ventral intestinal artery*, and a dorsal branch, the *intra-intestinal artery*.

As far as its origin is concerned, this vessel clearly answers to the anterior mesenteric artery of the skate, but the further course of the latter vessel rather corresponds with that of the dorsal intestinal artery of *Mustelus* (*vide infra*, p. 700).

The ventral intestinal artery.

Duodenal artery (in *Scymnus* and *Raja*), Parker (20, 22, and 23).

This trunk evidently corresponds, as far as its general relations are concerned, with that I have called duodenal in the skate and in *Scymnus*, since it lies ventrad of the duodenum or *bursa entiana*, supplying that part of the intestine. As, however, it originates in *Mustelus* from the anterior mesenteric artery instead of from the

cœliac as in the other two genera, and as, further, it supplies the ventral region of the intestine generally, it seems hardly desirable to retain the name "duodenal" for it.

After separating from the main stem of the anterior mesenteric, the ventral intestinal artery (Plate 35, fig. 7; Plate 36, fig. 12, *Vent. intest. A.*) curves round the dorsal aspect of the pylorus, between it and the intra-intestinal vein (*vide infra*, p. 705) and then passes on to the ventral face of the intestine, and so caudalwards alongside the ventral intestinal vein (p. 706), sending off transverse branches to the walls of the gut and to the folds of the spiral valve.

The intra-intestinal artery.

Intra-intestinal artery, PARKER (22).

As far as I am aware, the existence of this artery (*Int. intest. A.*) was first indicated by myself in a paper on the anatomy of *Scymnus lichia*. Leaving the main trunk of the anterior mesenteric artery, it curves round to the dorsal side of the intra-intestinal vein, sends off a small branch to the dorsal wall of the duodenum, and then comes to be enclosed, along with the corresponding vein (*infra*, p. 705), in the free edge of the spiral valve, to which it sends branches (fig. 12).

6. *The anterior spermatico-mesenteric artery.*

This (Plate 34, fig. 1; Plate 35, fig. 10; Plate 36, fig. 12, *Ant. sperm. mes. A.*) is a small artery arising from the dorsal aorta about 3 or 4 cm. caudad of the cœliaco-mesenteric. It passes almost directly backwards, crossing the origin of the lieno-gastric artery (*Li. gast. A.*), gives off the *anterior spermatic artery* to the gonad, and is continued as the *dorsal intestinal artery* (fig. 12, *Dors. intest. A.*), beside the corresponding vein, along the whole length of the dorsal aspect of the intestine, to which it gives off transverse branches corresponding to those of the ventral intestinal artery. The course of this vessel is thus much the same as that of the three mesenteric arteries of *Scymnus* (23), and of the great anterior mesenteric artery of *Raja* (16, 11, 23).

7. *The lieno-gastric artery.*

This vessel (Plate 34, fig. 1; Plate 35, fig. 10; Plate 36, fig. 12, *Li. gast. A.*) arises from the dorsal aorta immediately caudad of the anterior spermatico-mesenteric, the origin of which it crosses. It passes backwards and downwards alongside the anastomotic trunk between the spermatic and anterior lieno-gastric veins (*infra*, p. 708, figs. 1, 10, 12, and 13, *v*), sends off one or two small branches to the larger or free lobe of the pancreas, and reaches the dorsal wall of the cardiac division of the

stomach at about the junction of its third and fourth fifths. It then, as the *dorsal gastric artery* (*Dors. gas. A.*), branches out and supplies the posterior two-thirds of the dorsal wall of the cardiac division, a large *splenic artery* (*Spl. A.*) supplying the spleen and the posterior portion of the greater curvature of the stomach.

8. *The oviducal arteries.*

These vessels, existing only in the female, are large paired trunks (Plate 34, fig. 1, *Ovid. A.*) arising from the dorsal aorta about 6 cm. caudad of the cœliaco-mesenteric artery. Each passes at first directly outwards, dorsad of the corresponding cardinal vein, then turns sharply backwards, divides into an anterior and a posterior branch, and ramifies in the wall of the oviduct.

9. *The posterior spermatico-mesenteric artery.*

Figured but not named (in *Raja*), by MONRO (16).

Arteria mesenterica posterior, HYRTL (11).

This is a comparatively small vessel (Plate 34, figs. 1 and 3, *Post. sperm. mes. A.*). It passes backwards and downwards, pierces the posterior azygos lobe of the gonad (ovary or testis) * which it supplies, and reaches the anterior or free end of the rectal gland (figs. 3 and 12, *Rect. Gl.*) as the *posterior mesenteric artery* (*Post. mes. A.*) in the strict sense. It then extends caudalwards along the whole dorsal border of the rectal gland, sending out numerous branches right and left.

10. *The spinal (segmental) arteries.*

From the whole length of the dorsal aorta, including its hyoidean section, small arteries are given off right and left, corresponding to the vertebral segments, and supplying the dorsal muscles.

11. *The renal arteries.*

These, like the preceding, are given off segmentally from the dorsal aorta. In the male the parorchis (WOLFFIAN body or mesonephros) is similarly supplied.

12. *The iliac arteries.*

These arteries (Plate 34, figs. 1 and 3, *Il. A.*) arise from the dorsal aorta a short distance cephalad of the level of the pubic cartilage. Each passes outwards and

* In *Mustelus antarcticus* there is a single ovary in the female, but the male has two testes which coalesce posteriorly into a single lobe.

backwards, and divides into a *femoral artery* (*Fem. A.*) distributed to the pelvic fin, and a *posterior lateral artery* (*Post. lat. A.*) which extends forwards, accompanying the lateral vein, and contributing to the supply of the abdominal walls.

13. *The caudal artery.*

At the commencement of the hæmal canal of the caudal vertebræ, the dorsal aorta becomes the caudal artery (Plate 34, fig. 1; Plate 37, fig. 25, *Caud. A.*), which passes to the end of the tail, supplying its various tissues.

II. The Veins.

The investigation of the veins of Elasmobranchs is attended with so much more difficulty than that of the arteries, that it is hardly surprising to find the published accounts of them generally wanting in detail, besides being actually inaccurate in many particulars.

The veins of *Mustelus* may be classified, mainly on embryological grounds, as follows:—

A. System of the sub-intestinal vein.
  i. Renal portal section.
    1. The caudal vein.
    2. The renal portal veins.
      a. The posterior oviducal veins.
      b. The posterior spinal veins.
  ii. Hepatic portal section.
    3. The hepatic portal vein.
      a. The intra-intestinal vein.
      b. The ventral intestinal vein.
      c. The posterior lieno-gastric vein.
      d. The gastro-intestinal vein.
        α. The dorsal intestinal vein.
        β. The anterior lieno-gastric vein
        γ. The anterior gastric vein.
      e. The ventral gastric vein.
    4. The hepatic veins and sinus.

B. System of the cardinal veins.
5. The precaval sinuses.
a. The jugular veins.
α. The orbital sinuses.
i. The anterior facial veins.
ii. The anterior cerebral veins.
β. The hyoidean sinuses.
i. The posterior facial veins.
ii. The nutrient hyoidean veins.
b. The inferior jugular veins.
α. The nutrient branchial veins.
c. The cardinal veins.
α. The renal veins.
β. The spermatic veins.
γ. The anterior oviducal veins.
δ. The anterior spinal veins.
ε. The posterior cerebral veins and the myelonal vein.
ζ. The subscapular veins.
C. System of the lateral veins.
6. The lateral veins.
a. The brachial veins.
b. The iliac veins.
α. The femoral veins.
β. The cloacal veins.
D. System of the coronary veins.
7. The coronary veins.
E. System of the cutaneous veins.
8. The dorsal cutaneous vein.
9. The anterior ventral cutaneous vein.
10. The posterior ventral cutaneous vein.
11. The lateral cutaneous veins.

A. i. *System of the sub-intestinal vein: renal portal section.*
1. *The caudal vein.*
Vena caudalis, Stannius (25).
Veine caudale, Milne Edwards (14).
Caudal vein, Huxley (10), Claus (4), &c., &c.

The caudal vein (Plate 34, figs. 1 and 3; Plate 37, fig. 25, *Caud. V.*) lies in the hæmal canal immediately ventrad of the caudal artery. It receives the lesser

veins from the tissues of the tail, and, on leaving the hæmal canal, divides, at the level of the posterior extremity of the kidneys, into the symmetrical (right and left) renal portal veins.

2. *The renal portal vein.*
Veine porte rénale, Veine de Jacobson, JOURDAIN (12).
Renal portal vein, ROLLESTON (24), PARKER (21, 23).

It is somewhat remarkable, considering that the elaborate researches of JOURDAIN (12) appear in so well-known a journal as the "Annales des Sciences Naturelles," that the existence of a true renal portal vein in Elasmobranchs is either ignored or expressly denied by the writers of all the more important zoological text-books with the single exception of ROLLESTON (24). STANNIUS (25), MILNE EDWARDS (14), on the authority of HYRTL, HUXLEY (10), GEGENBAUR (7), MACALISTER (13), and CLAUS (4) all state explicitly that in Elasmobranchs, as in Cyclostomes, the caudal vein opens directly into the cardinals. MACALISTER certainly states that branches from the cardinal veins enter the kidneys, forming a reno-portal capillary plexus, and GEGENBAUR mentions the existence of similar *venæ renales advehentes* as branches of the caudal. But neither author gives a sufficiently detailed account to enable the reader to understand what the exact relations of the renal portal system are supposed to be.

As a matter of fact JOURDAIN, following up the earlier researches of JACOBSON, ROBIN, STEENSTRA TOUSSAINT, and others,* has clearly proved the existence of a renal portal system in *Raja, Squatina, Squalus,* and *Spinax,* in all of which he shows there to be no direct passage for blood between the caudal and the cardinal veins in the adult. I have confirmed JOURDAIN'S results in *Raja nasuta* (21, 23), and now find that they hold good also for *Mustelus antarcticus.*

The renal portal veins (Plate 34, figs. 1 and 3; Plate 37, figs. 22, 23, 24, *Ren. port. V.*) formed, as stated above, by the bifurcation of the caudal vein, pass forwards each along a groove on the dorsal aspect of the corresponding kidney, sending off numerous *afferent renal veins* (venæ renales advehentes) into the substance of the gland, and gradually diminishing in calibre towards its anterior end (*cf.* figs. 24, 23, and 22, *Ren. port. V.*).

The *spinal* (segmental) *veins* from the posterior abdominal region of the vertebral column and the posterior *oviducal veins* from the hinder moiety of the oviduct

* JACOBSON, 'MECKEL'S Archiv,' vol. 3, 1817, p. 147; and "De systemate venoso peculiari in permultis animalibus observato," Hafniæ, 1821. STEENSTRA TOUSSAINT, "Comment. de syst. uropoët. piscium," 'Ann. Acad. Lugd.-Batav.,' 1834–35; and "De syst. uropoët. Squali glauci," 'Hoeven en Vriese, Tijdschrift,' vol. 6, 1839, p. 199. ROBIN, "Note sur quelques portions du système veineux des Raies," 'Revue Zool.,' vol. 9, 1846, p. 5. OWEN, "Lectures on the comparative Anatomy and Physiology of the Vertebrate Animals," I., 1846, p. 284. (Quoted by JOURDAIN in 12.)

discharge into the renal portal veins: the vein of the left side also receives the dorsal cutaneous vein (*infra*, p. 720).

A. ii. *System of the sub-intestinal vein: hepatic portal section.*

3. *The hepatic portal vein.*

Pfortaderstamm, STANNIUS (25).
Veine porte hépatique, MILNE EDWARDS (14).
Portal arterial vein, OWEN (19).
Vena portæ, HUXLEY (10).

The hepatic portal vein, or portal vein as it is more usually called (Plate 36, figs. 12 and 13, *Hep. Port. V.*), is a large vessel, fully 12 mm. in diameter when distended with injection, and lying in the gastro-hepatic omentum side by side with the anterior mesenteric artery. It is formed, at about the level of the pylorus, by the confluence of the intra-intestinal (*Intr. intest. V.*), ventral intestinal (*Vent. intest.*), and posterior lieno-gastric (*Post. li. gast. V.*) veins; somewhat caudad of the cardia it receives the gastro-intestinal (*Gast. intest. V.*) and ventral gastric (*Vent. gas. V.*) veins; on reaching the liver it divides into two main branches, one for each lobe, besides sending off one or two lesser branches into the central portion of the gland.

The intra-intestinal vein.

Tronc veineux mésentérique (in *Thalassorhina* and *Zygæna*), DUVERNOY (6).
Partie postérieure du tronc principal de la veine porte, ou veine mésentérique (in *Zygæna*, &c.), MILNE EDWARDS (14).
Main root of the portal vein ("in Plagiostomes with the longitudinal spiral valve"), OWEN (19).
Darmvene ("bei Petromyzon und einigen Squaliden"), STANNIUS (25).
Intestinal vein (in Cyclostomes), MACALISTER (13).
A special vein which lies in the fold of the spiral valve, BALFOUR (2).
Intra-intestinal vein, PARKER (22).

This, one of the most interesting vascular trunks from a morphological point of view, was discovered fifty years ago by DUVERNOY in *Galeus* (?) *thalassinus* (? *Thalassorhinus*) and *Zygæna malleus*, and was afterwards shown to exist also in *Carcharias* and *Galeocerdo;* all four genera of Selachians possessing a "scroll-valve" (20) instead of the ordinary spiral valve. DUVERNOY showed that the principal intestinal vein in these cases was a large vessel, with thick muscular walls, enclosed within the free border of the longitudinally rolled scroll-valve, and forming the chief factor of the portal vein.

Special significance was given to this vein by BALFOUR'S discovery (1) that it is

formed from the pre-caudal portion of the sub-intestinal vein or primitive ventral vascular trunk of the embryo. It is usually stated, for instance by Müller (17), Stannius (25), and Milne Edwards (14), to be absent in those genera in which a true spiral valve is present, but has been shown by myself (22) to exist as a large and conspicuous trunk both in the species now under consideration and in *Callorhynchus antarcticus*.

The intra-intestinal vein of *Mustelus* (Plate 36, figs. 12 and 13; Plate 37, figs. 22 and 30, *Int. intest. V.*) is a large vessel, fully 8 mm. in diameter at its anterior end when moderately distended with injection. It lies imbedded in the free edge of the spiral valve, having therefore itself a spiral twist, and is accompanied by the intra-intestinal artery. It receives veinlets from the inner or central portions of the various turns of the spiral valve, emerges through the wall of the intestine (fig. 12) immediately dorsad of the pylorus, receiving as it does so a small *duodenal vein* (*Duod. V.*) from the dorsal wall of the duodenum, and finally joins the ventral intestinal and posterior lieno-gastric veins about 2–3 cm. to the right, and dorsad of the pylorus.

The ventral intestinal vein.

Duodenal vein (in *Raja* and *Scymnus*), Parker (20–23).

This, the second large vein of the intestine (Plate 36, figs. 12 and 13; Plate 37, figs. 22, 30, *Vent. intest. V.*), accompanies the artery of the same name, receiving transverse feeders from the walls of the gut, and from the outer or peripheral portions of the successive turns of the spiral valve. In the greater part of its course it is double, the two divisions lying one on each side of the artery. Anteriorly it passes to the left, between the intestine and the narrow pyloric division of the stomach, and finally turns immediately caudad of the pylorus to join the intra-intestinal and posterior lieno-gastric veins.

The posterior lieno-gastric vein.

This (Plate 36, figs. 12 and 13, *Post. li. gast. V.*) is a moderate-sized vessel lying between the pyloric division of the stomach and the adjacent right lobe of the spleen, and receiving feeders from both. Anteriorly it passes dorsad of the pylorus to join with the two preceding vessels as already described.

The gastro-intestinal vein

(Plate 36, figs. 12 and 13, *Gast. intest. V.*) is a large vessel lying parallel to and dorsad of the main portal vein, and joining the latter about 6 cm. cephalad of the pylorus. Posteriorly it is constituted by the confluence of the dorsal intestinal

(*Dors. intes. V.*) and anterior lieno-gastric (*Ant. li. gast. V.*) veins, while immediately before joining the portal it receives the anterior gastric vein (*Ant. gast. V.*).

The dorsal intestinal vein.

Mesenteric vein (in *Raja* and *Scymnus*), PARKER (20–23).

The remarks made above (p. 698), as to the absence of any close correspondence between the splanchnic arteries of different Elasmobranchs, apply equally well to the veins. In *Raja* (21, 23) nearly the whole of the blood from the intestine is returned by a large mesenteric vein, lying dorsally and somewhat to the left, this being supplemented by small duodenal (ventral intestinal) and intra-intestinal veins; these, together with the splenic and pancreatic veins, unite to form a common trunk which is afterwards joined by the gastric veins. The mesenteric vein, from its position, evidently answers to the vessel now under consideration in *Mustelus*. In *Scymnus* (22) the duodenal (ventral intestinal) vein has become as large as the mesenteric (dorsal intestinal); the two unite with one another, the mesenteric previously receiving a large lieno-gastric vein, and into the common portal vein thus formed the remaining gastric veins are discharged.

In *Mustelus* the blood is returned from the intestine by no fewer than three veins, each as large proportionally as the single mesenteric vein of *Raja*. Two of these, as already seen, unite with one another and with the posterior lieno-gastric; the third, the dorsal intestinal vein now under discussion (Plate 37, figs. 12 and 13, *Dors. Intest. V.*) commences on the ventral or free border of the rectal gland, where it anastomoses with the posterior oviducal, spermatic, and cloacal veins (Plate 34, fig. 3), and apparently receives most of the blood from the posterior extremity of the gonad. As far as my observations go, I am disposed to consider this relation with the rectal gland as constant for the dorsal intestinal vein of Plagiostomes.

Passing forwards, the dorsal intestinal vein receives transverse factors from the walls of the intestine and outer regions of the turns of the spiral valve; these probably anastomose both with those of the corresponding feeders of the ventral intestinal and with those of the intra-intestinal vein.

In the position actually assumed by the intestine in the adult fish the dorsal intestinal vein, through a considerable part of its course, lies on the left side, the ventral intestinal on the right (Plate 37, fig. 22). But as the former trunk is a continuation, forwards, of the line of the rectal gland and mesorectum, *i.e.* the line along which the mesentery, if present, would be attached, it is morphologically dorsal in position.

Arrived at the posterior boundary of the duodenum, or in other words at the junction between the first and second turns of the spiral valve, the dorsal intestinal vein leaves the wall of the gut (fig. 12), and, still accompanied by the corresponding artery, passes dorsalwards, supported by a fold of mesentery, to the posterior

end of the free or dorsal lobe of the pancreas. From the latter it receives small *pancreatic veins*, and finally unites with the anterior lieno-gastric to form the gastro-intestinal vein at about the level of the pylorus.

The anterior lieno-gastric vein

(Plate 36, figs. 12 and 13, *Ant. li. gast. V.*) accompanies the lieno-gastric artery, receiving the blood from the posterior two-thirds of the dorsal wall of the cardiac division of the stomach and from the larger or left lobe of the spleen. As mentioned above it unites with the dorsal intestinal vein at about the level of the pylorus, forming the gastro-intestinal vein.

In connexion with this vein there is a very remarkable anastomosis placing the portal circulation in direct and open communication with the ordinary systemic veins. This consists of a vessel (Plate 34, fig. 1; Plate 35, fig. 10; Plate 37, figs. 12 and 13, *v*), accompanying the proximal portion of the lieno-gastric artery, opening ventrally into the anterior lieno-gastric vein close to its junction with the dorsal intestinal, and communicating dorsally with the spermatic vein near its junction with the right cardinal.

The anterior gastric vein

(Plate 36, figs. 12 and 13, *Ant. gast. V.*) accompanies the artery of the same name, bringing blood from the proximal third of the dorsal wall of the cardiac division of the stomach. It enters the gastro-intestinal vein shortly before its junction with the portal.

The ventral gastric vein.

This (Plate 36, figs. 12 and 13, *Vent. gast. V.*) is the largest of the gastric veins, receiving the blood from the whole ventral region of the cardiac division. It accompanies the ventral gastric artery and opens into the portal vein about 2 cm. cephalad of the junction with it of the gastro-intestinal vein.

4. *The hepatic veins and sinus.*

a. Hepatic veins.

Venæ cavæ hepaticæ, Monro (16).

Veines hépatiques, Milne Edwards (14).

b. Hepatic sinus.

Receptacle formed by the venæ cavæ hepaticæ, Monro (16).

Hepatic sinus, Parker (21 and 23).

The blood supplied to the liver by the hepatic arteries and portal vein is collected into two immense hepatic veins (Plate 35, fig. 9; Plate 37, figs. 21 and 22, *Hep. V.*),

imbedded one in the right the other in the left lobe, and opening anteriorly into the hepatic sinus (*Hep. S.*).

The latter is a capacious chamber lying immediately cephalad of the liver and ventrad of the œsophagus. When distended with blood or injection it completely fills the antero-ventral region of the cœlome, abutting in front against the pericardio-peritoneal septum (*Peric. perit. Sept.*), while posteriorly it adapts itself to the anterior border of the liver. Its general relations to surrounding parts are best seen in transverse and horizontal sections of a frozen fish (Plate 35, fig. 9 ; Plate 37, figs. 19, 20, and 21).

Internally the hepatic sinus is divided into right and left compartments by an incomplete vertical partition (*p*) formed of an irregular network of fibrous trabeculæ. Each compartment is also traversed by numerous irregular fibrous bands. The hepatic veins open each into the compartment of its own side by a large aperture.

The hepatic sinus opens into the sinus venosus by two small circular apertures (*Hep. S'.*), placed close together in its anterior wall one on each side of the median partition.

It will be seen from the above description that the relations of the "system of the sub-intestinal vein" to the embryonic trunk from which it takes its origin is remarkably clear in *Mustelus antarcticus*. We have, first, the caudal vein, the relations of which are only disturbed by the atrophy of the post-anal gut; this is followed, after a short interruption of continuity, by the intra-intestinal, which is directly continued into the main portal vein; then the continuity of the original vessel is again interrupted, this time by the capillaries of the liver, following which we have the paired hepatic veins opening by the hepatic sinus into the heart. There is thus a successive series of longitudinal trunks, all unpaired, except the hepatic veins, indicating, in the adult, the course of the originally continuous sub-intestinal vein of the embryo.

In this, as in so many points, the Sharks exhibit a more generalized type of structure than the Rays. Besides the retention of a large intra-intestinal vein in many Selachians, the hepatic sinus in *Raja* has no direct communication with the sinus venosus, but opens by two widely separated apertures into the two precaval sinuses (*see* 16 and 21, or 23).

B. *System of the cardinal veins.*

5. *The precaval sinus.*

Described and figured, but not named, by Monro (16).
Truncus transversus, Ductus Cuvieri, Stannius (25).
Canal de Cuvier, Milne Edwards (14).
Precaval vein, Owen (19).
Ductus Cuvieri, Huxley (10), Rolleston (24), Gegenbaur (7).
Cuvierian duct, Macalister (13).
Precaval sinus, Parker (21 and 23).

The paired precaval sinuses are the homologues of the great precaval veins or Cuvierian ducts of Teleosts and of the precaval veins or venæ cavæ anteriores *s.* superiores of the higher Vertebrata. Owing to their form the name of sinus is more applicable to them than that of vein in Elasmobranchs.

The precaval sinus of *Mustelus* (Plate 34, figs. 1 and 2; Plate 35, figs. 9 and 10; Plate 37, figs. 19, 27, and 28, *Pr. cav. S.*) is a tubular chamber, about 15 mms. long and 5 mms. wide, situated immediately laterad of (external to) the sinus venosus, of which, on anatomical grounds alone, it might be looked upon as merely an extra-pericardial portion. From its junction with the sinus venosus it takes a direction upwards and outwards (fig. 19), its antero-lateral wall lying in close apposition to the fifth branchial arch (fig. 10) at the junction of the epi- and ceratobranchial, while its inner wall is closely applied to the œsophagus. Its precise relations are best seen in transverse and horizontal sections of frozen specimens (figs. 9 and 19), in which also the apertures of the various veins into the sinus are well displayed.

There is no constriction between the precaval sinus and the sinus venosus, the boundary between the two being marked only by a very low inconspicuous ridge in the lining membrane and by the position of the pericardial wall. Dorsally the precaval sinus passes with but slight change of diameter into the jugular vein, the entrance of which (figs. 2, 9, and 19, *Jug. V'.*) is guarded by a pair of semilunar valves; these act so perfectly that I have never known even a drop of injection to pass from the sinus into the vein. In the posterior wall of the precaval sinus is the large valveless aperture of the cardinal sinus (*Card. S'.*); on its ventral wall, close to its junction with the sinus venosus is a small aperture communicating with the lateral vein (*Lat. V'.*), and, lastly, in its antero-ventral region in the somewhat larger opening of the inferior jugular vein (*Inf. Jug. V'.*).

The jugular vein.

Internal jugular vein, MONRO (16).
Vena jugularis superior, MÜLLER (17).
Vena vertebralis anterior s. jugularis, STANNIUS (25),
Veines jugulaires, MILNE EDWARDS (14).
Vena jugularis, OWEN (19), HUXLEY (10), ROLLESTON (24).
Jugular vein, MACALISTER (13), GEGENBAUR (7).

As in the case of the carotid arteries (*supra*, p. 699) there is no precise correspondence between the cephalic veins of fishes and the "internal" and "external" jugulars of the higher animals. In position the vein now under consideration agrees, as MONRO saw, with the internal jugular, from which, however, it differs in receiving the main part of the blood from the whole head, and not only that from the brain.

The jugular vein (Plate 34, figs. 1 and 2; Plate 35, figs. 9 and 10; Plate 37, figs. 18, 26, and 27, *Jug. V.*) is a very large vessel, about 1 cm. in diameter, and triangular in section (fig. 18), situated immediately dorsad of the gills and laterad of the dorsal muscles. Anteriorly it lies in the horizontal plane of the auditory capsule (fig. 10), posteriorly in that of the œsophagus (fig. 19), so that it takes a direction from its cephalic end backwards and slightly downwards.

At its posterior extremity, as already stated above, the jugular turns downwards, caudad of the fifth branchial arch (fig. 10), to enter the precaval sinus, the junction between the two being marked by a pair of transversely-placed semilunar valves (figs. 2 and 19).

At its cephalic end the jugular becomes a somewhat irregular channel (fig. 10), and passes laterad of the auditory capsule, narrowing considerably in vertical height as it does so, and at the same time becoming wider from side to side. It is this portion of the jugular which receives the blood from the orbital sinus anteriorly, and that of the hyoidean sinus ventrally (*vide infra*). Throughout its whole length the jugular receives feeders from the dorsal region of the gills and other neighbouring parts.

The orbital sinus.

Situated within the orbit and surrounding the eye-muscles is a large irregular sinus (Plate 35, fig. 10, *Orb. S.*), always containing more or less blood in the freshly-dissected fish. When the eye and its muscles are removed, there is seen, in the posterior wall of the orbit, a small depression, bounded externally by a ligament, alongside and parallel with which is the hyomandibular nerve or posterior division of the seventh; this depression leads by a narrow oblique passage from the orbital sinus into the jugular vein, the aperture between the two being guarded by a distinct valve.

The orbital sinus receives the anterior cerebral vein (*infra*), and at the anterior (*inner*) canthus of the orbit a vein from the anterior and external region of the head

(*anterior facial vein, Ant. fac. V.*) is apparently discharged into it; but I have never been able to get a satisfactory injection of this vessel.

The only reference to the orbital sinus I have met with is the following somewhat obscure statement given by MILNE EDWARDS (14) on the authority of ROBIN, whose paper* I have unfortunately been unable to consult. "Les veines jugulaires communiquent entre elles par un tronc anastomotique assez large, et constituent en général, derrière les orbites, un sinus plus ou moins vaste ..... Il est extrêmement développé chez les Squales, et constitue de chaque côté des branchies un vaste reservoir qui s'étend dans les cavités orbitaires." This paragraph would seem to intimate that the anterior ends of the jugular veins communicate directly with one another, which they certainly do not in *Mustelus*; indeed, from their position, any such union is anatomically impossible.

The anterior cerebral vein.

This (Plate 35, fig. 8, *Ant. cereb. V.*) is a paired vein receiving the blood from the olfactory lobe, and from its own side of the prosencephalon, diencephalon, and mesencephalon. At the level of the diencephalon it turns directly outwards, passes through a foramen in the side wall of the skull, and discharges itself into the orbital sinus.

The blood from the rest of the brain is returned by the posterior cerebral veins which will be considered later (p. 715).

The hyoidean sinus.

This (Plate 34, fig. 2; Plate 35, fig. 10, *Hyoid. S.*) is a wide irregular vessel lying on the outer face of the hyoid arch, parallel to the first afferent branchial artery (*Af. br. A.* 1). In the middle part of its course it is double, consisting of two parallel trunks which unite with one another above at about the level of the junction of the epi- and cerato-hyal, below at that of the junction of the cerato- and hypo-hyal.

The hyoidean sinus discharges into the interior end of the jugular vein by an aperture on its ventral wall, guarded by a single valve. Ventrally it is continued forwards to the symphysis mandibulæ, parallel to and mesiad of the sub-mental artery, this portion of the sinus being the equivalent of a *sub-mental vein.*

The right and left hyoidean sinuses communicate with one another by two wide anastomotic trunks (fig. 2, *u*), one immediately cephalad, the other immediately caudad, and both of them ventrad of the common stem of the first and second afferent branchial arteries. Close to the junction of the hindermost of these anastomotic trunks the inferior jugular vein (*Inf. jug. V.*) communicates with the hyoidean sinus.

* ROBIN, "Sur le système veineux des poissons cartilagineux," 'Compt. Rend.,' vol. 21, 1845, p. 1282.

The chief tributaries of the hyoidean sinus are—(*a*) veins from the mandible entering its ventral or submental portion; (*b*) several nutrient veins from the hyoidean hemibranch (fig. 10, *Nu. hy. V.*); and (*c*) a vein (*Post. fac. V.*) entering near the junction of the hyomandibular and cerato-hyal which I have been able to trace only for a very short distance; probably it is safe to call it the *posterior facial vein.*

The inferior jugular vein.

A small vein somewhat like to our anterior external jugular, Monro (16).
Vena jugularis inferior, Müller (17), Stannius (25).

The inferior jugular veins (Plate 34, figs. 1 and 2; Plate 35, fig. 10; Plate 37, figs. 18, 26, and 27, *Inf. jug. V.*) are moderately-sized vessels, lying, one on each side, at the base of the gills, about 1–2 cm. laterad of the ventral aorta. Each anastomoses in front with the corresponding hyoidean sinus, and passing backwards, receives the nutrient veins of the four holobranchs. After receiving the last of these, the inferior jugular is deflected somewhat outwards and comes to lie in the roof of the pericardial cavity immediately mesiad of and parallel with the fifth cerato-branchial, finally entering the precaval sinus by the aperture already noticed.

In an embryo of *Scymnus lichia*, referrable to "Stage O" of Balfour (1, Plate VIII., figs. O, O'), the inferior jugulars are already large vessels, and, along with the jugulars, open by wide valveless apertures into the short precaval sinuses (Plate 37, figs. 27 and 27).

The nutrient branchial veins.

Venæ bronchiales, Müller (17).
Veines de Duvernoy, Milne Edwards (14).

Of these there are four on each side, one for each holobranch (Plate 34, fig. 2; *Nu. Br. V.*, 1–4). The first three are similarly situated, being parallel to the ventral portion of the extra-branchial cartilage and to the nutrient branchial artery; in each case the vein is anterior, the cartilage posterior, and the artery intermediate in position. The fourth nutrient vein takes a somewhat different course, lying mainly on the posterior face of the gill. They are connected with a rich plexus of veins.

The cardinal veins.

Abdominal cava, vena cava, Monro (16).
Vertebralvene, Müller (17).
Vena vertebralis posterior, Stannius (25).
Veine abdominale, Milne Edwards (14),
Vena cardinalis, Owen (19).
Cardinal vein, Huxley (10).

The cardinal veins (Plate 34, fig. 1; Plate 35, fig. 10; Plate 37, figs. 22, 23, and 24, *Card. V.*) lie immediately ventrad of the vertebral column. They are unsymmetrically developed, the right trunk (*R. Card. V.*) only being complete, and the left (*L. Card. V.*) uniting with it a short distance caudad of the anterior end of the kidneys. Posterior to the point of union with its fellow the right cardinal is situated in the middle line imbedded in the fused kidneys (figs. 23 and 24), and therefore not visible in the ordinary course of dissection without partly removing the latter; traced backwards it is found to undergo a gradual diminution in calibre, and to arise at the posterior end of the kidneys by the confluence of the efferent veins from that part of the renal tissues.

It has already been shown (*supra*, p. 703) that the caudal vein, contrary to the statements usually made, divides into two renal portal veins which pass forwards, gradually diminishing in calibre. It is now shown that the cardinal vein, so far from being continuous with the caudal, commences as a very small vessel in the posterior region of the kidneys, and extends forwards, gradually increasing in calibre. These facts are clear enough from ordinary dissection; if an injected specimen be hardened in alcohol, the cardinal vein slit open longitudinally and the contained injection-mass removed, the non-extension of the vein caudad of the kidneys is perfectly evident. But the matter becomes clearer still by the examination of a series of sections, as shown in Plate 37, figs. 22–24. In fig. 22, taken through the middle of the first dorsal fin and a short distance cephalad of the union of the left cardinal with the right, the renal portal veins are very small; in fig. 23, taken 4 cms. cephalad of the pubis, the right cardinal is seen to have taken up a median position and the renal portals have increased in size; in fig. 24, through the symphysis pubis, the renal portals have still further increased in calibre and are now distinctly larger than the right cardinal, which, in the next section, had disappeared.

It will be for future investigations to determine the age at which the formation of the renal portal system takes place, all that can be said at present being that it occurs during comparatively late embryonic life. In the embryo of *Scymnus*, already referred to ("Stage O," Balfour), the caudal vein is directly continued into the cardinals as in Cyclostomes (Plate 37, figs. 31 and 32); on the other hand, in a ripe fœtus of *Mustelus* I have ascertained, from a complete series of

thin sections of the pelvic region, that the adult condition of things is already fully established.

Tracing the cardinals forward from the point of union of the left with the right, each is seen to dilate into a large cavity, the *cardinal sinus* (Plate 34, figs. 1 and 2; Plate 35, figs. 9 and 10; Plate 37, figs 19–21, *Card. S.*) which opens into the corresponding precaval sinus as already described (p. 710). The two cardinal sinuses communicate with one another by a wide oval aperture (fig. 10, *ap.*) close to the entrance of the spermatic vein (*Sperm. V.*).

The cardinal veins receive the *efferent renal veins* from the kidneys, and, in the anterior half of the abdomen, the *oviducal* and the *spinal* (segmental) *veins*. The posterior oviducal and spinal veins, as already stated, open into the renal portal veins.

The spermatic vein.

This (Plate 34, fig. 1, Plate 35, fig. 10, *Sperm. V.*) is a very large and capacious vessel, deserving rather to be called a sinus than a vein in the adult. It receives the blood from all but the posterior extremity of the ovary or testes (see p. 707), and opens into the conjoined portion of the cardinal sinuses. Close to its entrance the spermatic vein receives the large anastomotic trunk (*v*) from the anterior lieno-gastric vein (*supra*, p. 708).

The posterior cerebral and myelonal veins.

The veins from the anterior part of the brain, *i.e.*, as far back as the mesencephalon, have been seen (p. 712) to pour their blood, by the symmetrical anterior cerebral veins, into the orbital sinuses. The veins from the cerebellum and medulla oblongata unite to form on each side a *posterior cerebral vein* (Plate 35, fig. 8, *Post. cereb. V.*), which lies dorso-laterad of the medulla. About 0·5 cm. caudad of the calamus scriptorius the two posterior cerebral veins unite with one another, on the dorsal surface of the spinal cord, to form the *myelonal vein* (*Myel. V.*) which extends backwards along the whole length of the cord. In each vertebral segment this vessel forms a small simple rhomboidal plexus, from the lateral angles of which commissures are given off, and, passing outwards and downwards, discharge into either the jugular, the cardinal, the renal portal, or the caudal vein. There is thus a series of segmentally arranged commissures between the myelonal vein on the one hand, and one of the sub-vertebral series of veins on the other. The commissures receive also feeders from the dorsal muscles.

The subscapular sinus.

The subscapular sinus (Plate 34, fig. 1, Plate 36, fig. 16, Plate 37, fig. 21, *Subscap. S.*) is a capacious vessel situated immediately mesiad of the scapula, *i.e.*, between it and

the underlying muscles; it receives vessels from the neighbouring parts, as well as the lateral cutaneous vein (*infra*, p. 721), and, passing forwards and inwards, discharges into the cardinal sinus.

The œsophageal veins.

These form a close plexus, resembling that of the mammalian gravid uterus; they open on each side, by small apertures in its inner wall, into the cardinal sinus.

C. *System of the Lateral Veins.*

6. *The Lateral Vein.*

(Anterior part only.) *A large vein from the abdominal muscles and side of the fish*, Monro (16).

Gros vaisseau lymphatique sous-péritonéal, vaisseau latéral, Robin, quoted by Milne Edwards (15).

Ilio-hæmorrhoidal plus *Epigastric vein*, Parker (21 and 23).

Lateral vein, Parker (22).

The lateral vein (Plate 34, figs. 1–3; Plate 35, figs. 9 and 10; Plate 37, figs. 20–24, *Lat. V.*) is a vessel of considerable size lying in the side wall of the abdomen immediately external to the peritoneum, and passing from the level of the pelvic fin forwards and upwards to the level of the pectoral. Posteriorly it unites with its fellow across the dorsal face of the pubic cartilage (figs. 1, 3, and 24); anteriorly it turns inwards or towards the middle line, and then forwards, outwards, and upwards, to open into the precaval sinus (*supra*, p. 710, figs. 1, 2, 9, and 19). It receives veins from the abdominal walls, as well as from the pectoral and pelvic fins (*infra*, p. 717), and anastomoses with the anterior ventral cutaneous vein (p. 720).

The anterior moiety of the lateral vein of the Skate is figured by Monro (16, Plate III.), who, however, only traced it a short distance backwards. Robin* seems to have been the first to notice the veins in their entirety, but he apparently missed their connexion with the brachial and femoral veins, as well as with the precaval sinus; he also originally described them as lymphatics, but afterwards, finding them to contain blood, was led to consider them as veins.† They were re-discovered by myself in the Skate (21), and were subsequently found to exist in *Scymnus*, *Acanthias*, *Mustelus*, and *Chiloscyllium* (22). In recording their presence in the last named genera I remarked that, although I had found no account of them, it was difficult to believe such large vessels to have escaped notice; it had not occurred to me at that time to consult the section on the lymphatic system in Milne Edwards's "Leçons," and no mention of the lateral veins is made under the head of the venous system.

* Robin, "'Revue Zoologique' de Guérin," 1845, p. 225.—Quoted by Milne Edwards (15, p. 477).

† Robin, "Note sur le système sanguin et lymphatique des Raies et des Squales," 'L'Institut,' vol. 13, 1845, p. 452.—Quoted by Milne Edwards (15, p. 472).

The brachial vein.

The brachial vein (Plate 34, figs. 1 and 2, Plate 35, fig. 10, Plate 37, figs. 20 and 21, *Brach. V.*), receiving the blood from the pectoral fin, lies parallel with and immediately mesiad of the metapterygium; at its anterior end it turns inwards, and opens into the lateral vein close to its junction with the precaval sinus.

The iliac vein.

This (Plate 34, figs. 1 and 3, Plate 37, fig. 24, *Il. V.*) is a short trunk formed by the union of the cloacal and femoral veins (*infra*); it receives an anastomotic branch from the anterior ventral cutaneous vein (figs. 3 and 14), and opens into the lateral vein near the articulation of the basipterygium with the hip-girdle (fig. 24).

The femoral vein

(Plate 34, figs. 1 and 3, *Fem. V.*) lies immediately dorsad of the basipterygium. It receives factors from the substance of the fin, and, in the male, from the clasper.

The cloacal vein

(Plate 34, figs. 1 and 3, *Clo. V.*) takes a course nearly parallel to the preceding. It receives several large vessels from the cloaca, the anterior of which are also fed by factors from the rectum, oviduct, and ovary. Thus the cloacal veins anastomose with the mesenteric, spermatic, and oviducal veins. Posteriorly the two cloacal veins receive the blood from the posterior ventral cutaneous vein (figs. 3 and 14).

I think there can be no doubt that the lateral veins, which may now be fairly considered as characteristic of Plagiostomes, are homogeneous with the epigastric or anterior abdominal veins of Amphibians and Reptiles. When first suggesting this homology (21) I was unaware of the support afforded to the determination by the facts of embryology. In Amphibia (2, p. 539) there are in the embryo two epigastric veins which open in front into the sinus venosus, while posteriorly they are connected with the iliac veins, and also receive factors from the allantoic bladder and rectum. Reference to fig. 1 shows that a very slight shifting of the proximal end of the brachial vein would make it open separately into the precaval sinus (anterior vena cava), while a displacement mesiad of not more than one or two millimetres would cause the lateral vein to open directly into the sinus venosus.

I have elsewhere (21 and 22) advanced the hypothesis that the lateral vein may be looked upon as derived from the vein of the continuous lateral fin of the ancestral vertebrate, from which, according to the Balfour-Thacher theory, the limbs of vertebrates are evolved. On this hypothesis the discovery of the veins and of their relation to those of the pectoral and pelvic fins may be looked upon as an indirect

confirmation of the lateral fin theory. BALFOUR himself considered that, if they could be found in the embryo or in some less specialized form than the skate, their presence would furnish an argument in favour of the theory. In quoting this opinion of BALFOUR'S (22, p. 232) I have pointed out that the vein in question is now known to exist in at least four genera of Selachians, and that in an embryo of *Scymnus lichia*, corresponding very nearly with BALFOUR'S "Stage O" (1, Plate VIII., figs. 0, 0'; 22, Plate XXXII., fig. 13), it is a large and important trunk. Figs. 29–32 (Plate 37) show the relations of the lateral vein in various transverse sections of this embryo; from its conspicuous character one is rather surprised to find that BALFOUR makes no mention of it in any of the forms investigated by him. Mr. SEDGWICK, however, informs me that a re-examination of some of his sections has shown the vein to be present.

In the paper last referred to (22) I have also hazarded the suggestion that the lateral veins may be genetically connected with the lateral vessels of a vermian ancestor. As BALFOUR has shown, the dorsal aorta is the primary dorsal trunk of the vertebrata, the sub-intestinal vein their primary ventral trunk, the cardinal veins being quite secondary structures. I have, unfortunately, no Selachian embryos of an earlier stage than "O," and am therefore unable to say at what period the lateral vein makes its appearance, whether before or after the cardinals. If further investigation should show it to arise before the latter it will have to be definitely reckoned as one of the primary vascular trunks of the *Protochordata*; if shown to have a later origin its phylogenetic significance will be left still uncertain.

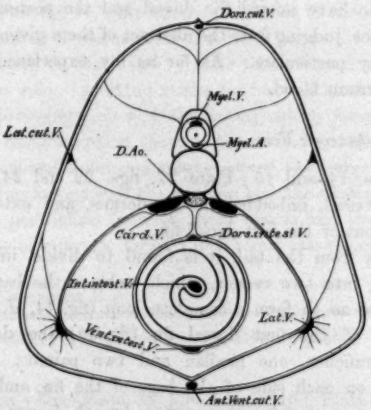
In any case there can be no doubt that the blood-vessels must not be left out of consideration in instituting comparisons between the Vertebrata and the lower types. For instance, according to HUBRECHT'S hypothesis (9), the Chordata are derived from Nemertean-like forms, in which the lateral nerve-cords have coalesced dorsad of the proboscidean sheath, which latter has been modified into the notochord. Comparing the vascular trunks of a Nemertean with those of an embryo Selachian, we find in both a dorsal and a pair of lateral vessels; the lacunæ described by OUDEMANS (18), in *Valencinia*, as lying symmetrically ventro-laterad of the proboscidean sheath may also, perhaps, be compared with the cardinal veins, but the ventral vessel, so characteristic of vertebrates, is wholly absent in the worm.

Again, if with DOHRN, SEMPER, and others, we compare a vertebrate with an inverted Annelid, we find certain interesting agreements between the blood-vessels of the two. In Chætopods there is a dorsal vessel in which the blood flows forwards (from tail to head), and which is intimately connected with the dorsal wall of the intestine. This, on the inverted Annelid theory, answers to the sub-intestinal vein of the Selachian embryo, in which the blood also flows forwards, and which arises in the splanchnopleure. The dorsal aorta would then correspond with the supra-neural vessel of the worm, and the lateral vessels of the two types would agree with one another; the myelonal vein of Selachians might also be compared with the sub-neural

vessel of some Annelids. Thus the study of the vascular system would seem to lend more support to the Annelid than to the Nemertean theory, a strong point against the latter hypothesis being the absence of a ventral vessel in any known Nemertean.

The most recent speculation of this sort is that of BATESON (3), who proposes to include *Balanoglossus* in the Chordate Phylum. This theory clearly receives no support from a comparison of the blood-vessels of the Enteropneusta with those of the Vertebrata. *Balanoglossus* certainly possesses dorsal, ventral, and lateral trunks, but the blood in the dorsal vessel—which, by the hypothesis, answers to the dorsal vessel of vertebrates—flows from behind forwards, and both dorsal vessel and heart lie dorsad of the supposed notochord.

Fig. A.



Diagrammatic transverse section of the trunk region of a Selachian, showing the principal longitudinal blood-vessels. The arteries are distinguished by dotting, the veins by vertical shading. (For explanation of reference letters see p. 726.)

The diagram, fig. A (woodcut), will probably be useful as showing the position of the chief longitudinal vessels—arteries and veins—in the trunk region of a typical Selachian. Of these the dorsal aorta and the intra-intestinal (sub-intestinal) vein are of primary importance in any phylogenetic speculations which avoid favouritism towards particular organs or sets of organs; the lateral veins are probably, and the cardinal veins certainly, sub-primary, while the remaining vessels, with the possible exception of the myelonal vein, are, in all probability, of altogether secondary significance.

D. *System of the Coronary Veins.*

7. *The Coronary Veins.*

There are two coronary veins in *Mustelus* (Plate 34, fig. 4, *Cor. V.*), a right and a left, situated one on either side of the furrow between the auricle and the ventricle. Each receives the veins from its own side of the ventricle and conus arteriosus, and, passing backwards, opens into the sinus venosus (fig. 9, *Cor. V'.*) in the immediate neighbourhood of the sinu-auricular aperture (*Sin. aur. ap.*).

E. *System of the Cutaneous Veins.*

The cutaneous, like the lateral veins, were first described by Robin * as lymphatics. He, however, seems to have missed the dorsal and the posterior ventral cutaneous veins, and his researches, judging from the abstract of them given by Milne Edwards, are imperfect in many particulars. As far as my experience goes the vessels in question invariably contain blood.

8. *The Dorsal Cutaneous Vein.*

This (Plate 36, figs. 14 and 15; Plate 37, figs. 22 and 24, *Dors. cut. V.*) is a median longitudinal vessel, imbedded in the dermis, and extending from the tail to some distance in front of the first dorsal fin.

Tracing it forwards from the tail, it is found to divide, immediately caudad of the second dorsal fin, into two vessels, which embrace the base of the fin, uniting again in front of it so as to form a complete loop (fig. 14, *d, f,* 2). On reaching the posterior border of the first dorsal fin (fig. 15) the dorsal cutaneous vein divides into three branches, one median and two paired; the latter (*d, f,* 1) extend forwards, one on each side of the base of the fin, and unite in front of it in an azygos subcutaneous trunk; the median branch leaves the skin, and passes downwards and forwards in the median fibrous septum dividing the muscles of the right and left sides (fig. 22); on reaching the vertebral column it turns to the left of the latter—in the single specimen in which it was traced out—and, continuing its downward course, opens into the left renal portal vein.

9. *The Anterior Ventral Cutaneous Vein.*

> *Tronc lymphatique médian abdominal*, Robin, quoted by Milne Edwards (15).

This vessel (Plate 36, fig. 14; Plate 37, figs. 21, 22, and 24, *Ant. vent. cut. V.*) lies in the middle ventral line of the abdomen, extending from the shoulder

* Robin, 'Revue Zoologique,' 1845.—Quoted by Milne Edwards (15).

girdle in front to the pubis behind. It is situated mesiad of the skin, *i.e.*, between it and the abdominal muscles. It receives numerous tributaries which form a very beautiful plexus on the abdominal wall immediately beneath the skin. Anteriorly the vein trifurcates, the three branches uniting again in the form of a rhomboid, the lateral angles of which are connected with the lateral veins (fig. 14) close to the entrance of the brachial veins. Posteriorly the anterior ventral cutaneous vein sends off two branches, a right and a left, almost at right angles; these pass along the ventral aspect of the pubic cartilage, turn round the outer border of the latter, and then, passing inwards parallel to and dorsad of their former course, anastomose with the iliac veins (figs. 3 and 14).

10. *The Posterior Ventral Cutaneous Vein*

(Plate 34, fig. 3; Plate 36, fig. 14, *Post. vent. cut. V.*) lies in the middle ventral line of the caudal region. Passing forwards from the tail it forms a loop round the base of the ventral (so-called anal) fin (fig. 14, *V. f.*) like those formed round the dorsal fins by the dorsal cutaneous vein. Arriving at the hinder wall of the cloaca it bifurcates, the two branches anastomosing each with the corresponding cloacal vein.

There is thus direct communication, through the cloacal and iliac veins, between the anterior and posterior ventral cutaneous veins; indeed, they might almost be described as a single vessel forming loops round the anal fin and the cloaca. It will be noticed that there is a curious parallelism between these vessels and the sub-intestinal vein, which, in its early state of entirety, forms a loop round the cloaca.

11. *The Lateral Cutaneous Vein.*

Tronc lymphatique latéral sous-cutané, Robin, quoted by Milne Edwards (15).

The lateral cutaneous veins (Plate 36, figs. 14 and 16; Plate 37, figs. 22 and 24, *Lat. cut. V.*) are paired longitudinal vessels, extending along the side of the trunk and tail at the junction of the dorsal and ventral muscles. Like the ventral cutaneous veins, they lie immediately beneath the skin. They anastomose posteriorly both with the caudal and with the dorsal cutaneous vein (fig. 14); anteriorly each discharges into the corresponding subscapular sinus (figs. 14 and 16).

General Considerations.

One of the most striking features of the blood-vessels of Selachians, as represented by *Mustelus antarcticus*, is the number of large anastomotic trunks between important arteries and veins. In the arteries there are commissures uniting the anterior and posterior carotids (Plate 35, fig. 6, *w*), others placing the efferent arteries of each

holobranch in communication (*com.* 2), and others uniting the whole series of efferent branchial arteries with the subclavians (Plate 34, fig. 2; and Plate 35, fig. 6, *com.* 1).

But it is in the venous system that we find the most striking cases of anastomosis. The hyoidean sinuses communicate with the jugulars, with the inferior jugulars, and with one another (Plate 34, fig. 2); the cardinals are placed in communication with one another (Plate 35, fig. 10) and with the portal vein (figs. 10 and 12); the lateral veins unite with one another (fig. 1), as also do the cloacal veins (Plate 36, fig. 14); the anterior ventral cutaneous vein opens in front into the lateral, behind into the iliac veins (fig. 14); the lateral cutaneous is connected with the subscapular vein in front and with the caudal and dorsal cutaneous veins behind.

The diagram, fig. B (woodcut), is intended to bring out these peculiarities of the venous system. There can be little doubt that, in many instances, alternative courses are open to the returning blood, so that the venous portion of the circulation is by no means so definite as in the higher animals.

Another point of interest is the tendency of the veins to enlarge into great irregular sinuses. The cardinals dilate anteriorly into capacious cardinal sinuses, and the hepatic veins into an immense hepatic sinus; the blood from the head is poured partly into the orbital, partly into the hyoidean sinuses. In fact the majority of the venous channels, as will be seen by an inspection of figs. 1 and 2, are of a calibre quite disproportionate to that of the corresponding arteries.

Thus the veins of a Selachian have more or less of a lacunar character, reminding one of what is found in many of the higher Invertebrata. It is a question how far this state of things is primary or ancestral, and how far secondary.

Other matters of general interest are discussed in earlier portions of the paper. The most important of these are: the relation of the arteries of the adult to the embryonic aortic arches (pp. 689–691); the renal portal system (pp. 704 and 714); the derivatives of the embryonic sub-intestinal vein (p. 709); the homology of the lateral vein with the epigastric vein of Amphibia (p. 717); and the relation of the blood-vessels of Selachians to the primary vascular trunks of the Chordata (p. 718).

Methods of Investigation.

The arteries may be conveniently injected from the ventral gastric artery, the cannula being directed towards the dorsal aorta; if the operation is conducted successfully the whole arterial system is filled, with the exception, of course, of the branches of the ventral gastric artery itself. Fine (dental) plaster of Paris coloured with vermilion or carmine forms an excellent injecting material.

The injection of the veins is more difficult. By injecting from the anterior lieno-gastric or from the dorsal intestinal vein, the cannula being directed towards the main portal vein, the whole of the portal system is filled, the injection usually also passing through the spermatico-portal anastomosis into the cardinals, sinus

Fig. B.

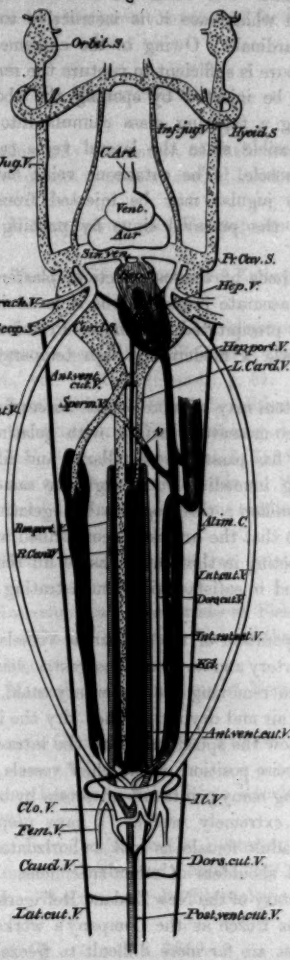


Diagram of the venous system of *Mustelus antarcticus*.

The renal portal section of the system of the sub-intestinal vein is shaded transversely, the hepatic portal section longitudinally; the system of the cardinal veins is dotted, that of the lateral veins is left unshaded; the cutaneous veins are represented by black lines. The liver (*Lr.*), alimentary canal (*Alim. C.*), gonad (*Gon.*), and kidneys (*Kd.*) are represented by conventional outlines. For explanation of reference letters, see p. 726.

venosus, and hepatic sinus. The renal portal veins may be separately injected from the caudal vein, in which case it is instructive to use a different colour to that employed for the cardinals. Owing to the extreme thinness of their walls a comparatively slight pressure is sufficient to rupture the renal portal veins.

The lateral veins may be injected by opening the abdomen, cutting across one of the veins and pushing a tapering glass cannula into it far enough to prevent escape; tying is inadmissible since the lateral veins cannot readily be dissected from the surrounding muscle. The cutaneous veins have to be separately filled in the same way. The jugular may be injected from the ventral end of the hyoidean sinus or from the precaval sinus by pushing the cannula beyond the valves.

For all the above methods of venous injection, plaster of Paris may be used as for the arteries, but for accurate investigations it is advisable to employ a solution of gelatine coloured with precipitated carmine or Prussian blue. The vessels, unlike those of Teleosts, will bear a moderately high temperature (40°–50° C.) without injury.

The whole vascular system may be more or less successfully filled by injecting from any branch of the cœliaco-mesenteric artery with gelatine coloured with Prussian blue, and, when the mass has passed the capillaries and filled the veins, withdrawing the syringe and injecting immediately, through the same cannula, with plaster of Paris coloured with vermilion; this replaces the gelatine as far as the smallest arteries, but no farther, so that the arteries become filled with red and the veins with blue.* It is only by injecting in this way, so as to fill the veins from the capillaries, that I have ever succeeded in satisfactorily demonstrating the cerebral and myelonal veins.

A very instructive preparation of the splanchnic vessels may be made by injecting from the ventral gastric artery and anterior lieno-gastric vein with red and blue plaster of Paris respectively, then removing the alimentary canal, and, after washing out its contents, distending with air and drying. When dry the intestine should be opened in one or two places to show the spiral valve and the intra-intestinal artery and vein.

For making out the precise position of the chief vessels in relation to surrounding parts, as well as for settling many points left uncertain by ordinary dissection, sections of frozen specimens are extremely useful. I have prepared a complete series of transverse sections of an adult female, as well as horizontal and vertical longitudinal sections of the head and shoulders of other specimens. I have to thank Mr. W. Cunningham Smith, Secretary of the New Zealand Refrigerating Company, for allowing me to have my specimens frozen at the Company's works, Burnside, near Dunedin. I may mention that fishes are far more difficult to freeze than mammals; after six days' exposure to a temperature of 20° F. (about −7° C.), my dog-fishes, which were

* This method has been recommended by my friend Prof. H. F. Osborn, of Princeton ('American Naturalist,' vol. 19, 1885, p. 526, and 'Microsc. Soc. Journ.,' vol. 5, 1885, p. 905).

not more than 3 feet long and 5 inches in diameter, were less thoroughly frozen than the carcases of sheep exposed to the same temperature for forty-eight hours.

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Description of Plates.

The Heart, Ventral Aorta, and Afferent Branchial Arteries are coloured *purple*, the remaining arteries *red*, and the veins *blue;* cartilaginous structures are *dotted*.

Reference Letters.

The following list refers only to the vascular system ; other structures are referred to in the description of the figures in which they occur :—

The letter *R* or *L* prefixed to an abbreviation indicates *right* or *left*.

When there is a series of vessels to which the same name is applied (*e.g.*, the afferent and efferent branchial arteries), the individual trunks are numbered from before backwards.

The aperture of an artery or vein into a vessel of higher order is indicated by the abbreviation for the vessel in question with the addition of a *dash*—*e.g.*, *Jug.* ′. in fig. 2 points to the aperture of the jugular vein into the precaval sinus.

Heart.

Aur. Auricle.
C. art. Conus arteriosus.
Sin. ven. Sinus venosus.
Sin. aur. ap. Sinu-auricular aperture.
Vent. Ventricle.

Arteries.

Add. mand. Branch of mandibular to M. adductor mandibulæ.
Af. br. A. Afferent branchial.
Ant. car. A. Anterior carotid.
Ant. cereb. A. Anterior cerebral.
Ant. gast. A. Anterior gastric.
Ant. lat. A. Anterior lateral.
Ant. mes. A. Anterior mesenteric.
Ant. sperm. mes. A. Anterior spermatico-mesenteric.
Brach. A. Brachial.
Buc. A. Buccal.
Caud. A. Caudal.
Cereb. A. Cerebral.
Cœl. A. Cœliac.
Cœl. mes. A. Cœliaco-mesenteric.
Com. 1. Commissural trunk uniting ventral ends of efferent branchials.
Com. 2. Commissure uniting the two efferent branchials of each holobranch.
Com. 3 and 4. Commissures uniting *com.* 1 with hypobranchials.
Cor. A. Coronary.
Cor. mand. Branch of mandibular to M. coraco-mandibularis.
D. Ao. Dorsal aorta.
Dors. gast. A. Dorsal gastric.
Dors. intest. A. Dorsal intestinal.
Ef. br. A. Efferent branchial.
Epbr. A. Epibranchial.
Ext. rect. Branch of orbital to M. rectus externus.
Fem. A. Femoral.
Hep. A. Hepatic.
Hypbr. A. Hypobranchial.
Il. A. Iliac.
Inf. Obl. Branch of orbital to M. obliquus inferior.
Inf. rect. Branch of orbital to M. rectus inferior.
Int. intest. A. Intra-intestinal.
Int. rect. Branch of orbital to M. rectus internus.
Mandib. A. Mandibular.
Max. na. A. Maxillo-nasal.
Myel. A. Myelonal.
Nu. br. A. Nutrient branchial.
Nu. hy. A. Nutrient hyoidean.
Ophth. A. Ophthalmic.
Orbit. A. Orbital.
Ovid. A. Oviducal.
P. card. A. Pericardial.
Post. car. A. Posterior carotid.
Post. cereb. A. Posterior cerebral.
Post. lat. A. Posterior lateral.
Post. mes. A. Posterior mesenteric.
Post. sperm. mes. A. Posterior spermatico-mesenteric.
Psbr. Arterioles of pseudobranch.
Psbr. A. Pseudobranchial.
Pyl. A. Pyloric.
Rost. A. Rostral.
Spl. A. Splenic.
Subcl. A. Subclavian.
Sup. obl. Branch of orbital to M. obliquus superior.
Sup. rect. Do. to M. rectus superior.
V. Ao. Ventral aorta.
Vent. gast. A. Ventral gastric.
Vent. intest. A. Ventral intestinal.
w. Commissural trunk between anterior and posterior carotid.
x. Point of origin of *y* from posterior carotid.
y. Hyoidean epibranchial.
z. Anterior or inter-hyoidean portion of dorsal aorta.

Veins and Sinuses.

Ant. cereb. V. Anterior cerebral.
Ant. fac. V. Anterior facial.
Ant. gast. V. Anterior gastric.
Ant. li.-gast. V. Anterior lieno-gastric.
Ant. vent. cut. V. Anterior ventral cutaneous.
A.p. Aperture between right and left cardinals.
Brach. V. Brachial.
Card. S., *Card. V.* Cardinal.
Caud. V. Caudal.
Clo. V. Cloacal.
Cor. V. Coronary.
D. f. 1. Loop formed round base of 1st dorsal fin by dorsal cutaneous.
D. f. 2. Do. round 2nd dorsal fin.
Dors. cut. V. Dorsal cutaneous.
Dors. intest. V. Dorsal intestinal.
Duod. V. Duodenal.
Fem. V. Femoral.
Gast. intest. V. Gastro-intestinal.
Hep. port. V. Hepatic portal.
Hep. S., *Hep. V.* Hepatic.
Hyoid. S. Hyoidean.
Il. V. Iliac.
Inf. jug. V. Inferior jugular.
Int. intest. V. Intra-intestinal.
Jug. V. Jugular.
Lat. cut. V. Lateral cutaneous.
Lat. V. Lateral.
Nu. br. V. Nutrient branchial.
Nu. hy. V. Nutrient hyoidean.
Orbit. S. Orbital.
Pancr. V. Pancreatic.
Post. cereb. V. Posterior cerebral.
Post. fac. V. Posterior facial.
Post. li.-gast. V. Posterior lieno-gastric.
Post. vent. cut. V. Posterior ventral cutaneous.
Pr. cav. S. Precaval.
S. scap. S. Subscapular.
V. Anastomotic trunk between portal and spermatic.
V. f. Loop formed round base of ventral (=anal) fin by posterior ventral cutaneous.
Vent. gast. V. Ventral gastric.
Vent. intest. V. Ventral intestinal.

PLATE 34.

Mustelus antarcticus.

Fig. 1. General view of the vascular system from the ventral aspect (half nat. size).
The afferent branchial arteries are removed on the left side (right in the figure) and the greater part of the efferent branchials on the right; the left efferent branchials have their ventral ends reflected outwards so as to bring them into one plane.

Fig. 2. The heart with the chief arteries and veins of the sub-mental and sub-branchial regions; from the ventral aspect (nat. size).
The arteries are for the most part removed on the right side (left in the

figure) and the veins on the left; the precaval sinus has its ventral wall removed.

Fig. 3. The vessels in connexion with the cloaca and adjacent parts; from the left side (nat. size).

Clo., cloaca; *Kd.*, left kidney; *Rect.*, rectum; *Rect. gl.*, rectal gland; *Ov.*, ovary; *Ovd.*, left oviduct.

Fig. 4. The heart; from the right side (nat. size).

Fig. 5. Transverse section of the second holobranch (nat. size).

Br. fil., branchial filaments; *Br. r.*, cartilaginous branchial ray; *C. br.*, cerato-branchial; *Ex. br.*, extra-branchial; *M. ep. c. br.*, M. epi-cerato-branchialis; *M. rad.*, radiating muscle; *No.* 1, *No.* 2, branches of vagus.

PLATE 35.

Mustelus antarcticus.

Fig. 6. The anterior portion of the dorsal aorta with the efferent branchial, cephalic, &c., arteries; from the ventral aspect (nat. size).

The efferent branchial arteries are brought into one plane by their ventral ends being reflected outwards.

Fig. 7. The cœliaco-mesenteric artery; from the ventral aspect (nat. size).

Fig. 8. The brain and anterior part of the spinal cord, showing the arteries on the left side, the veins on the right; from the dorsal aspect (nat. size).

Dien., diencephalon (=thalamencephalon); *Epen.*, epencephalon or cerebellum; *Mesen.*, mesencephalon (optic lobes); *Meten.*, metencephalon or medulla oblongata; *Myel.*, myelon or spinal cord; *Prosen.*, prosencephalon; *Rhinen.*, rhinencephalon or olfactory lobe.

Fig. 9. Part of a longitudinal horizontal section of a frozen specimen, showing the connexion of the chief veins with the sinus venosus and precaval sinus; seen from the dorsal aspect (nat. size).

Lr., liver; *p.*, partition dividing hepatic sinus into right and left compartments; *Peric. perit. Sept.*, pericardio-peritoneal septum.

Fig. 10. Diagrammatic view of the head with the chief blood-vessels; from the left side (nat. size).

Aud. caps., auditory capsule; *C. br.* 5, dorsal end of fifth cerato branchial; *C. hy.*, dorsal end of cerato-hyal; *Ep. br.* 5, ventral end of fifth epi-branchial; *Hy. Mn.*, ventral end of hyomandibular; *Mck.*, proximal end of lower jaw; *Pl. Qu.*, proximal end of upper jaw; *Spir.*, spiracle.

Fig. 11. The left orbital artery, showing the branches to the eye muscles (nat. size).

PLATE 36.

Mustelus antarcticus.

Fig. 12. The alimentary canal and its blood-vessels; from the dorsal aspect (half nat. size).

Card. st., cardiac division of stomach; *Pyl. st.*, pyloric division; *Intest.*, intestine, partly cut open to show *Spir. V.*, spiral valve; *Rect.*, rectum; *Rect. gl.*, rectal gland.

Fig. 13. The hepatic portal vein; from the ventral aspect (nat. size).

Fig. 14. The cutaneous veins; from the ventral aspect (half nat. size).

All four veins are shown as interrupted a short distance caudad of the vent, so as to bring them within the compass of the plate; the dorsal vein is also interrupted, for the sake of clearness, where it passes behind the cloacal, &c., veins.

Fig. 15. The first dorsal fin and adjacent parts, showing the relation of the dorsal cutaneous vein to the fin and to the renal portal vein; from the left side (half nat. size).

Kd., the left kidney.

Fig. 16. The left subscapular sinus, supposed to be seen through the scapula, showing the entrance of the lateral cutaneous vein; from the left side (nat. size).

Scap., scapula.

Callorhynchus antarcticus.

Fig. 17. The efferent branchial and cephalic arteries with part of the dorsal aorta, for comparison with fig. 6; from the ventral aspect (nat. size).

PLATE 37.

Mustelus antarcticus.

Transverse sections of adult female (all $\frac{3}{5}$ nat. size).

Fig. 18. Through the branchial region; posterior face of section.

Br. 1, *Br.* 2, first and second holobranchs; *Hy.*, hyoidean hemibranch; *Ph.*, pharynx.

Fig. 19. Taken immediately caudad of the dorsal moiety of the pericardio-peritoneal septum and through the posterior extremity or apex of the ventricle; on the right side the pericardio-peritoneal septum is cut away and the sinus venosus and precaval sinus are opened; posterior face of section.

Cor., coracoid; *Œs.*, œsophagus; *Peric. perit. Sept.*, pericardio-peritoneal septum.

Fig. 20. Anterior face of the next section, showing the apex of the ventricle and the ventral moiety of the pericardio-peritoneal septum.

The anterior end of the liver (*Lr.*) is cut through, showing the hepatic sinus with its vertical partition (*P.*). In the cardinal sinus is seen the aperture of the subscapular sinus (*S. scap. S'.*), and the lateral vein is cut through close to the entrance of the brachial vein (*Brach. V'.*); *Cor.*, coracoid; *Ovd.*, oviduct.

Fig. 21. Posterior face of the same section, showing the cardinal sinuses at their widest. Letters as before.

Fig. 22. Through the first dorsal fin, showing the principal abdominal viscera and their blood-vessels; posterior face of section.

Card. st., cardiac division of stomach; *Pyl. st.*, pyloric division; *Intest.*, intestine showing spiral valve; *Lr.*, liver; *Spl.*, spleen; *Ov.*, ovary; *Ovd.*, oviducts, the left abnormally situated to the right of the stomach and intestine; *Kd.*, kidneys.

Fig. 23. Portion of section through the posterior abdominal region (4 cms. cephalad of pubes), showing the median position of the right cardinal.

Fig. 24. Through the symphysis pubis, showing the diminished size of the cardinal, the increased size of the renal portal, and the anastomosis of the lateral veins; anterior face of section.

Rect., rectum; *Ovd.*, oviducts.

Fig. 25. Posterior face of part of the same section, showing the caudal vein bifurcating to form the renal portals.

Scymnus lichia.

Transverse sections of embryo, "Stage O" (Balfour); all fifteen times nat. size.

Fig. 26. Through the posterior branchial region. Various segments of the branchial arches, not yet chondrified, are indicated by dark shading.

Br. 2, *Br.* 3, the second and third holobranchs; *Ph.*, the pharynx.

Fig. 27. Through the cardiac region. The section is slightly oblique, the right side being caudad of the left, and showing the entrance of the jugular into the precaval sinus; on the left side the jugular and inferior jugular are still separate.

Œs., œsophagus.

Fig. 28. Taken a short distance (eight sections) caudad of the preceding, which it resembles in being slightly oblique. On the left side the entrance of the cardinal into the precaval sinus is shown; on the right the cardinal is separate, and the entrance of the lateral vein is shown. The ventricle is

cut through immediately caudad of the auriculo-ventricular and arterial apertures.

Scap., scapula; *Cor.*, coracoid.

Fig. 29. Through the pectoral fins.

St., stomach; *Lr.*, liver; *Ovd.*, oviduct; *Som. stk.*, somatic stalk.

Fig. 30. Through the middle abdominal region.

Intest., intestine, with spiral valve; *Spl.*, spleen; *Ov.*, ovary; other letters as before.

Fig. 31. Through the posterior abdominal region. The cardinal veins have united with one another immediately cephalad of their union with the caudal vein

Int. ren., inter-renal body; other letters as before.

Fig. 32. Through the pelvic fins.

Rect., rectum; *Rect. gl.*, rectal gland.

XVIII. *Contributions to the Anatomy of the Central Nervous System in Vertebrate Animals.*

By Alfred Sanders, *M.R.C.S., F.L.S.*

Communicated by Dr. Günther, *F.R.S.*

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[Plates 38—41.]

Part I.—Ichthyopsida. Section I.—Pisces. Subsection II.—Plagiostomata.

Introduction.

Among the numerous writers who have turned their attention to the nervous system of Fishes, a list of whom comprises most of the great anatomists of the present and past century, nearly all have confined their investigations to the brain of the Teleostei, to which their attention was almost exclusively directed, and only to a small extent was the nervous system of the Plagiostomata referred to. The names of these writers were given in the first of this series of papers.* Busch † was the first who devoted a treatise entirely to the nervous system of the Plagiostomata, with which he combined the Ganoids. Written in Latin, this is a plain and, upon the whole, accurate description of the external or macroscopic appearance of the brain of the Plagiostomata and Ganoids.

Miklucho-Maclay's ‡ contribution, also macroscopic, appeared in 1870. In it he propounded an entirely new theory as to the arrangement of the various parts of the central nervous system. I have made some remarks on this in a former paper; suffice it here to mention that he describes as the thalamencephalon ("Zwischenhirn") that part of the brain which the majority of anatomists consider to be the optic lobe, and the lobe which almost unanimously has been described as the cerebellum he maintains to be the mesencephalon, while he restricts the term hind brain to the small posterior and inferior tuberosity of the cerebellum.

* 'Phil. Trans.' (vol. 169, pp. 769—772), 1878, part ii.

† "De Selachiorum et Ganoideorum Encephalo." Berolini, 1848.

‡ "Beiträge zur vergleichende Neurologie der Wirbelthiere." Leipzig, 1870.

The next writer, VIAULT,* turned his attention principally to the microscopic features of the brain of these animals; he was almost the first to do so exclusively to the Plagiostomata. In a general way he was preceded by STIEDA† in the Teleostei, but the investigations of this writer in the Plagiostomata were confined to the spinal cord only.

LEYDIG‡ had long previously conducted researches into the microscopic anatomy of the Plagiostomata, but they were principally restricted to other organs of the body and were very slightly turned to the brain, in which, however, he succeeded in making a noteworthy discovery.

VIAULT'S treatise, although very voluminous, contains no new facts, and the illustrations leave much to be desired, being stiff and apparently semi-diagrammatic. He did not neglect homology, and refuted MIKLUCHO-MACLAY on one side and ROHON and FRITSCH, in advance, on the other; against the former he asserted the claim of the cerebellum to that name, and also showed how the optic lobes could not be the cerebrum, but were homologous with the corpora bigemina ("tubercules jumeaux"), and therefore belonged to the mesencephalon. The hypoarium he identified with the tuber cinereum.

This writer, then, has adopted the ordinary interpretation of the various segments of the brain in these animals; far otherwise is it with ROHON.§ It seems as if the more recent German anatomists were unable to put pen to paper without trying to suggest some new and more or less untenable views. With regard to the cerebrum and the cerebellum he agrees with the majority of the older writers, but in the battle-ground of homology—the optic lobes—he publishes some new views. He considers these lobes not as the mesencephalon alone but as a combination of the mesencephalon with the thalamencephalon; he remarks that this segment of the brain considered histologically consists of two completely differentiated masses, although macroscopically they appear intimately united. The first mass, "Zwischenhirn," according to him, consists of a dorsal part which begins over the posterior part of the "regio ventriculi tertii," and lies as a cover upon the "Mittelhirn," and, further, of a ventral part, which is formed from the infundibulum and lobi infundibuli and their ventricles. Between and behind these two masses the Mittelhirn is inserted. This consists of the roof of the aqueduct of Sylvius (on which the dorsal part of the "Zwischenhirn" is applied), the floor of the same, and the nervous substance beneath.

These authors complete the list of those whom I have been able to discover whose work has been confined to the elucidation of the central nervous system of the Plagiostomata.

* "Recherches histologiques sur la structure des centres nerveux des Plagiostomes." 'Archives Zool. Expér.,' vol. 5, 1876.

† "Ueber den Bau des Rückenmarks der Rochen u. d. Haie." 'Zeitschr. Wissensch. Zool.,' vol. 23, 1873.

‡ "Beiträge zur Mic. Anat. u. Entwickelungsgeschichte d. Rochen u. Haie." Leipzig, 1852.

§ "Das Centralorgan des Nervensystems der Selachier." 'Wien, Akad. Denkschr.,' vol. 38, 1878 (Abthg. 2), p. 96.

Macroscopic Anatomy.

The species which have been investigated for the purposes of this essay are *Raja batis* and *clavata*, *Rhina squatina*, *Scyllium catulus* and *canicula*, and *Acanthias vulgaris*.

As the object of this paper is principally the microscopic anatomy, and only in a subordinate way is it devoted to the macroscopic appearance, a short description only of the external aspect of the brain will suffice, especially as good figures of many species are given in the works of ROHON and MIKLUCHO-MACLAY.

The Rays are distinguished by the greater solidity and size of the central nervous system, particularly of the cerebellum. The ventricles are nearly obliterated, while in Acanthias, on the other hand, the ventricles are of great size and have comparatively thin walls, so that the brain has a deceptive appearance of great size. In the Rays the olfactory lobe is attached to the anterior external angle of the cerebrum by a peduncle, which becomes longer as the size, and presumably the age, of the animal increases. In Scyllium and Acanthias the corresponding peduncle is short, and the olfactory lobe is nearly sessile and projects but a short distance in front of the anterior end of the cerebrum; in the former of these two species the olfactory lobe is of large size, almost equalling the cerebrum itself, while in Acanthias and Rhina it is comparatively small, and in the latter it is connected to the hemispheres by a somewhat long but thin peduncle.

The cerebrum in Acanthias and Rhina presents both dorsally and ventrally a longitudinal furrow, which is well developed anteriorly; in Scyllium there is a dorsal furrow, but none on the ventral surface. In the Rays the surface is smooth, but in Scyllium the upper surface presents a pair of tuberosities.

The cerebrum is attached to the anterior end of the optic lobes by a gutter-shaped neck of greater or less length, the thalamencephalon, of which the nervous tissue forms the floor and sides only; the dorsal portion is covered by the pia mater, and gives access to processes from the same which form a veritable plexus choroides. The posterior edge of the thalamencephalon is marked by the attachment of the pineal gland (epiphysis) which in Scyllium is carried at the end of a long stalk, and when in position is placed in contact with the interior surface of the skull above the anterior end of the cerebrum. In Rhina the pia mater forms a vascular sac which covers over and projects above the level of the thalamencephalon.

The mesencephalon, optic lobes, or corpora bigemina are two rounded prominences placed between the thalamencephalon in front and the cerebellum behind; they are separated from each other by a median longitudinal furrow. In Scyllium they are completely covered by the anterior lobe of the cerebellum; in the Rays this lobe projects only partially, and in Acanthias and Rhina scarcely at all over these segments of the brain.

On the ventral or hæmal aspect the hypoaria or lobi infundibuli are seen. They

may be considered as an appendage to the thalamencephalon; they are not so well developed, comparatively speaking, as in the Teleostei. In front of them is seen the chiasma nervorum opticorum, while at their posterior end in the furrow between them and the anterior end of the medulla oblongata is situated the hypophysis cerebri or pituitary body; this is connected with the inferior end of the infundibulum by a glandular tube. On each side of the hypophysis cerebri is a membranous vesicle, the saccus vasculosus, which differs from the part with the same name in the Teleostei in not being glandular. The saccus vasculosus of the Teleostei is probably included in the pituitary body in the Plagiostomata, since the posterior part has an arrangement here, corresponding to that in those fishes. The communicating tube lies between the two lobes of the hypoaria, a position which is the same as that occupied by the sac in question in the Teleostei. If this is the case the saccus vasculosus in Plagiostomata is a new formation, and has no homologue in the osseous fishes.

The hæmal surface of the medulla oblongata presents a shallow, longitudinal, median furrow, the anterior prolongation of a similar one in the spinal cord, which is the representative of the anterior median fissure of the higher vertebrates.

The cerebellum in Scyllium is cylindrical in its central part, but terminates anteriorly and posteriorly in pointed extremities; its external surface is smooth, but presents in the midline a longitudinal furrow.

In the Rays, also, there is a longitudinal furrow, and several subordinate furrows give the external surface a more complicated appearance.

In Acanthias and Rhina the cerebellum is comparatively small, oval in shape, and is marked by a longitudinal furrow, which in the former is supplemented by another, placed transversely, forming the figure of a cross.

At the sides the cerebellum is connected to the medulla oblongata by the crura ad medullam; at the point of junction where these become the restiform bodies,* and for some distance behind, some folds are developed in Scyllium, Acanthias, and Rhina. In the Rays these folds form quite a complicated arrangement.

The medulla oblongata forms a wide shallow trough, the fourth ventricle, or sinus rhomboideus, which contracts gradually behind until it becomes merged in the spinal cord at the calamus scriptorius. In the Rays the fourth ventricle is much narrower than in the other species. The open dorsal part is covered by a process of pia mater, which is devoid of nervous tissue, but develops tortuous folds lined by columnar epithelium, enclosing both arteries and veins, and altogether has a glandular appearance.

The spinal cord in Raja has a quadrangular section, its largest diameter being at the posterior end of medulla oblongata; it gradually diminishes in size towards the posterior end of the body, and extends to within one inch from the extreme end of the tail. In specimens which I have dissected I did not find the enlargement at its anterior extremity as stated in Owen's 'Anatomy of Vertebrates.'† In Scyllium, Rhina, and Acanthias, the shape of this part is oval, and slightly depressed on the

* Busch, *loc. cit.*, p. 23. † Vol. 4, p. 271.

ventral surface, on which is a longitudinal furrow, the representative of the anterior median fissure.

Apparent Origin of the Cranial Nerves.

First pair.—The olfactory nerves pass out from the anterior end of the bulbi olfactorii, and passing through the dense connective tissue partitions, are distributed on the pectinated mucous membrane of the olfactory organ; their course and terminations do not come within the scope of this work.

Second pair.—The optic nerves present a well marked chiasma which projects beyond the anterior end of the lobus infundibuli or hypoarium.

Third pair (figs. 1 and 2)—Oculomotorii—rise from the usual position on the inferior surface of the medulla oblongata. Their origins are covered here by the hypophysis cerebri, and there is a slight interval between them and the posterior end of the hypoaria. This fact seems to show that the posterior lobes of these bodies, which are well developed in the Teleostei, are not represented in the Plagiostomata.

Fourth pair—Trochleares—have also a well-defined and constant superficial origin; they invariably emerge from a furrow between the optic lobes and the cerebellum, marking the point of demarcation between the mid-brain and the hind-brain.

Fifth pair (figs. 1, 2, and 3)—Trifacial—arise by two roots in the Rays, one the anterior from the side of the crura cerebelli, the other more dorsal from the folds of the restiform bodies; these two join together and pass through the foramen into the orbit, and immediately join the Gasserian ganglion.

The course of the ophthalmic branch is rather peculiar; it is derived from the upper root in the foramen, and divides into two branches, one of which passes over and the other under the external rectus muscle of the eyeball; these, uniting again into one trunk, pass over the olfactory sac, and emerging through a passage in the cartilage go on to supply the anterior pointed angle of the body.

In Acanthias (fig. 3) the trifacial arises by one root only from the lower edge of the restiform bodies. It passes out of the skull through its own foramen, and immediately joins the Gasserian ganglion. The three usual divisions arise from this ganglion; the superior of the three, the ophthalmic, seems here to consist almost entirely of the frontal branch, which passes horizontally through the orbit, crosses above the peduncle of the olfactory lobe, and is distributed to the skin as far as the anterior extremity of the snout. The next division, the largest of the three, emerges from the anterior end of the inferior side of the ganglion, having the origins of the recti muscles between itself and the last branch, and passes downward and forward through the deep part of the orbit to the ventral edge, where it divides into two branches; of these the inferior supplies the region opposite to and below the foramen of the optic nerve; the other branch passes forward beneath the peduncle of the olfactory lobe, and supplies the inferior surface of the snout behind the distribution of the frontal. The third division, the smallest of the three, arises from the inferior angle of the

ganglion, and passing directly downward supplies the integument about the angle of the mouth.

In Rhina (fig. 2) the trifacial has a corresponding origin and distribution to that in Acanthias, except that it arises by two roots.

In Scyllium (fig. 1) the arrangement of this nerve is somewhat different, it arises by a single root opposite the posterior end of the crura cerebelli ad medullam. While still within the skull it gives off two branches, each of which passes through its own separate foramen; outside these join together and pursue a course comparable to that of the ophthalmic in Acanthias and Rhina. The root, having given off these branches, passes on through another foramen, and immediately outside it forms the Gasserian ganglion, which gives off the superior maxillary branch from its distal extremity; from the proximal end of the ganglion is given off a trunk which at once divides into two branches, of which one passes beneath the trunk of the superior maxillary nerve, and piercing the floor of the orbit supplies the muscles situated beneath that part, viz., the masticatory muscles. The other branch follows the usual course of the facial; it passes obliquely backward and outward behind the spiracle to the outer margin of the body in front of the branchial chamber.

Sixth pair—Abducentes—arise near the central line of the medulla oblongata on the ventral surface by four bundles, which soon unite into a single strand on each side. AHLBORN* describes a small nerve in the Petromyzon, which he thinks is this nerve, but it is evidently only an unusual branch of the trifacial. If it were the abducens, Petromyzon would be an exception to all other vertebrata.

Seventh pair.—Facial.—In the Rays this is inseparably united to the under part of the upper root of the trifacial; it emerges through a separate foramen immediately in front of the ear capsule and, passing over the muscles of the jaws, supplies the cheek.

In Acanthias (fig. 3) also the facial has a separate root, which arises in close juxtaposition to the posterior side of the trifacial. It passes through a foramen separated from that of the former by a very thin plate of cartilage; it supplies the region of the spiracle. In Rhina and Scyllium this nerve is a branch from the Gasserian ganglion. ROHON† describes it as a branch of the acusticus, but in all specimens that have been examined by me there have been none in which the two nerves could be said to be joined either at their origins or elsewhere.

Eighth pair—Acusticus.—In all these species this is in juxtaposition to the posterior side of the facial, and arises beneath the upper root of the trifacial.

Ninth pair (fig. 3).—Glossopharyngeal.—In Acanthias this nerve arises from the side of the medulla oblongata on about the same level as the trifacial. It is a small nerve, but is quite independent of the trifacial in front and of the vagus behind; it

* "Untersuchungen über das Gehirn der Petromyzonten." 'Zeitschr. f. Wissensch. Zool.,' vol. 39, 1883, p. 211.

† *Loc. cit.*, p. 63.

pierces the auditory apparatus obliquely, passes backward and outward through a channel in the cartilage of the skull, and supplies the first branchial arch.

In Scyllium (fig. 1) also this nerve arises independently of the vagus, but nearer to it than the corresponding one in Acanthias; it pursues a course through the auditory apparatus resembling that in Acanthias, and like it is distributed to the first branchial arch.

In the Rays it has a similar distribution, but arises beneath the anterior roots of the vagus.

In Rhina (fig. 2) this nerve has a somewhat different arrangement. It arises in connexion with the anterior side of the anterior root of the vagus, and after having passed through the chamber containing the otolithic vesicle, enters a long channel in the postero-external angle of the skull, in which it enlarges into an elongated fusiform ganglion, and continuing in the same outward and backward direction eventually divides into two branches, of which the anterior supplies the membrane bounding the gill cavity, and runs along the anterior surface of the branchia attached to it; the other supplies the anterior border of the first branchial arch and passes round its outer edge towards the pharynx.

Tenth pair (fig. 2).—Vagus.—In Rhina this nerve rises by seven roots. The two anterior roots are thickest; they arise close together and join immediately. The third root, which was only found on one side, comes out close behind the posterior of these. Then follows an interval, after which occur three fine roots, which immediately join together to form a single fine cord, which after a short course joins the third root in the foramen of exit; so that the nerve emerges from the skull in two divisions, which close outside and, under cover of the projecting process containing the glosso-pharyngeal, develop a large ganglion. Close outside the foramen a large branch, which appears as a continuation of the two anterior roots, is given off from the outer edge of the proximal end of the ganglion, and passing outward and backward, bifurcates at the edge of the branchial arches; one branch goes along the posterior border of the first and the other to the anterior border of the second branchial arch.

The ganglion at its distal end divides into three branches; the outer of these passes backward and divides into two nerves, which are distributed in the same manner as the last, one going to the posterior border of the second, and the other to the anterior edge of the third branchial arch. The middle one of the three terminal branches of the ganglion is the largest, and passes backward to form the ramus intestinalis of the vagus; before entering the abdominal cavity it gives rise to a branch, which goes towards the outside and, dividing into two, supplies the posterior border of the third branchial arch and the anterior border of the fourth, precisely in the same manner as the others. The ramus intestinalis gives off another branch beyond, but this I failed to trace to the posterior border of the fourth branchial arch, which it probably supplies. The remaining division of the ganglion passes down the back between the muscles and forms the ramus lateralis.

This arrangement corresponds to that described by GEGENBAUR* in Hexanchus, although there is a slight difference in detail in the distal distribution. In Hexanchus there appear to be a greater number of branchial arches, so there are a greater number of these rami branchiales. Moreover, the third branch, the ramus pharyngeus, seems to be absent in Rhina, as I did not find it.

In Scyllium (fig. 1) this nerve arises by six roots, which are placed close together with no interval between; the most anterior of these roots is the largest and folds over the others, forming nearly half the nerve. After passing through the foramen the nerve develops an elongated ganglion, from which four branches are given off, corresponding to those in Rhina. In the distal arrangement, however, the rami branchiales differ in not bifurcating; but each branch goes direct into the base of the posterior gill of its respective branchial arch, so that they are represented solely by the rami pharyngei; in other respects the distribution in both species is alike. Acanthias differs from the other species in this that the vagus, instead of arising by several small roots, arises by one large one placed some distance behind the origin of the glossopharyngeal.

Eleventh pair.—The hypoglossal does not appear to be present in any of the specimens examined by me.

The spinal nerves have the usual dorsal and ventral roots; in the Rays only do they present any peculiarity, the anterior nerves, to the number of seventeen or more, come off close together in a small space beyond the skull. This arrangement, of course, has reference to the largely developed lateral fins, which they supply; they are much larger in size and closer together than in the other species examined.

The Central Cavities.

The central cavities, canals, and ventricles are lined throughout by a layer of endothelium, which is also termed ependyma by some writers.

The shape of the central canal in the spinal cord (figs. 6, 7, and 8) varies somewhat in Scyllium, but usually it is oval in section with the long axis placed transversely; the transition to the fourth ventricle is indicated by a deficiency of the endothelium on the dorsal edge of the canal. The lumen gradually enlarges, in some cases becoming triangular or in others lozenge-shaped, the upper angle gradually extending to the dorsal surface, so that the canal at last becomes open at the top, forming the calamus scriptorius. There is often found a small quantity of granular material in the canal, which presents the appearance of being a coagulation; it corresponds to the rod occasionally found in the central canal in the Teleostei, and shows perhaps that the cerebro-spinal fluid coagulates more firmly in the latter than in the former.

The fourth ventricle (figs. 9, 10, 11, 12, and 13) is a broad shallow trough, extending from the calamus scriptorius to the anterior end of the crura cerebelli ad medullam,

* "Ueber die Kopfnerven von Hexanchus." 'Jenaische Zeitschrift,' vol. 6, 1871, p. 524.

where it passes into the aqueduct of Sylvius. The anterior end is continuous directly with the posterior extremity of the ventricle, the widest part being immediately behind the crura cerebelli ad medullam, thus differing from the arrangement in the Teleostei, in which the anterior end forms a distinct chamber beneath the cerebellum, communicating with the posterior part by a narrow passage. Behind the cerebellum the fourth ventricle is covered by the processes of the pia mater. BUSCH* terms it the arachnoid, but adds that it differs essentially from the arachnoid of the human brain, inasmuch as it abounds in numerous vessels. He speaks of a third membrane as the pia mater dipping into all the cavities. It does not appear from my dissections to be different, but both seem to be the same membrane.

The walls of the fourth ventricle (figs 1, 2, 3, and 5) are formed on each side by an elongated and anteriorly-folded ridge which becomes merged into the lateral parts of the cerebellum. These may be looked upon as the restiform bodies. On the floor (figs. 1 and 5) may be seen two longitudinal cylindrical prominences separated by a furrow occupying the central line; these are the funiculi teretes. They indicate the position of the ventral longitudinal columns, which may be considered to correspond to the anterior pyramids. BUSCH † also considered them to be so, but in the figure he makes them extend as visible protuberances quite up to the posterior commissure, an arrangement which was not found in my specimens, although they are traceable thus far, but are not apparent on the surface. Laterally the floor is occupied by a row of hemispherical beadlike tubercles, which form the origins for the roots of the vagus. Immediately behind the crura cerebelli ad medullam the funiculi teretes disappear, and the place of each is taken by a shallow longitudinal trough, the external margin of which corresponds to the external limit of the ventral longitudinal column. Towards the anterior end of the crura a deep furrow appears in the central line of the floor of the ventricle; it extends into the aqueduct of Sylvius and gradually deepens until it joins the third ventricle. In the optic lobe the aqueduct of Sylvius becomes somewhat enlarged and forms a ventricle of a concavo-convex form, the concavity being directed upward; near to its anterior end a passage, the continuation of the furrow on the floor above mentioned, leads into the third ventricle beneath the posterior commissure.

The third ventricle, or thalamencephalon, is not closed in superiorly by nervous tissue, but gives entrance to processes from the pia mater which intrude through its roof to form the choroid plexus. It communicates by a gutter-shaped passage with the ventricle of the cerebrum, which ventricle at the posterior end is single, the foramen of MONRO, but at the anterior end becomes double, the lateral ventricles; each lateral ventricle communicates by a small round foramen with a passage leading into the peduncle of the olfactory lobe; this passage passes forward in the tuberosity situated on the outer side of the cerebrum, close to the outer wall, and soon becomes a mere slit, which enlarges when the peduncle is free of the cerebrum, and takes on a

* *Op. cit.*, p. 20, tab. iii., fig. 8.

† *Op. cit.*, p. 23.

rounded form; it penetrates only a short distance into the posterior end of the olfactory lobe so that the ventricle of this is but small.

Prof. Burt. G. Wilder* gives a different interpretation to these facts. He considers that the ventricles in the olfactory lobes are the true lateral ventricles, that the apertures by which they communicate with the ventricles of the cerebrum are the foramina of Monro, and that the walls of the cerebrum are what he proposes to call the "prothalami," connected by the crura thalami with the optic lobes, and that the ventricles which they contain are simply the forward continuation of the third ventricle. There are many objections to this mode of interpretation; the ventricles in the cerebrum and their continuation backward into the third ventricle fulfil all the requirements of the case; but principally the structure of the olfactory lobes would forbid their being interpreted as cerebral hemispheres, which would have to be the case if their cavities were to be looked upon as lateral ventricles.

The third ventricle extends inferiorly into the ventricles of the hypoaria, which in the species under consideration, viz., Scyllium, is broad from side to side, and narrow from above downward; posteriorly it ends in two lateral pouches, between which, in the midline, is a communication with a cavity in the interior of the hypophysis cerebri, which cavity is lined by a continuation of the endothelium lining the ventricle of the hypoarium.

The arrangement in Rhina and Acanthias resembles that of Scyllium, but in the former all the cavities are more extensive. On the other hand, in the Rays all the ventricles are smaller and more contracted; the fourth ventricle is a comparatively narrow deep fissure, the ventricle of the optic lobe is a small triangular enlargement of the aqueduct of Sylvius, and communicates by a narrow passage with the third ventricle. The lateral ventricles are non-existent, only a single very small cavity extending for rather less than one quarter of its length into the parenchyma of the posterior inferior margin of the cerebrum remains as the representative of the foramen of Monro (fig. 18).

Microscopic Anatomy of the Brain.

Lobi Olfactorii.

The olfactory lobe may be looked upon as consisting of two parts, the lobe proper and the peduncle, but this classification is merely superficial, as in structure they pass gradually into each other. The lobe itself is more or less pear-shaped, broader at the anterior end, where it abuts on to the olfactory organ, and narrower behind, where it is attached to the peduncle.

The olfactory lobe may be described as consisting of three layers placed one in front of the other. The internal, which is also the posterior layer, occupies more than half of the lobe at the posterior extremity, and consists entirely of a mass of small cells embedded in a network of fibrillæ and granular neuroglia. In front of these is seen

* 'Philadelphia, Acad. Nat. Sci. Proc.,' 1876.

the middle layer which consists of glomeruli olfactorii. These occupy all the anterior side of the mass of small cells, and extend a short distance both above and below along the sides; the posterior contour of this middle layer is not however regular, but is rather concavo-convex in outline.

The glomeruli olfactorii (fig. 23) were first discovered by LEYDIG.* He gives a figure which differs considerably from mine, and shows only granular neuroglia and no fibrillæ; at one end is a compact bundle of unseparated fibres going to the olfactory organ, at the other end a group of uncleated cells attached to the glomerulus by fibrils. The difference in the mode of treatment usual at that time would account probably for the non-separation of the fibrillar network.

The anterior layer consists of interlacing bundles of fibres passing from the anterior end of the glomeruli in a more or less curved course to the connective tissue framework which supports the follicles of the olfactory organ.

The most conspicuous difference to be observed in this lobe, as compared with the corresponding part in the Teleostei, is the much higher development of the glomeruli, which in the osseous fishes are represented by merely an undifferentiated mass of neuroglia.

The peduncle has also three layers. The interior is occupied by a layer of small cells which surround the lumen of the peduncular passage. Outside these is a stratum of glomeruli which extends round the internal layer on all sides, except one very small gap. They are placed thicker at the anterior end of the peduncle, and gradually thin out on approaching the cerebrum; they cease entirely within one section from that body. The outer circumference of the peduncle is occupied by the fibres derived from these glomeruli, which also become more numerous, and occupy a larger space on proceeding more forward.

The description above given is taken from specimens of Scyllium; in the other species examined essentially the same disposition of the layers is found.

We now go more particularly into the minute structure (fig. 23), commencing with the third or internal layer. This layer is composed of a granular neuroglia, in which ramify fibrillæ of extreme tenuity forming an extensive network. In this are embedded numerous small cells; these are oval, pear-shaped, or spherical, and in many cases show a nucleus and nucleolus; they give off processes which join the network. In Scyllium and Acanthias they are of about the same size. The larger ones measure about 9μ long by 7μ broad; from these to the smaller ones of 5μ there is every variety of dimensions. In the Rays also they are of about this size. In Rhina are a trifle larger, varying from 13μ by 8μ to 8μ in diameter. Cells of these different sizes are seen scattered thickly throughout the layer.

The glomeruli (fig. 23) are elongated pear-shaped or oval masses arranged generally with their long axes in the direction of the nerve fibres; that is to say, one end is turned toward the internal and the other toward the first or external layer. They

* 'Beiträge z. Mikr. Anatomie u. Entwickelungsgeschichte d. Rochen u. Haie,' Leipzig, 1852.

consist of a central core of granular material, which in fortunate preparations is seen to be composed of closely intertwined fibrillæ of very fine dimensions; externally fibrils of a larger size are to be observed running parallel to the long axis of the glomerulus, in the course of which elongated cells are developed. These fibrils occupy the outside of and the space between the glomeruli; they converge towards the outer or anterior end of these bodies, and go to form a flat bundle of fibrils emerging from that end of the glomerulus; at the inner extremity a fibrillar network, a continuation of the central core, connects these structures with the internal or third layer.

The course of the fibrillæ would seem to be from the cells occupying the internal layer into the centre of the glomeruli, where they form a much closer network; then by some means not quite apparent they would seem to join the fibrils, enveloping the exterior surface of those bodies to form the bundles which pass out to join the nerves which supply the Schneiderian membrane.

In Scyllium the cells which occur in the course of the external fibrils of the glomeruli are rather larger than in the other species, and the cells of the internal layer which are found in the neighbourhood of the glomeruli are larger (18μ by 14μ) than those in the interior of the internal layer, which is not the case in the other species. With this exception the above account applies equally to all the species investigated.

The ventricle of this lobe is lined by an endothelium, consisting of a single layer of cells; this is a continuation forward of the layer which lines the other ventricles of the brain and the central canal of the spinal cord.

The structure of the peduncle gradually merges into that of the cerebrum without any distinct or obvious line of demarcation; neither are there to be seen any bundles of fibres or any crura connecting the one with the other.

Cerebrum.

On examining a transverse section through the cerebrum of Rhina, it will be observed that externally beneath the pia mater the cerebral parenchyma is composed almost entirely of a granular neuroglia, through which a few small cells are sparsely dispersed; these measure generally from 7μ to 8μ in length and from 5μ to 7μ in width; they give off processes which join the network of fibrillæ which pervades the neuroglia. This layer occupies about one-sixth of the whole thickness of the cerebrum between the external surface and the lateral ventricle. Many capillaries penetrate the brain from the pia mater through this part.

Internal to this layer the whole extent of the remaining five-sixths of the cerebral wall is occupied by a mass of cells; these, in addition to cells more or less spherical or oval, resembling those in the outer layer in size and appearance, consist of a great number of larger cells, measuring from 10μ to 12μ in diameter, together with others more or less oval in shape and measuring from 13μ to 8μ in long diameter and from

8μ to 6μ broad. Towards the internal surface the smaller cells again predominate in number and are more scattered than in the central portions occupied by the larger cells, but are yet closer together and more numerous than in the outer layer. The ventricles are lined by an endothelium composed of a single layer of columnar cells.

In Scyllium the cells of the inner layer appear to average a smaller size than in Rhina, being from 5 μ to 6 μ in diameter, the smaller ones predominating; they are arranged in elongated groups parallel to the endothelium. The fibrillæ have principally a tendency to a radial direction. On the outer side the cells are more sparingly distributed, they are generally about 7μ in diameter; in the central portions some cells have a tendency to be arranged in groups of four or five; sometimes so many as nine occur in one cluster, but these are of the smaller kind. Large cells resembling those of Rhina are not so numerous as in that species; they usually occur singly, and measure about 10μ or 11μ; some of them are elongated or pear-shaped. The neuroglia is of the same nature as in Rhina.

In Acanthias the parenchyma of the cerebrum resembles that of Rhina. On the outer side there is the usual space of granular neuroglia and network of fibrillæ, in which very small cells are scattered. Then comes the layer of cells of greater size, which are generally more elongated than in Rhina; their dimensions vary from 14μ to 7μ in length and from 10μ to 6μ in width. Towards the inner surface smaller cells predominate; as a rule the cells occur singly, but small groups, consisting of two or three cells, are sometimes to be found. This brain, therefore, forms a transition between Scyllium and Rhina.

At the base of the cerebrum (fig. 19), even with or slightly above the floor of the ventricles, there are four elongated groups of the larger cells agglomerated together with little if any neuroglia between; they are placed one in each outer wall and one in each inner wall, all running longitudinally; they are situated at one-third of the thickness from the outer and inner wall respectively. The remainder of the section is composed of neuroglia, with cells scattered through it resembling the arrangement at the dorsal part of the brain. In the space between the outer margin and the groups above described the neuroglia is more compact and presents a smaller number of cells, which, however, are more abundant on the side nearer the ventricles, where the neuroglia is also of a looser texture. The anterior commissure is principally derived from the outer edge of the group situated in the external wall, but some few fibres are derived from the inner edge of the same; they cross over to the opposite side of the brain and bear the same relation to the group on the other side.

This commissure, which I think must homologize with the anterior commissure of mammalia, appears to correspond to what Bellonci* terms the olfactory commissure in *Anguilla vulgaris*; but in these fishes there are no visible bundles connecting it with the olfactory lobe.

* "Intorno alla struttura e alle connessioni dei Lobi Olfattorii, &c. 'Roma, R. Accad. Lincei Mem.,' vol. 13, 1882.

The crura cerebri are derived from both edges of the groups situated in the inner wall, the two origins being more equal than is the case with the anterior commissure. The nucleus in each half of the cerebrum gives out a separate column, which goes to form the crus of its own side, and has no communication with that of the opposite side.

The anterior commissure is confined to the ventral region of the cerebrum, and is not found above the level of the nuclei above described. A third bundle of fibres is mentioned by Rohon * as descending from the dorsal and middle cerebral regions obliquely towards the crura cerebri in Raja; these he considered to represent the "fasciculi longitudinales tegmenti posteriores." These bundles were not found in any of my specimens.

In the Rays there is no continuous belt of neuroglia existing round the circumference; in some places, the cells extend close to the external surface, in others the neuroglia is deficient in cells. In the interior of the cerebrum the cells are arranged still more decidedly in groups than in Scyllium; there are a larger number of cells collected together, generally from nine to twenty-one, but the individual cells are smaller, not exceeding 4 μ or 5 μ, and most of them are spherical. These clusters are separated from each other by small masses of neuroglia, and in some parts they are in contact forming an irregular network. At the base of the cerebrum there is a continuous stratum of cells which follows the curve of the posterior margin. These correspond to the groups described in Acanthias, and, as in that species, the crura cerebri are in relation to them, and pass through the posterior end of the group.

Hypoaria.

In Rhina the parenchyma of the hypoaria consists of an irregular network of minute fibrillæ, resembling that found in the corresponding parts in the Teleostei. In this parenchyma numerous cells are scattered irregularly throughout the mass; they are of various shapes, spherical, pyriform, oval, or fusiform; they give off one, or more, generally several processes, which join the above mentioned network; they differ in size from 13μ by 7μ to 6μ in diameter. They are not so regularly pear-shaped as in the Teleostei, neither do they present a distinct chamber or space surrounding them; they also give off a greater number of processes.

The absence of a chamber surrounding the cells here, and also the presence of a greater number of processes, may possibly have been conditioned by a difference in the mode of treatment. Chromic acid was used in preparing the brains of the Teleostei, this probably caused a greater contraction of the parenchyma, and so may have broken off the processes, while at the same time it formed the space round each cell, and as the parenchyma is softer in the hypoaria, so the effect would be greater. The process followed in preparing the brains of the Plagiostomata, which was the

* *Op. cit.*, p. 71.

same as that employed for the nervous system of Mormyridæ, avoided this contraction of the tissues.

The ventricle in Rhina is of great size, and occupies nearly the whole of the lobe, leaving only very narrow walls. This ventricle is lined by an endothelium which consists for the greater part of its extent of one layer of columnar cells, the internal end of which sends processes into the interior of the parenchyma; they are backed up by a layer one, two, or three deep of spherical cells of an endothelial nature, which sometimes project more or less into the substance of this lobe. Next to these cells is found a layer comparatively free from cells; here the general direction of the fibrillæ is longitudinal, but the direction of the processes from the endothelial cells is radial, crossing the former at right angles; the result being that this layer has a squared striation; the cells, which are occasionally present in this layer, are spherical in contour, resembling in this respect the inner cells of the endothelium. At the posterior portion, where the hypophysis cerebri is attached, the endothelium is composed of the rounded cells only, which are extended from four to seven deep; they are continuous with the endothelial cells of the hypophysis cerebri. The processes from the endothelial cells resemble those from the "Stiftzelle" of the cerebellum; they are present even when the endothelium consists only of the spherical cells, but they are not so well marked as when the columnar cells are found. In the layer of parenchyma, which is next to the endothelial lining of the ventricles, there are more cells to be found in that part where the endothelium consists of spherical cells than where it is composed of columnar cells only; which seems to show some connexion between the two kinds of cells, that is to say, between the spherical cells which go to form the endothelium, and those which are dispersed through the ventricular part of the parenchyma.

Scyllium and Acanthias resemble Rhina in the arrangement of the endothelium and parenchyma in this lobe. The cells in both these species are on an average smaller, and the smaller cells predominate.

In the Rays (fig. 18) the parenchyma is more compact, the fibrillæ form a closer network, and the cells are scattered with larger intervals between them. There are no ventricles in the hypoaria in the Rays, and consequently no endothelial layer; the shape of the cells is more generally fusiform than in the other species, and their average size is greater.

In order to conclude the description of the region of the third ventricle (fig. 18), a tuberosity may be mentioned which exists in the dorsal part of the wall of this ventricle and in front of the anterior end of the optic lobe. In it are found numerous small cells embedded in a granular neuroglia, but no ganglion cells are to be seen. A well-defined bundle of fibres proceeds from it backwards and slightly downwards, but disappears at the ventral side of the longitudinal columns. This tuberosity corresponds to the tuberculum intermedium of Gottsche.* Ahlborn† describes similar structures

* Müller's 'Archiv,' 1835.

† "Unters. über das Gehirn d. Petromyzonten." 'Zeitschr. Wissensch. Zool.,' vol. 39, 1883.

in Petromyzon; the ganglion he looks upon as corresponding to the ganglion habenulæ, and the bundle of fibres to MEYNERT's fibres. This interpretation may possibly be correct; the relation of the parts is not antagonistic to, but rather favours this view, but the ganglion interpedunculare is not to be found in any specimens examined.

Lobi Optici.

The ventricle of the optic lobe in Scyllium (figs. 15 and 18) is partially divided into two by a longitudinal depression of the roof, corresponding internally to the external longitudinal furrow, which gives to the outer contour the appearance of a double lobe (the corpus bigeminum), an appearance which is not verified by the internal construction, the ventricle being single; in this respect the optic lobe of the Scyllium resembles the tectum lobi optici of the Teleostei. There are, however, considerable differences between the two, more especially in the macroscopic aspect. The torus longitudinalis is absent, the transverse commissure which crossed its base alone remaining; the thickness of this lobe exceeds that of the tectum actually and proportionately. On the other hand, in compensation, the structure is simplified.

The tori semicirculares, which are well developed in the bony Fishes, are wanting in the Plagiostomata, so that in the latter the floor of the ventricle is slightly concave; the furrow in the central line is common to both.

There are three layers to be made out more or less clearly in these lobes. The first or external layer contains numerous cells embedded in a granular neuroglia; they attain their maximum in this layer, but extend also into the next in diminishing numbers. Toward the outer edge a few scattered cells only are to be seen, but they gradually increase in number, and in the centre they attain their maximum. These cells are mostly spherical in shape, and measure on an average about 8μ in diameter; in addition to these, there are found in this layer, in much smaller numbers, fusiform cells placed with their long axes in a radial direction. These cells correspond in every respect with the fusiform cells which are found in the tectum lobi optici of the Teleostei. Like all the bodies here distinguished by the term cell, they have a nucleus and a nucleolus; the nucleus is not coloured so deeply as the remainder of the cell contents by the staining fluid. This shows their correspondence in structure with the cells of the cerebrum, hypoarium, and the granular and molecular layers of the cerebellum, and differentiates them from the motor cells of the spinal cord and the medulla oblongata and other ganglion cells, in which the nucleus is stained of a deeper tint than the remainder of the cell contents. They are generally broader, but not so long as those of the Teleostei; one end gives off a comparatively thick fibre going toward the inner surface of the lobe, the other end, being pointed, probably gives off a fine process, judging from the analogy with the cells in Teleostei, but if so it escaped notice from being broken off in the sections. The position of the nucleus varies, sometimes being near the pointed, at others near the broad end of the cell.

In Acanthias, cells which have a decidedly fusiform shape are rare, the greater number being oval or rhomboidal. They occur in or on the radial fibres as in Scyllium. Most of the cells, however, are spherical, and are arranged in longitudinal groups, separated by intervals of granular neuroglia, in which are found fibres from the optic tract running from before backward; but in the second layer they are divided by bundles of fibres belonging to the transverse commissure. Occasionally, but very rarely, a fusiform cell is found in the course of a longitudinal fibre. The fusiform cells are generally 20μ long by 6μ broad, from this they vary to 12μ by 7μ; they are thus larger than the corresponding cells in Scyllium. The oval or round cells are usually about 8μ by 7μ. There is a gradation between the extremely elongated cells, and those which are nearly spherical.

In Rhina (fig. 22) the fusiform cells are more numerous, they are also larger than in Acanthias. The rounded cells are also very numerous; as in the other species they give off processes to the surrounding network.

The greater scarcity of cells in the optic lobe of the Rays is compensated for by its greater thickness. The fusiform cells are very scarce, most of those present being rounded, but some oval ones occur. They are on an average smaller than those which are found in Acanthias, and about equal in size to those present in the optic lobe of Scyllium.

The second layer consists principally of bundles of transverse fibres, derived partly from the commissura ansulata, and to a small extent from the lateral columns of the medulla oblongata; these transverse fibres form many bundles, which are separated from each other by granular neuroglia, into which the rounded cells from the first layer penetrate; so that these two layers may be looked upon as overlapping each other, the cells of the first layer encroaching on to the territory of the second; thus the limits of the two divisions are not strictly defined. In this stratum there occur also cells intermediate in character between the fusiform and the spherical cells.

The third layer lies immediately internal to the endothelial lining of the ventricle; in it are found at the posterior end only some fibres which come forward from the cerebellum; they do not extend far, and are not visible at the anterior end of the lobe. The small cells from the first and second layers also penetrate into this one, and occur quite close to the endothelium at parts. In addition to these elements, large cells (fig. 22, *Dachkerne*, Rohon) occur, which are nearly equal in size to the motor cells of the ventral horns in the spinal cord, but they differ in shape and in the number of processes, and also probably in composition, since they much more easily become crumpled and lose shape. They are spherical or oval, and sometimes pear-shaped. They are provided with a conspicuous nucleus and nucleolus; the cell contents are usually smooth, or only finely granular, the nucleus has a coarser appearance, and contains large granules, and the nucleolus is seen as a large granule with a dark border and lighter centre. They generally give off one process only, or very rarely

two. ROHON * imagined that he saw several processes cut off close to many of these cells, and therefore came to the conclusion that they were multipolar cells, but such is not the case. I have examined them very minutely, and in good preparations the contour has always appeared perfectly smooth, but in preparations that have not succeeded well their outline becomes irregular, and this no doubt misled ROHON.

The single process, when only one is present, passes towards the outside, so that of the cells on each side of the mid-line each sends its process in an opposite direction; they always join the transverse commissure which constitutes the second layer. The average length of these cells is 55 μ, and the width 35 μ. VIAULT does not appear to have seen these cells, as no mention is made of them in his account of the histology of the optic lobes.†

In Scyllium these cells first appear rather suddenly as a large circular group in that part of the roof of the aqueduct of Sylvius which corresponds to the valvula cerebelli opposite the exit of the fourth nerve; in the next section in advance the number of cells grouped together is not so large, and still farther forward they spread out into a single row laterally, while still keeping more closely clustered together in the mid-line. They extend as far as the anterior end of the optic lobe, where they terminate with a few scattered cells.

In the Rays these cells commence behind also opposite the exit of the trochlearis, not, as in Scyllium, by a large group, but by a single cell on each side close to the anterior termination of the granular layer of the cerebellum. These two cells are rather isolated; in the next three sections in advance only one is to be found; in front of this they gradually increase in number until they form a layer, one or two, and, in some places, three cells in thickness, extending some distance laterally. Anteriorly they diminish in number and terminate with two outlying ones some distance behind the transition of the optic lobe into the third ventricle. In these fishes occurred the only instance of an exception to the rule of each cell having but one process; in this case a process was seen emerging from each end. There was also a single instance only of a cell sending its process toward the mid-line instead of toward the outside.

In Rhina these cells are restricted to the mid-line and do not spread out laterally. They commence as a group of four or five cells immediately in front of the exit of the trochlearis; they rather suddenly increase in number, and are restricted to the projection of the roof into the ventricle; on the edge of the group one or two outlying cells are occasionally found. They extend forward for about half the length of the ventricle. In another specimen there was a slight variation. They commence in this specimen in the valvula cerebelli by three groups, one on each side and one situated superiorly in the mid-line; this latter disappears, and the two former coalesce centrally and extend forward nearer the anterior end of the ventricle than in the last specimen, and terminate in a few stragglers.

* *Op. cit.*, p. 78.

† *Op. cit.*, p. 501.

In Acanthias the arrangement resembles very much that found in Rhina; these cells begin suddenly some distance in front of the trochlearis and are restricted as a group in the centre of the roof of the optic ventricle; anteriorly they terminate over the posterior edge of the posterior commissure, which in this species is some distance behind the anterior termination of the ventricle.

These cells occur also in turtles, in which their arrangement resembles that described in the last two species; in these animals they are grouped together in the projecting roof of the ventricle of the optic lobe in precisely the same manner as in Acanthias or Rhina.

The function of these cells is not at first sight apparent; according to the theory that assigns motor functions to the larger cells and sensory functions to the smaller cells, these ought to be motor cells; but their position is adverse to this view, the motor cells being usually, though not always, placed on the ventral side of the nervous centres. They may be probably looked upon as corresponding to the large cells in the nerve cell layer of the retina. These latter have been considered by some authors as refuting the above-mentioned theory, but, according to Professor M. Foster,* there is a residual motion in the eye after all the nerves have been severed, which he attributes to these cells in the retina. The cells in the optic lobe have probably some such function, and it would be interesting to know whether the retina in the Plagiostomata differ in any respect from the retina of those animals, *e.g.*, Teleostei, which do not possess these cells in the optic lobe.

Cerebellum.

Two lobes may be distinguished in the cerebellum of Scyllium: one, anterior, projecting over the surface of the optic lobe for quite half its length, and ending in front in a pointed extremity; and one behind, which in like manner extends over the fourth ventricle; dorsally these coalesce and form one indistinguishable substance. Corresponding to the external shape, there is internally a ventricle which occupies the whole length of the lobe. Slightly in front of the central point there is a communication below with the general cavities of the brain through the dorsal wall of the aqueduct of Sylvius.

In structure the cerebellum (fig. 17) presents the four layers found in the brain of the Teleostei, that is to say, externally the molecular layer characterized by the striation directed perpendicularly to the external surface, then the layer of Purkinje cells or intermediate layer, internal to which occur the stratum of fibres, principally longitudinal, but in some places transverse, the fibrous layer, derived from the crura cerebelli ad medullam to a great extent; finally the granular layer is the most internal and projects into the ventricle.

The aggregate foldings of these four layers constitute the cerebellum, and in a

* 'Textbook of Physiology,' p. 408.

longitudinal section are seen to be as follows. Commencing at the posterior end of the optic lobe, with which they are connected by a narrow isthmus (the sole representative of the structure so highly developed in Teleostei), they all pass forward to form the ventral wall of the anterior lobe; arrived at the anterior point they turn back and constitute the dorsal wall of the anterior and posterior lobes, between which there is often found a small fissure; from the posterior end they again pass forward and form the ventral wall of the posterior lobe. Arrived at the foramen of communication with the fourth ventricle they turn towards the ventral surface, and then backward, supplying the posterior part of the roof of that passage, and finally turn upward to end close beneath the lower surface of the posterior lobe; here they form a curved free process which may be termed the inferior lobe. BUSCH* did not recognise the cerebellar structure of this lobe, and considered it as merely an appendage to the restiform bodies, "Fimbria ex corpore restiforme orta."

The four layers are not equal either in superficial extent or in mass (fig. 17). The external or molecular layer is deficient both at the anterior and posterior end for a small space; behind it continues from the under surface of the posterior lobe into the upper surface of the inferior lobe; at the point of junction of the two it forms a fold of its own, not participated in by the other layers (except the intermediate), which fills up the angle between the lower surface of the one and the upper surface of the other lobe.

The intermediate or PURKINJE cell layer is scarcely separable from the molecular, and may be looked upon as part of the same, it follows all its windings, and is deficient where that is deficient.

The third or fibrous layer may be traced at the junction of the anterior end of the cerebellum with the optic lobe into the posterior end of the latter, where it enters the third layer close to the endothelium; it consists of a longitudinal bundle of fibres, which passes along the ventral wall of the posterior lobe, and is dispersed and disappears in the anterior end of the granular layer. It may, perhaps, be looked upon as the representative of the crura cerebelli ad cerebrum. In the dorsal wall this layer forms two subordinate strata, the external of which consists of transverse or oblique, while the internal is formed of longitudinal fibres; in some parts there is a longitudinal layer external to as well as internal to the transverse fibres; these latter are derived directly from the crura cerebelli ad medullam, whereas the longitudinal gain their position more indirectly; they disappear in the granular layer anteriorly and posteriorly. Both sets of fibres are traceable into the inferior lobe.

On inspecting a transverse section of the cerebellum (fig. 13), it is seen that the granular layer consists of four longitudinal tuberosities or tori, which project into the ventricle of the cerebellum; they are placed one on each side of the central line of the dorsal wall, and in a corresponding position on the ventral wall. In the interval between these tori this layer is absent, and the wall at these parts is formed only by

* *Op. cit.*, Tab. 1, fig. 4.

the molecular layer with the PURKINJE cells and the fibrous layer. Anteriorly each torus commences by a rounded projection into the roof of the aqueduct, then passes dorsad along the anterior wall of the channel forming the communication between the ventricle and the aqueduct; and, passing forward along the floor of the anterior lobe, amalgamates at the anterior end of the cerebellum with the dorsal tuberosities, forming a mass of granular layer occupying the whole thickness of the lobe at that point; the two tori then pass along the whole length of the dorsal wall to the posterior end of the cerebellum, where an arrangement is found resembling that at the anterior extremity; hence the tori are to be traced into the ventral wall of the posterior lobe, and from there downward along the posterior wall of the channel above mentioned, then backward along the roof of the sinus rhomboidalis; finally, they turn upward, and end in a club-shaped free extremity as it appears in a longitudinal section. A consideration of the description just given would seem to show that, properly speaking, there are only two tori in the granular layer, since the dorsal and ventral are continuous at each end of the cerebellum.

Transverse sections (fig. 13) show that the inferior lobe is the medium of communication between the cerebellum and the medulla oblongata through the restiform bodies. The junction is effected at the lateral part of this lobe. Through the connecting link the fibres of the crura cerebelli ad medullam pass from one to the other. Owing to the folding of this lobe the section passes through the granular tori twice, giving the appearance as if there were four, two projecting into the ventricle of the cerebellum and two into the roof of the fourth ventricle. At the point of junction of this lobe with the restiform columns, the crura cerebelli part into two divisions, one, the external, passes upward along the outer wall of the cerebellum to the dorsal angle, then turning inward joins that of the other side in the mid-line, crossing through the bases of the two dorsal tori of the granular layer. In the central line, at the bottom of the fissure which divides the two tori from each other, the wall of the cerebellum consists only of the commissure in question, and a modified molecular layer, much narrower than at the other parts of the cerebellum. The internal division of the crura goes partly upward, and forms a commissure through the bases of the upper tuberosities of the inferior lobe, and partly downward to make a corresponding commissure through the lower tuberosities. At both the upper and lower central points the wall is composed only of the transverse fibres, and a small portion of the molecular layer.

The inferior lobe (fig. 17) is the only part which MIKLUCHO-MACLAY allows to belong to the hindbrain, all the remainder of the cerebellum belonging in his view to the midbrain or mesencephalon. But this view is untenable, the inferior lobe is clearly nothing more than a fold of the cerebellum with no independent characteristic whatever; it has no ventricle, nor any indication of relationship to the primitive vesicles of the brain, as a glance at the figures given by BALFOUR* would show.

* 'Development of Elasmobranch Fishes,' Plate xv., fig. 7a.

In Raja (fig. 12) the cerebellum has the same fundamental arrangement as in Scyllium, the four layers, the molecular, the PURKINJE cells, the fibrous and the granular, are all present. Externally there is a deep transverse furrow formed by the molecular layer; in the lateral walls also there are two involutions of the molecular layer all followed by that of the PURKINJE cells. Internally the ventricle is occupied by the four tori of the granular layer, which nearly fill it, leaving only a comparatively small cruciform fissure between them. The inferior lobe is well developed, and the fold of the molecular layer therein is larger than in Scyllium.

The total outcome of the difference is that in Raja the cerebellum is more solid and massive; there is a larger quantity of material. The molecular layer is more extensive, being folded into several furrows, and the granular layer forms larger tuberosities. These facts seem to show that the greater activity of life, in conjunction with the lighter and less muscular but more agile body, of the dogfish, does not require the quantity of nervous substance to start it that is necessary to move the more massively muscular and heavier carcase of the sluggish Ray, so that it would appear that quantity of muscle requires more nervous activity than quickness of motion.

The case of Rhina seems to contradict this theory, for with a massiveness of body less, certainly, than that of the Rays, but much greater than that of Scyllium, it is provided with a brain possessing less nervous tissue than that of Scyllium. How to reconcile this incongruity is more than I can say. Both in Rhina and Acanthias the cerebellum resembles that of Scyllium, except that it is lighter and thinner, and contains a larger ventricle.

The tuberosities, the arrangement of which has been just described, correspond to the granular layer in the solid cerebellum of the Teleostei; the form here, as has just been explained, is that of four, or rather two, discrete longitudinal, more or less cylindrical, tuberosities or tori. The fundamental structure is the same both in the Plagiostomata and the Teleostei; there is the network of fibrillæ (fig. 20), in which minute cells are thickly scattered in an irregular manner; these cells are rather larger than those found in the corresponding part in the Teleostei; they average 6μ by 4μ, and 5μ or 4μ in diameter, their outline is more irregular, and they give off on all sides processes which join the network of the neuroglia.

The arrangement of the network differs somewhat from that found in the Teleostei. The general direction of the fibrils is longitudinal; but in addition to this the fibrillæ form small knots or little masses of neuroglia in which the fibrils are inextricably wound together so as to be almost indistinguishable. These knots might almost be compared, on a small scale, to the glomeruli of the olfactory lobe. There is every gradation of closeness, between some in which the fibrils are scarcely more compacted together than in the remainder of the layer, to others which have almost the appearance of cells. They are best seen in the Acanthias, in which they are more developed than in the other species. The cells are, on an average, of larger size in Rhina than in either Acanthias, Scyllium, or Raja, in all of which they are about equal.

The intermediate layer of PURKINJE cells follows the folds and involutions of the molecular layer, except a small tract along the central line of the dorsal portion. In it are seen not only the true characteristic PURKINJE cells, but also smaller cells; between the two there is a gradation in size.

The larger PURKINJE cells (fig. 21) are about 27μ long by 14μ broad, and the smaller 15μ by 10μ; they are generally elongated and triangular in shape; two molecular processes are given off from the side turned towards the molecular layer, and one finer process from the narrower end is directed towards the granular layer. Occasionally these cells are broader at the end turned towards the granular layer, as if they had been truncated; sometimes fusiform cells are found with a process emerging from each extremity, resembling those found in Teleostei. The processes which pass into the molecular layer may often be traced nearly as far as the external surface, their course is nearly direct, and they give off branches which preserve the same general direction as the main trunk; they thus contribute to a greater extent to the striation of the molecular layer than in Mormyrus. The smaller cells mentioned above are found in the neighbourhood of the PURKINJE cells, and their processes join the fibrillar network of the molecular layer.

The outer or molecular layer of the cerebellum (fig. 21) consists, in addition to the processes of the PURKINJE cells, also of a fibrillar network, the general tendency of the striation of which is in a longitudinal direction; the fibrils composing this are not so broad as those which go in a course perpendicular to the surface. Near the external edge the network becomes irregular. Cells of small size are found sparingly scattered through the central and outer portions of the layer; they are more numerous than in Mormyrus, and still more so than in Mugil Cephalus, in which they were seldom found; they are rather larger in size than in the Teleostei, averaging 6μ or 7μ in diameter; they are spherical or sometimes slightly triangular in shape, and give off processes on all sides which join, some the longitudinal and some the radial fibrils. In Rhina they appear to be fewer in number, but somewhat larger than in the other species examined. In other respects this description applies equally to all.

The ventricle of the cerebellum is lined by a layer of endothelium derived from the general endothelial lining of the brain cavities.

Restiform Bodies.

The crura cerebelli (fig. 12) in their descent on to the medulla oblongata, are accompanied by the molecular and granular layers, which extend in Raja nearly as far as the posterior end of the fourth ventricle; so that the tuberosities which bound that ventricle laterally appear as if they were the continuation of the cerebellum; the absence, however, of the PURKINJE cells marks a substantial difference in organization; for these cells may be considered as characteristic of the cerebellum, structures more

or less resembling the granular or molecular layers are found in other parts, but PURKINJE cells occur nowhere else. In Scyllium the granular layer does not extend so far back as the molecular. The same may be said of Rhina and Acanthias.

Spinal Cord.

The grey matter of the spinal cord (figs. 6, 7, and 8) has somewhat the shape of a goblet with its foot. The dorsal horns spread out towards the dorsal surface in a way to make it resemble a Lucernaria; this resemblance is more apparent in Acanthias than in the figure. The ventral horns are extremely well defined and distinct, they extend horizontally on each side of the central canal, in shape like a pair of clubs. Surrounding the central canal there is a single-celled layer of endothelium to be observed; and external to that a space of connective tissue which may be compared to the substantia gelatinosa centralis. In a transverse section one may reckon four columns of longitudinal fibres. Those beneath the ventral horns of grey substance may be termed the ventral columns, situated between the central canal and the ventral surface of the cord, and bounded on each side by the ventral roots of the spinal nerves. The lateral columns extend from the outer sides of the ventral roots to the external side of the dorsal horns. The remaining two columns are the dorsal situated above the dorsal cornu, and a small pair of fasciculi placed immediately dorsad of the central canal above the connecting link between the two ventral cornua. In addition to these there are a few small bundles running longitudinally through the dorsal cornua, which are the dorsal roots of the spinal nerves. The ultimate destination of these columns will be seen presently, as far as I have been able to unravel their course. Each ventral horn of grey substance contains about nineteen or twenty cells in one section. These cells are of a larger size than those found in the ventral horns in the Teleostei; they are generally elongated, and give off several processes; they are arranged in an imbricated manner, with their long axes directed obliquely from the ventral to the dorsal surface. As the spinal cord passes on to merge into the medulla oblongata, the ventral cornu becomes depressed towards the ventral edge of the cord, so that instead of being placed in a horizontal position, it gradually comes to form an angle of about forty-five degrees. Immediately behind the posterior end of the fourth ventricle it begins to disappear, but some of the cells may be traced in the corresponding position for a few sections farther forward, and they finally disappear at the posterior end of the medulla oblongata. At this point another ganglion becomes apparent, which is not a continuation forward of the ventral cornu, but is situated nearer the central line and the dorsal surface in the grey matter of the floor of the fourth ventricle, rather to its outer side. It forms part of the origin of the vagus. The fibres of the ventral column vary in size, but they are larger than those of the lateral and dorsal columns; their average diameter is less than that of the corresponding fibres in the spinal cord of the Teleostei.

Although in the Plagiostomata the gigantic fibres discovered by MAUTHNER in the Teleostei are not present, and thus a difference is found to exist between these groups in the structure of the spinal cord, yet in Ceratodus, which perhaps presents an archaic form of nervous system, a form of nerve fibre is seen, which is in all likelihood the primitive form of this kind of fibre. In this nerve fibre, instead of one very large axis cylinder, there are several, which are all enclosed in a separate and distinct medullary sheath; they occupy a position in the spinal cord and medulla oblongata corresponding to MAUTHNER'S fibres.

The ventral columns on arriving at the fourth ventricle form two distinct longitudinal projections in the central line of the floor of the same; anteriorly these projections become smaller and disappear at the anterior end of the fourth ventricle; they disappear as such, but the columns themselves are traceable farther forward much diminished in size. Along the floor of the aqueduct of Sylvius they gradually break up into a number of smaller bundles which recede from the floor of the aqueduct toward the outer side. They can be traced, much diminished in size, into the ventral part of the posterior commissure. The lateral columns also diminish greatly in the number of fibres which they contain in their course forward. Those bundles which are internal and nearer the ventral columns become lost in the region of the posterior commissure, while some of the more external seem to join the transverse fibres which form the second layer of the optic lobe. Those between the two disappear in the region above the hypoarium, some crossing the fibres of the crura cerebri, which disappear in the same region. ROHON * describes these columns as the continuation backward of the pedunculi cerebri, with which he thinks they are continuous, and he identifies the ventral columns with a system that he has traced in the cerebrum,† and which he has homologized with the fasciculus longitudinalis tegmenti posterior. This sort of thing is a mere *tour de force* of the imagination; these columns may homologize in a general sort of way, but to pick out the particular fasciculi in the human brain to which they correspond is going a little too far. I am unable to confirm his assertions, as neither of these connexions were to be traced in my sections (fig. 18).

Deep Origin of the Cerebral Nerves.

First pair.—The olfactory nerves are derived from the olfactory lobe, a description of which has already been given.

Second pair.—The optic nerves form a distinct and well-formed chiasma. They do not show the pectinated arrangement characteristic of the Teleostei. At the lower portion of the chiasma the fibres of the nerve of one side intersect those of the other in layers as they cross, like the intertwined fingers of the two hands; but dorsad, more than half of the nerve of one side crosses over that of the other, forming a simple

* *Op. cit.*, pp. 72 and 77, fig. 51.

† *Op. cit.*, p. 79, fig. 42.

decussation. I could discover no fibres that remained on the same side, neither are there any which traverse the chiasma from one nerve to the other without passing on into the brain; neither does the commissura transversa HALLERI, which is stated by GOTTSCHE * to be present behind the chiasma in Teleostei, appear to be present here.

With regard to the deep origin of the nerve (figs. 18 and 22), the principal part is derived from the anterior end of the optic lobe, where the fibrils of origin occupy nearly two-thirds of the width, comprising in fact the whole of the first or external layer. The fibrils appear to be derived from the cells mentioned in the description of the lobe in question. In addition to this origin some of the fibres are derived from the hypoarium. BELLONCI † denies this origin, and is of opinion that bundles belonging to the anterior and posterior commissura transversa have been credited to the chiasma by mistake; but, unfortunately for this idea of BELLONCI, these structures do not exist in specimens of Plagiostomata that I have examined. My reason for including the hypoarium as part of the origin of the optic nerve is this; if very thin horizontal sections of the chiasma be examined at the point where its posterior margin abuts on the hypoarium, several fibres are to be seen passing into it, dispersing through that body, and not returning into the chiasma. If they were simply passing through to some other part they would retain their formation as bundles and would not separate into single fibrillæ. I have not actually traced any of them into a cell, but, nevertheless, as it is, I think this arrangement shows that they end, or begin rather, in the hypoarium.

Third pair (fig. 15).—The oculomotorii nerves are derived from a small group of nerve cells situated on each side of the fissure on the floor of the ventricle of the optic lobe; from these ganglia a well-defined bundle passes downward and outward, and emerges from the inferior surface, one on each side of the raphé. AHLBORN ‡ however describes a decussation, or, as he prefers to call it, a chiasma of the roots of this nerve in the Petromyzon. It seems hardly probable that a decussation should be present in this animal and not in the Plagiostomata, but of course it is just possible that he is right; if so, the Petromyzon is differently organized to the other members of the class Pisces. It is true that there is a decussation of fibres in the neighbourhood of this nerve, which might have deceived AHLBORN, but it is a distinct formation, and has nothing to do with the oculomotorii; it extends for some distance both before and behind them, and appears to homologize with the commissura ansulata § in the Teleostei. In the region between the roots of this nerve there are apparently two sections of this commissure: a dorsal part the fibres of which decussate, and a ventral part the fibres of which simply pass over from one side to

* *Op. cit.*, p. 442, fig. 8.

† "Ueber den Ursprung des Nervus Opticus, &c." 'Zeitschr. f. Wissensch. Zool.,' vol. 35, 1880, p. 24.

‡ "Untersuchung über das Gehirn der Petromyzonten." 'Zeitschr. f. Wissensch. Zool.,' vol. 39, 1883, p. 271.

§ GOTTSCHE, *op. cit.*, p. 439.

the other; they all cross the roots of the oculomotorii at right angles and plainly do not join them. The greater number of the fibres entering into the composition of this commissure are derived from the second layer of the optic lobe. In fact it may be considered as the commissure of the optic lobe, but in addition to these fibres some few appear to be derived from the lateral columns of the cord.

In the Rays this commissure is not so well developed as in Scyllium and Acanthias. In the latter it extends farther behind the oculomotorii than in the others. It seems as if, in these latter, the extent of these association fibres compensates for the size of the ventricles and the consequent thinness of the walls of the optic lobe, cerebrum, and the other parts of the brain.

Fourth pair.—The trochleares are very difficult to trace; they have a distinct decussation in the region of the valvula cerebelli, which is represented by a narrow neck of tissue between the cerebellum and the optic lobe, and is not nearly so well developed as in the Teleostei. I have not been able to trace the roots of these nerves to any ganglion, but in one specimen they were seen turning forward by the side of the aqueduct of Sylvius. In the region of the ganglion of the oculomotorius, there are no collections of cells to be found from which they might be supposed to have been derived. Rohon,* in his account of the origin of this nerve, appears to have fallen into a slight mistake; he describes it as arising from a swelling which is in direct connexion with the cortical part of the anterior tuberosity of the cerebellum. A consideration of his figure convinces me that the section has been made through the restiform bodies behind and not in front of the crura cerebelli ad medullam; if so, the bundle of fibres which he looks upon as the root of the trochlearis must have some other signification. Acanthias is a very good subject for this investigation, because its cerebellum being small does not extend forward and obscure the relation of the parts between it and the optic lobe. Fig. 14 is taken from this animal, and shows the narrow valvular portion extending as a tent over the anterior part of the fourth ventricle at the point of its transition into the aqueduct of Sylvius. It shows also the decussation of the fibres and the commencement of their course along the side of the aqueduct.

Fifth pair (fig. 11).—The trifacial is derived from three roots. One comes forward from the posterior part of the medulla oblongata from the lateral columns of the cord. This evidently corresponds to the ascending root in Mammalia. Another comes backward from the anterior part of the tuberosity of the trifacial, into which it penetrates from the formatio recticularis; these two bundles pass in opposite directions until they arrive at the point of exit of the nerve, when they turn directly outward and cross each other to join the emerging root of the nerve. The third origin comes from a group of cells situated on the lateral part of the floor of the fourth ventricle; in addition to the fibres from the ganglion, others are derived from the raphé.

* *Op. cit.*, p. 80, fig. 49.

There are extensive commissures or arcuate fibres in the region of this nerve, which cross the ventral surface and other parts in the formatio reticularis; they appear to join the restiform bodies.

Sixth nerve.—The abducentes arise by about four small bundles from each side of the raphé. They can be traced from the external surface of the ventral side of the medulla oblongata at about the middle of its length to the lower edge of the ventral columns, where they are lost. They resemble in their arrangement the oculomotor nerves, but unlike them they could not be traced to any nerve cells, unless they have anything to do with some large cells which are sparingly scattered through the formatio reticularis in this region.

Seventh (figs. 6, 7, 8).—The facialis, which in Rhina and Scyllium is a branch of the trifacial, and in Acanthias has a separate root, arises from a bundle which comes forward from the spinal cord. This fasciculus appears above the central canal, between the dorsal and ventral cornua; in its course forward it lies internal to the ascending root of the trifacial. This bundle is described by Rohon* (who apparently missed its connexion with the facial) as the fasciculus longitudinalis lateralis, which he considers as a new formation especially confined to the Plagiostomata, but it corresponds to the root of the facial in the Teleostei; it was described in my paper on the brain of the Teleostei† under the name of the upper column of the trifacial.

Eighth.—The Acusticus appears to arise from the grey matter of the floor of the fourth ventricle.

Ninth pair.—The glossopharyngeal also appears to arise from the grey substance of the floor of this same ventricle.

Tenth pair.—The vagus arises from the series of rounded tubercles which are situated on the floor of the fourth ventricle. Gegenbaur‡ compares these tubercles to the swellings on the surface of the medulla oblongata in Trigla, and explains the fact that they are situated outside, by supposing that the sinus rhomboidalis in that species is too small to receive them, whereas in Hexanthus the presence of a larger ventricle affords space for them within it.

These tubercles contain a few cells of medium size, from which the roots of the nerve appear to be derived; each tubercle seems to give rise to a separate root, which, passing out of the medulla, joins the others to form the nerve in a manner formerly described.

Rohon,§ who has gone minutely into the relations of the roots of this nerve, has come to the following conclusions. The lobus vagi corresponds to a collection of nerve ganglia, and each root, arising from a single ganglion, corresponds to a posterior (dorsal) root of a spinal nerve; and, further, the roots that emerge behind the lobus

* *Op. cit.*, p. 101.

† 'Phil. Trans.,' 1878, vol. 169, p. 765, fig. 9.

‡ *Op. cit.*, p. 520.

§ "Ueber den Ursprung des Nervus Vagus." 'Wien, Zool. Instit. Arbeiten,' Heft 1, 1878.

vagi represent a mixed system of anterior and posterior roots, which correspond partly to the anterior and partly to the posterior roots of spinal nerves: thus confirming a conclusion previously arrived at by GEGENBAUR.

Spinal nerves (figs. 6 and 7).—The ventral roots of these nerves emerge as a bundle on each side of the raphé, from the internal inferior surface of the ventral horn of grey substance, and, taking a direction nearly downwards, pass out to form the ventral roots of the spinal nerves.

The dorsal roots enter the cord obliquely, and there divide into two cords: one cord from the anterior part of the root passes backward, the other cord from the posterior part of the root passes forward; they cross over the next nerve both in front and behind, and join the lateral columns of the cord.

I am unable to do more than confirm the account which STIEDA* (who was the first to describe this arrangement) gave.

Conclusion.

A comparison of the brain of the Plagiostomata with that of the Teleostei brings out several differences in detail; but the general scheme is the same in both.

In the Plagiostamata the lobi olfactorii have a well marked ventricle, communicating by a passage through the peduncle with a ventricle in the cerebrum, which anteriorly is double, but behind is single. There can be no doubt that these correspond to the lateral ventricles, while the posterior single part, which is continuous with the third ventricle, can be no other than the Foramen of Monro. The position of the anterior commissure, which connects the two sides of the ventral part or floor of these ventricles, would seem to indicate that the corpora striata should be looked for here. The Teleostei possess none of these structures, with the sole exception of the anterior commissure; in them the olfactory lobe and the cerebrum are solid.

The theory of RABL-RÜCKHARD† is scarcely applicable to the Plagiostomata. His conclusions are that the cerebrum in Teleostei is homologous to the island of Reil and the corpus striatum only; while a layer of endothelium which lines the pia mater and forms a shut sac, communicating with the third ventricle, is genetically homologous with the unthickened wall of the cerebral hemispheres, "Grosshirnbläschen," and the space it encloses is the "ventriculus communis." Other writers have compared the hemispheres in the Teleostei with the corpora striata, *e.g.*, TIEDEMANN,‡ but of course this does not detract from the merit of this writer, whose reasoning and detailed exposition of the homology of these structures are worthy of attention. It remains only to be considered how far a layer of endothelium may be taken to represent

* 'Zeitschr. Wissensch. Zool.,' vol. 23, 1873.

† "Das Grosshirn der Knochenfische u. seine Anhangsgebilde." 'Archiv Anat. Physiol.' (Anat. Abth)., 1883, p. 307.

‡ "Anatomie und Bildungsgeschichte des Gehirns im Fœtus des Menschen." 1816.

nervous substance. It is not to be overlooked that Von Baer* described these hemispheres in the embryo of the Teleostei, as being hollow and containing a small tubercle occupying the position of the corpora striata. In the Plagiostomata, however, the walls of the cerebral hemispheres are thickened, and two ventricles are present, so that we have here a considerable advance in structure. The theory that these represent the lateral ventricles of Mammalia is not universally received. Prof. Burt. G. Wilder† looks upon them as implying an extension forward of the third ventricle, and the real Foramina Monroi he finds in the openings on each side which lead into the ventricles of the olfactory lobes, which latter he considers to be the true lateral ventricles. This view seems to me to be scarcely tenable; not only is the structure of these lobes incompatible with this opinion, but the position of the so-called Foramina Monroi themselves is against it, the true foramen being a single passage, and not one in each lateral wall, which would be the case if this opinion should be received.

With regard to the homologies of the remaining tuberosities of the brain, except the optic lobes, there are no great difficulties.

The hypoaria, lobi infundibuli or inferiores may be considered to belong to the thalamencephalon, which may be defined as the space between the anterior commissure in front and the posterior commissure behind, and includes the above-mentioned lobe together with the infundibulum. Thus the thalamencephalon is pushed under the mesencephalon to a great extent in the Plagiostomata.

To the question, how much if any part of the thalamencephalon is represented in these fishes? I should be inclined to reply, probably no part of it. The only consideration which would seem to favour the idea of its presence would be the position of the posterior commissure, the homology of which is generally conceded; but this does not necessarily carry with it the homology of the part which it connects, and even if it does so, the posterior commissure does not solely nor even principally connect the two thalami optici. But it is probable that the territory between the anterior end of the optic lobe and the chiasma nervorum opticorum corresponds to the corpus geniculatum externum, or, as Mayser‡ suggested, the corpus geniculatum internum, for the torus semicircularis in Teleostei.

Rohon, whose ideas generally seem worthy of respect, has rather gone beyond the mark in his theory of the optic lobe; a combination of the thalamencephalon and the mesencephalon in this lobe is an idea that will not bear inspection. In the first place there is not the slightest appearance in any sections that I have seen of a difference in structure in the optic lobes; on the contrary, the difficulty was to distinguish any separate layers for description, the characteristic cells extended without a break in greater or less numbers from the external to the internal surface.

* 'Ueber Entwickelungsgeschichte der Thiere.' Zweiter Theil, p. 307.

† 'Philadelphia Acad. Nat. Sci. Proc.,' 1876.

‡ 'Zeitschr. Wissensch. Zool.,' vol. 36, 1881.

Von Baer's theory, although subsequent researches proved it to be untenable, was nevertheless much more reasonable, for there was some ground for it, in the projection of a part of the cerebellum into the ventricle of the optic lobe.

Next the dilemma, which our author propounds on p. 99, as to the trochlearis, has no existence except in the imagination. This nerve does not arise so far back as he supposes; if one inspects his fig. 49 it will be at once seen that one has to do with a section taken behind the crura cerebelli ad medullam, and that the structure represented could not be the roots of this nerve. The true point of decussation of this nerve has been described in the account of the origin of the nerves.

There is no question here of any physiological consequences, either good or bad, in consequence of its fibres passing through the optic tract, since the case does not occur, the two sets of fibres being some considerable distance apart.

In addition to these considerations I may cite Rabl-Rückhard,* who states that at no time during the development of the embryo does the thalamencephalon pass or project behind the point of attachment of the epiphysis cerebri. Neither does Balfour† give any indication of any such occurrences.

If we should abandon the landmarks of homology as Rohon and Fritsch would have us do, we shall be plunged into a sea of doubt; science would be reduced to a mere system of guesswork, and we should become like those natives in South Africa whom a recent traveller described, as amusing themselves over the camp fire by discussing the question as to which fingers of their hand the toes of an ostrich corresponded.

Explanation of Plates.

The following letters have the same signification throughout:—

a.c. 5	Ascending root of trifacial.
a.c.ce.	Anterior commissure.
a.l.	Anterior lobe of the cerebellum.
aq.Sy.	Aqueduct of Sylvius.
b.	Nervi branchiales.
c.a.	Commissura ansulata.
c.c.	Crura cerebri.
c.ca.	Central canal of the spinal cord.
c.cbl.	Crura cerebelli ad medullam.
c.cbl.c.	Crura cerebelli ad cerebrum.
cbl.	Cerebellum.
ce.	Cerebral lobes.

* *Loc. cit.*

† 'A Monograph on the Development of Elasmobranch Fishes.' London, 1878.

ch.op. Chiasma nervorum opticorum.
ch.pl. Choroid plexus.
d.h. Dorsal horn of grey substance.
d.r. Dorsal roots of the spinal nerves.
f.a. Fibræ arcuatæ.
f.f. Fasciculus of the facial nerve.
f.l. Fibrous layer of the cerebellum.
f.m. Foramen of Monro.
gl. Glomerulus olfactorius.
g.l. Granular layer of the cerebellum.
gn.ce. Collections of cells at the base of the cerebrum.
gn.mo. Ganglion of the oculomotorius nerve.
gn.r.op. Ganglion of the roof of the optic lobe.
gn.tri. Ganglion of the trifacial nerve.
gn.v. Ganglion of the vagus.
hy. Hypoarium.
i.l. Internal layer of the olfactory lobe.
i.m. Inferior maxillary nerve.
i.n. Infundibulum.
l.c. Lateral columns of the cord.
l.i.cbl. Inferior lobe of the cerebellum.
l.op. Lobi optici.
l.v. Lateral ventricles.
m.f. Meynert's fibres.
m.l. Molecular layer of the cerebellum.
m ob. Medulla oblongata.
n. 2. Optic nerve.
n. 3. Oculomotorius nerve.
n. 4. Trochlearis nerve.
n. 5. Trifacial nerve.
n. 6. Abducens nerve.
n. 7. Acusticus nerve.
n. 9. Glossopharyngeal nerve.
n. 10. Vagus nerve.
o. Ophthalmic nerve.
ol. Olfactory lobe.
op.tr. Optic tract.
p.c. Posterior commissure.
p.i. Hypophysis cerebri.
p.l. Posterior lobe of the cerebellum.
p.m. Pia mater.

r.c.	Restiform column.
r.i.	Ramus intestinalis.
r.l.	Ramus lateralis.
s.g.c.	Substantia gelatinosa centralis.
s.m.	Superior maxilliary nerve.
s.r.	Sinus rhomboidalis or fourth ventricle.
t.c.op.	Tranverse commissure of the optic lobe.
th.	Thalamencephalon.
t.i.	Tuberculum intermedium, GOTTSCHE.
t.tri.	Tuberosity of the trifacial.
t.v.	Tuberosity of the vagus.
v.cbl.	Ventricle of the cerebellum.
v.h.	Ventricle of the hypoarium.
v.h.g.	Ventral horn of grey substance.
v.l.c.	Ventral longitudinal column.
v.op.l.	Ventricle of the optic lobe.
v.r.	Ventral root of the spinal nerve.
v.th.	Third ventricle.
v.v.	Valve of Vieussens, valvula cerebelli, or anterior medullary velum.

PLATE 38.

Fig. 1. Dorsal view of the brain of Scyllium Catulus, the pia mater covering of the third and fourth ventricle removed. Natural size.

Fig. 2. Dorsal view of the brain of Rhina Squatina, showing part of the distribution of the cranial nerves. Pia mater not removed. Natural size.

Fig. 3. Dorsal view of the brain of Acanthias vulgaris. Pia mater not removed. Natural size.

Fig. 4. Side view of the brain of Acanthias vulgaris. Natural size.

Fig. 5. Dorsal view of the fourth ventricle and the optic lobes. Pia mater and cerebellum removed. × 2.

Fig. 6. Transverse section through the spinal cord of Rhina Squatina. × 22.

Fig. 7. Transverse section through the spinal cord of Rhina a little more in advance. × 22

Fig. 8. Transverse section through the spinal cord immediately behind the fourth ventricle. Rhina × 22.

Fig. 9. Transverse section through the posterior end of the Medulla oblongata. Rhina. × 22.

Fig. 10. Transverse section through the middle of the fourth ventricle. Rhina. × 22.

PLATE 39.

Fig. 11. Transverse section through the fourth ventricle at the posterior end of the cerebellum. Rhina. × 22.

Fig. 12. Transverse section through the roots of the Acusticus. Raja Batis. × 22.

Fig. 13. Transverse section through the Crura cerebelli ad medullam. Rhina. × 22.

Fig. 14. Transverse section through the valvula cerebelli and the trochlear nerves, at the transition of the fourth ventricle with the aqueduct of Sylvius. Acanthias. × 22.

PLATE 40.

Fig. 15. Transverse section through the region of the oculomotorius nerve. Acanthias. × 22.

Fig. 16. Transverse section through the anterior end of the optic lobe and the posterior commissure. Acanthias. × 12½.

Fig. 17. Longitudinal section through the cerebellum. Scyllium. × 7.

Fig. 18. Longitudinal section through the mesencephalon and the third ventricle. Raja.

PLATE 41.

Fig. 19. Horizontal section through the third ventricle and the base of the cerebrum. Acanthias. × 12½.

Fig. 20. Transverse section through part of the granular layer of the cerebellum. Acanthias. × 450.

Fig. 21. Section through part of the molecular layer of the cerebellum. Scyllium. × 450.

Fig. 22. Longitudinal section through part of the optic lobe. The section was composed from two separate pieces. Rhina. × 65.

Fig. 23. A glomerulus olfactorius and the adjacent part of the internal layer. Acanthias. × 325.

XIX. *The Coefficient of Viscosity of Air.*

By Herbert Tomlinson, *B.A.*

Communicated, with the addition of two Notes, by Professor G. G. Stokes, *P.R.S.*

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[Plate 42.]

Origin and Purpose of the Investigation.

Three years ago I entered on a series of researches relating to the internal friction of metals, little calculating, when I did so, that the task which I had set myself would occupy almost the whole of my spare time from that date to this. So, however, it has been, and one of the many causes of delay has been the necessity of making a re-determination of the coefficient of viscosity of air; for the resistance of the air played far too important a part in my investigations to permit of its being either neglected or even roughly estimated. The coefficient of viscosity of air may, according to Maxwell, be best defined by considering a stratum of air between two parallel horizontal planes of indefinite extent, at a distance r from one another. Suppose the upper plane to be set in motion in a horizontal direction with a velocity of v centimetres per second, and to continue in motion till the air in the different parts of the stratum has taken up its final velocity, then the velocity of the air will increase uniformly as we pass from the lower plane to the upper. If the air in contact with the planes has the same velocity as the planes themselves, then the velocity will increase $\frac{v}{r}$ centimetres per second for every centimetre we ascend. The friction between any two contiguous strata of air will then be equal to that between either surface and the air in contact with it. Suppose that this friction is equal to a tangential force f on every square centimetre, then

$$f = \mu \frac{v}{r},$$

where μ is the coefficient of friction. If L, M, T represent the units of length, mass, and time, the dimensions of μ are $L^{-1}MT^{-1}$.

Several investigators have attempted to determine the coefficient of viscosity of air, and the following table shows how very widely the results obtained differ among each other:—

TABLE I.

Author.*	Coefficient of viscosity of air in C.G.S. units.	Temperature in degrees Centigrade.
G. G. STOKES, from BAILY'S pendulum experiments.	·000104	°
MEYER, from BESSEL'S experiments	·000275	
MEYER, from GIRAULT'S experiments	·000384	
MEYER	·000360	18
MEYER (second paper)†	·000333	8·3
,, ,,	·000323	21·5
,, ,,	·000366	34·4
MAXWELL	·000200	18

Further, MAXWELL finds the coefficient of viscosity of air to be independent of the pressure and to vary directly as the absolute temperature.‡ The above author gives the following formula for finding μ, the coefficient of viscosity, at any temperature $\theta°$ C.:—

$$\mu = ·0001878(1 + ·00365\theta).$$

MAXWELL offers an explanation of the difference existing between his own results and those of MEYER, but states that "he has not found any means of explaining the difference between his own results and those of Professor STOKES." Professor STOKES has, however, been good enough to inform me that, as at the time of making his deductions from BAILY'S experiments it was not known that the coefficient of viscosity of air was independent of the pressure, but, on the contrary, was assumed by him to vary directly as the pressure, the resistance offered by the residual air in BAILY'S partial vacua was underestimated, and, as a consequence, the deduced coefficient of viscosity was too small. It is to be hoped that Professor STOKES will at some future period apply the necessary corrections, but as this has not yet been done, and as we have still no explanation of the discrepancies existing between the other values of μ given in Table I., I wished to make some independent observations on the viscosity of air for the purpose of ascertaining how far these would agree with those of MAXWELL, in which I was inclined to place great confidence.

MAXWELL employed the method of torsional vibrations of disks placed each between two parallel fixed disks at a small, but easily measurable distance, in which case, when the period of vibration is long, the mathematical difficulties of determining the motion of the air are greatly diminished. This method appeared to be a very good one, but, as I wished to make my determinations under conditions similar to those

* For references see MAXWELL'S Bakerian Lecture, 'Phil. Trans.,' vol. 156, 1866, p. 249.

† MEYER has more recently made other determinations of the coefficient, for which see the end of the paper.

‡ This result does not seem to be confirmed by other experimenters. (See the end of the paper.)

which held in my experiments on the internal friction of metals, I have employed the torsional vibrations of cylinders or spheres attached to a horizontal cylindrical bar and moving in a sufficiently unconfined space. The mathematical difficulties connected with the use of vibrating spheres are not so serious, but those in which cylinders are concerned are very considerable. They both, however, have been surmounted by Professor G. G. Stokes in his valuable paper "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums,"* and to this paper I am indebted for the mathematics essential to the purpose of the present inquiry.

Description of Apparatus and Mode of Experimenting.

A wire, $a\,b$ (Plate 42, fig. 1), was suspended in the axis of an air-chamber, W, made of two concentric copper cylinders enclosing between them a layer of water. The outer diameter of the air-chamber was 4 inches, the inner diameter 2 inches, and the length $4\frac{1}{2}$ feet. Resting on the top of the air-chamber and wedged into it was a stout T-shaped piece of brass, C, to the lower extremity of which was clamped one end of the wire. The lower extremity of the wire was soldered or clamped at b to a vertical cylindrical copper bar b Q, which was in turn clamped at Q to the centre of a horizontal bar V V. The bar V V consisted of a piece of thin, hollow, drawn brass tubing, of which the length was 30·70 centims. and the diameter 1·420 centim. This bar was graduated into millimetres and carried two suspenders, S, S, which were clamped to it at equal distances from the centre (fig. 3). The suspenders were each provided with an index such that their positions on the bar V V could be readily estimated to one-tenth of a millimetre. The mean diameter of the cylindrical portion, S K, of each suspender was 0·3366 centim., and the length of this portion 8·50 centims. To the ends, K, of the suspenders could be screwed (fig. 3) hollow cylinders of stiff paper or metal, or spheres of wood; when the former were employed the suspenders were provided with disks, m, m, of the same diameter as the cylinders, and about 2 millims. in thickness. Two brass caps, D, D (fig. 4), provided with screws about 8 centims. in length and 2 millims. in diameter, fit one into each end of the hollow bar V V, and can be easily removed from or placed in the latter.

To begin with, two cylinders or two spheres were screwed on to the ends of the suspenders (in the former case right up to the disks m, m), and the logarithmic decrement and the time of vibration determined from a very large number of vibrations. The cylinders or spheres were now unscrewed, and, the brass caps, D, D, having been temporarily removed for the purpose, two brass cylinders, h, h (fig. 4), each of the same mass as either of the vertical cylinders or spheres which had just been removed, were, by means of companion-screws, cut along their axes, adjusted on to the screws attached to the caps D, D, and at such a distance from the latter as preliminary experiments had proved would give nearly the same vibration-period, when the caps should be replaced in the bar V V, as had existed before the vertical

* 'Camb. Phil. Soc. Trans.,' vol. 9, No. X. (1850).

cylinders or spheres had been removed. The caps D, D, were now replaced in V V, and the logarithmic decrement, together with the time of vibration, was once more carefully determined. Observations such as these, when certain corrections presently to be mentioned had been applied, enabled one to calculate the effect of the resistance of the air on the vibrating vertical cylinders or spheres as far as the diminution of the amplitude of vibration was concerned.

The bar V V with its appendages was protected by a wooden box B of sufficient size to permit of vibrations, which, as regards the resistance of the air, were practically as free as in the open.* This box was provided with a window, E E, and two side-doors, lined with caoutchouc so as to fit air-tight; these side-doors were kept shut, except when it was necessary to make fresh adjustments. The torsional vibrations of the wire were observed by means of the usual mirror-and-scale arrangement, which is sufficiently shown in fig. 1, where M is the light mirror reflecting an illuminated circle of light crossed by a vertical, fine, dark line on to a scale bent into an arc of a circle of 1 metre radius, and placed at a distance of 1 metre from the mirror.

My three years' experience of the internal friction of metals had taught me that this last is by no means constant unless the greatest care be taken to prevent slight fluctuations of temperature. The above-mentioned fact seems to have escaped the notice of MAXWELL and MEYER, probably on account of the internal friction of the metal having a considerably less damping effect than the resistance of the air in their experiments. With me, however, especially in some cases, changes in the internal friction of the metal would have rendered it very difficult, nay, impossible, to attain the accuracy which I aimed at, and I deemed it advisable to protect the wire still further, as follows:—The top of the air-chamber W was well covered with baize, and surrounding W, and concentric with it, was a larger air-chamber X, made of tinned iron. This air-chamber was 11½ inches in inner diameter, 15 inches in outer diameter, and 46 inches in height; the two concentric chambers of which it was composed enclosed between them a space 2 inches thick, stuffed with sawdust, whilst on the top of the chamber was placed a double cover A, also packed with sawdust. Passing through the outer air-chamber X, and through the walls of W, were two metal tubes in which were placed two thermometers T_1, T_2, with their bulbs near the wire; these thermometers were made by indiarubber tubing to slide air-tight in the metal tubes. A section of the two chambers X and W passing through one of the thermometers is shown in plan in fig. 2. The whole of this part of the apparatus rested on a stout wooden table, in which was pierced an aperture of a size just sufficient to allow the zinc tube Z, soldered to the air-chamber W, to pass through it and into the box beneath. A third thermometer T_3 served to give the temperature of the air in the

* In fig. 1 the cylinders appear to be closer to the sides of the box than they were in reality; the bar V V faced the window, but, for the sake of showing the arrangement of the cylinders better, it has been drawn facing the adjacent side of the box. The centres of the cylinders were at least six inches from the sides of the box.

box B, whilst the mean of the readings of T_1 and T_2 was used for the temperature of the wire. The thermometer T_3 was divided to one-tenth of a degree Centigrade, and had been tested at Kew; whilst the thermometers T_1 and T_2, which were graduated in degrees Centigrade, had been carefully compared by myself, degree by degree, with T_3.

The barometric pressure was registered by means of a delicate aneroid barometer, reading to $\frac{1}{100}$th of an inch, which has been in my possession for 15 years; this instrument I had recently compared with a standard mercury barometer.*

Before commencing the actual experiments on the viscosity of the air, it was found advisable to subject the wire to a preliminary training, in order not merely to diminish the internal friction of it, but also to make this last as constant as possible. In the first place, the wire was well annealed; this had the effect of reducing the internal friction of the hard-drawn metal to less than one-half of its previous amount.† In the next place, a load, equal to that of the cylinders or spheres to be used, having been suspended to VV, the wire was alternately heated to 100° C. and cooled again, this process being repeated for about a week, on each day of the week, until there was no further alteration of the internal friction of the wire when cool. This treatment still further reduced very considerably the damping of the vibrations due to the wire. The manner in which the heating was effected will be shown in a future paper, in which also will be recorded the results of experiments on the temporary effect of change of temperature on the torsional elasticity and internal friction of the metals used. When the wire had undergone this preliminary treatment, and all the arrangements were complete, the bar VV, with its appendant cylinders or spheres, as the case might be, was started by small impulses imparted by a worsted thread, until the arc of vibration, as reckoned from rest to rest, had reached about 400 divisions of the scale (about 10°, since 41·227 divisions represented 1°). After the arc of vibration from rest to rest had subsided to about 200 scale-divisions, the vibrator was again started, and this process was repeated until something like a thousand oscillations had been executed.‡ Finally the vibrator was re-started for the actual observations, through an arc of about 200 scale-divisions, and when about 50 oscillations had been executed after this last starting the readings were begun. Suppose that a_1, b_1; a_2, b_2; a_3, b_3; a_4, b_4; a_5, b_5, and a_6 are eleven consecutive

* In spite of the long period which has elapsed since this instrument was first made for me by the late Mr. Becker, of Elliott Bros., the spring still shows a slight amount of permanent yielding, which during the last two years has altered the reading by ·015 inch.

† Either silver, platinum, or copper wires, well annealed, may be used with advantage. I should not recommend unannealed piano-steel wire as used by Maxwell; the last metal possesses, it is true, great elasticity, but the internal friction of silver, platinum, or copper can, by annealing, be made considerably less than that of the unannealed steel.

‡ The object of this treatment was to reduce the internal friction to its permanent condition, since long rest, or sometimes even a comparatively short rest, always raised sub-permanently the internal friction.

readings,* the ten corresponding arcs from rest to rest will be a_1+b_1, b_1+a_2, a_2+b_2, b_2+a_3, a_3+b_3, b_3+a_4, a_4+b_4, b_4+a_5, a_5+b_5, b_5+a_6. The means of a_1+b_1, b_5+a_6; b_1+a_2, a_5+b_5; a_2+b_2, b_4+a_5; b_2+a_3, a_4+b_4, and of a_3+b_3, b_3+a_4 were written down, and if these agreed well with each other, which was almost invariably the case, the logarithmic decrement of the mean of the five means was taken. Now, say that n single vibrations have taken place between the end of this and the beginning of the next set of consecutive readings, the difference between the logarithms of the first and second total means will, when divided by $n+10$, give the mean logarithmic decrement for a single vibration. The logarithmic decrement was found to be constant in each experiment within the limit of probable error; the deviations from uniformity were sometimes in one direction and sometimes in the opposite, and there was no evidence of any law of increase or diminution of the logarithmic decrement as the amplitudes decrease. In the intervals between one set and another of the readings, taken in the manner mentioned above, other readings were taken for the purpose of determining the vibration-period; the time of transit of the light across the centre of the scale, first in one direction and then in the opposite, was recorded for ten successive passages by means of a good watch provided with a seconds-hand, a similar series being recorded after every 200 vibrations. These last observations enabled the period of vibration to be determined with such exactness that we may completely disregard any error arising from want of precision in this respect. From time to time, at regular intervals, the readings of all three thermometers and of the aneroid barometer were taken, so that the mean pressure of the atmosphere, the temperature of the wire, and the temperature of the air in the box B could be calculated with the necessary accuracy. The greatest care was taken that the cylinders or spheres suspended from the horizontal bar VV should hang vertically; also that there should be no appreciable pendulous motion of the wire; if such motion existed it was checked by the hand before any of the readings were taken. Very great care was also taken in determining the moments of inertia of the vibrator in the various experiments, these being each obtained by several different methods,† which gave very concordant results. I shall have occasion in a future memoir to dwell on the various sources of error to which determinations of moments of inertia are liable; so it will suffice, perhaps, here to mention that this part of the work alone occupied my entire attention for nearly two weeks. The following five experiments, or rather sets of experiments, were made:—

Experiment I.

The wire was of well-annealed copper, 97 centims. in length and 0·06272 centim. in diameter. Two cylinders, each having a mass of 70·19 grammes, were used. These

* This number was always taken.

† The moments of inertia could be calculated with sufficient accuracy from the dimensions and mass of the vibrating system; they were, however, determined also indirectly by the two methods employed by Maxwell.

cylinders were made of paper wrapped round a metal core a sufficient number of times to secure the requisite stiffness; the different layers of paper were pasted together, and when the whole was dry the metal core was withdrawn; the outside of each of the cylinders was also coated with French polish to prevent the absorption of moisture. The mean diameter of each of the cylinders was measured by calipers reading to $\frac{1}{1000}$th of an inch, and estimated to $\frac{1}{10000}$th of an inch. In obtaining the value of the mean diameter of each cylinder, twenty measurements were made, ten at equal intervals along the whole length, and ten at the same intervals, but in a direction at right angles to the first. The measurements showed a very fair uniformity of diameter throughout the whole length, the mean being 1·0079 inches for one cylinder and 1·0108 inches for the other. In the calculations subsequently made it was assumed that the diameter of each cylinder was the mean of the two last given, *i.e.*, was 1·0093 inches or 2·5636 centims. The lengths of the two cylinders were also very nearly the same, being 60·90 centims. and 60·85 centims. respectively; accordingly each cylinder was assumed to have a length of 60·875 centims. The ends of the cylinders consisted of wooden disks, into the centre of which was let a small brass disk provided with a screw, which was a companion to the screws at the ends of the suspenders S, K, so that the cylinders could be screwed right up to the disks M, M (fig. 3). The object of having the disks M, M, was to eliminate the effect of the friction of the air about the ends of the cylinders,* for Professor STOKES's mathematical investigations only apply strictly to cylinders of infinite length.

After the preliminary precautions previously mentioned had been taken the logarithmic decrement was determined from a great number of vibrations with the cylinders on; the cylinders were then each turned round their axes through a right angle, for the purpose of eliminating any error which might otherwise arise from the section of the cylinder being slightly elliptical instead of circular, and the logarithmic decrement was once more found. The cylinders were now unscrewed from the suspenders, and, the brass caps having been for the purpose removed from the hollow bar VV, the two brass cylinders *h*, *h*, were adjusted in the manner before mentioned, so that the vibration-period might remain very nearly unaltered; the caps were then replaced. All the adjustments alluded to above were performed very carefully so as to avoid jarring the wire, for if this precaution be not taken the internal friction will be temporarily increased, and will not come back to its previous value until the wire has been vibrated for a considerable time. A period of more than an hour was now allowed to elapse, the wire during this time being kept more or less in a state of vibration, but not through a greater arc than that represented by 400 scale-divisions from rest to rest, when the logarithmic decrement was again determined.

* It would have been well to have had these disks much thicker. As it is, the disks would only imperfectly serve the purpose for which they were intended; the effect about the ends of the cylinders was, however, completely eliminated in Experiment IV. It would appear, moreover, from the results that with the long cylinders here used the effect mentioned above is neglectable.

These processes were repeated during some eight or nine hours of each day through a period of three days, with the following *mean** results :—

PAPER Cylinders on.

Temperature of the air in degrees Centigrade.	Temperature of the wire.	Barometric height in inches.	Period of a single vibration in seconds.	Logarithmic decrement for one vibration.
12·02	12·43	29·872	6·8373	·0036476†
PAPER Cylinders off.				
12·25	12·31	29·817	6·8202	·0009103

The moment of inertia of the whole vibrator when the paper cylinders were on was 33773 in centimetre-gramme units.

Mathematical Formulæ necessary for the Investigation.

Before it can be shown how the results given above were made use of in finding the coefficient of viscosity of air, it will be necessary to point out how the requisite mathematical formulæ can be obtained. I will first take the case of a cylinder vibrating horizontally under the influence of the torsional elasticity of a wire attached to its centre and hanging vertically.

Conceive the cylinder divided into elementary slices by planes perpendicular to its axis. Let r be the distance of any slice from the middle point, θ the angle between the actual and the mean positions of the axis, $d\mathrm{F}$ that part of the resistance experienced by the slice which varies as the first power of the velocity. Then, calculating the resistance as if the element belonged to an infinite cylinder moving with the same linear velocity, we have by Art. 31 of Prof. STOKES's paper—

$$d\mathrm{F} = \frac{k\mathrm{M}'\pi}{\tau}\frac{d\xi}{dt},$$

where M' is the mass of fluid displaced by the slice, $\frac{d\xi}{dt} = r\frac{d\theta}{dt}$, τ = the vibration-period, and k is a constant, provided the vibration-period, the diameter of the cylinder, and the nature of the fluid remain unchanged.

* I have not thought it necessary to give here more than the mean values, as in a portion of a paper on the internal friction of metals, which I hope shortly to be able to offer to the Royal Society, I have entered fully into the details of experiments very similar to these.

† Mean of eight trials, each of 200 vibrations, the numbers varying from ·0036300 to ·0036969.

Let G be the moment of the resistance, l the whole length of the cylinder, a the radius of the cylinder, and ρ the density of the fluid; then

$$M' = \pi\rho a^2 dr,$$

and

$$G = \frac{\pi^3 k' \rho a^2 l^3}{12\tau} \frac{d\theta}{dt};$$

whence

$$\log_e \text{dec.} = \frac{\pi^3 k' \rho a^2 l^3}{24I},$$

I being the moment of inertia of the whole vibrator; thus

$$\log_{10} \text{dec.} = \frac{\pi^3 k' \rho a^2 l^3}{24I} \log_{10} e. \quad \ldots\ldots\ldots \quad (1)$$

When we have a pair of cylinders of equal mass and dimensions suspended vertically from points equally distant from the axis of the wire, we can easily prove in a manner similar to the above that the logarithmic decrement due to the resistance of the air on the cylinders is expressed by the formula

$$\log_{10} \text{dec.} = \frac{\pi^3 \rho \beta^2 l d^2 k'}{16I} \log_{10} e. \quad \ldots\ldots\ldots \quad (2)^*$$

If the logarithmic decrement be known, we can determine from (2) k', and hence, by interpolation, from the table given on p. 46 of Prof. STOKES's paper, m, this last being connected with μ, the coefficient of viscosity, by the formula—

$$m = \frac{\beta}{4} \sqrt{\frac{\pi\rho}{\tau\mu}}. \quad \ldots\ldots\ldots \quad (3)$$

Since β, τ, and ρ are known, we can from (3) find μ.

In the case of two spheres of equal mass and dimensions there is no difficulty in obtaining the following formulæ from the data on p. 32 of Prof. STOKES's paper:—

$$\log_{10} \text{dec.} = \frac{\pi k' M' d^3}{4(I + 2kM')} \log_{10} e, \quad \ldots\ldots\ldots \quad (4)$$

where I is the moment of inertia of the whole vibrator, M' the mass of fluid displaced by each sphere, and k and k' are connected with μ by the equations—

* In this equation and in equation (4) the effect of the rotation of the cylinders about the axes is neglected. For the necessary correction see the end of the paper.

$$k = \tfrac{1}{2} + \frac{9}{4a}\sqrt{\frac{2\mu\tau}{\pi\rho}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

$$k' = \frac{9}{4a}\sqrt{\frac{2\mu\tau}{\pi\rho}}\left\{1+\frac{1}{a}\sqrt{\frac{2\mu\tau}{\pi\rho}}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

in which a is the radius of each sphere.

Application of the Mathematical Formulæ to the Results of Experiment I.

It will be seen that the logarithmic decrement with the paper cylinders on is ·0036476, whilst with the paper cylinders off it is ·0009103; therefore the logarithmic decrement due to the resistance of the air on the cylinders only is approximately ·0027373. I write 'approximately' because there are certain corrections to be applied which I will now proceed to describe. In the first place, the vibration-period, when the paper cylinders were on, though nearly the same as when the cylinders were off, was not quite the same. I therefore determined approximately the value of μ, without making this or the other small corrections to be mentioned presently, and used this value to obtain approximately the logarithmic decrement which would be due to the resistance of the air on the cylindrical bar VV and the cylindrical portions S, K, of the suspenders. The logarithmic decrement due to the resistance of the air on the other portions of the suspenders and on the disks m, m, was obtained by making independent observations, in which the bar was vibrated first with the suspenders on the bar, and then with the suspenders off, but with cylinders of equal mass placed inside the hollow bar VV, so that the time of vibration should remain unaltered.

Suppose that λ represents the logarithmic decrement due to the resistance of the air on the bar and the suspenders, and that t_1, t_2, are the vibration-periods with and without the paper cylinders respectively, then, with a sufficient degree of approximation, provided t_1 does not differ much from t_2, we have the amount to be added to the uncorrected logarithmic decrement equal to

$$\lambda\left(1-\frac{t_2^4}{t_1^4}\right).$$

Again, the temperature of the air and the pressure of the atmosphere were not quite the same with and without the paper cylinders. It can, however, be shown that for the small differences of temperature and pressure which we have here the logarithmic decrement will be independent of the temperature * and vary directly

* The logarithmic decrement will not be independent of the temperature unless μ varies as the absolute temperature. If we adopt the results of recent experiments, the logarithmic decrement should approximately vary as $\sqrt{\frac{364+t}{273+t}}$, where t is the temperature in degrees Centigrade. The correction which this would entail I have neglected, as being inappreciable in these experiments.

as the square root of the pressure; the amount to be added to the uncorrected logarithmic decrement, owing to the above causes, will therefore be

$$\lambda\left(1-\sqrt{\frac{p_1}{p_2}}\right),$$

where p_1 and p_2 are the pressures with and without the paper cylinders respectively.

Further, when the cylinders were screwed on to the suspenders, about 4 mms. of the latter entered the former, so that the observed logarithmic decrement was less than it should be by an amount which would be nearly equal to the logarithmic decrement due to the resistance of the air on two vertical cylinders 4 mms. in length and 0·3366 cm. in diameter; this could be calculated to within a sufficient degree of approximation by using the approximate value of μ. The amount in this particular case was ·0000037.

Lastly, the temperature of the wire was not the same with and without the paper cylinders, but, as the effect of change of temperature had been determined previously, this difference could be allowed for.

No correction is required for any variation in the internal friction of the wire itself, arising from difference in the vibration-periods with and without the paper cylinders; for I had previously satisfied myself that the diminution of amplitude resulting from internal friction is nearly independent of the time of vibration.

Accordingly we have the following amounts to be added to the uncorrected logarithmic decrement:—

	Correction.
For difference of time of vibration with and without paper cylinders	+·0000008
For difference of pressure of air	−·0000005
For difference of temperature of the wire	−·0000002
For portions of suspenders which enter the cylinders	+·0000037
Total	+·0000038
Corrected logarithmic decrement	·0027411

In calculating ρ, the density of the air, I have assumed that the latter is half saturated with moisture, and that the mass of a cubic centimetre of dry air at 0° C., and under a pressure of 29·9217 inches of mercury, is ·0012930 gramme; thus, in the present instance,

$$\rho=\frac{29\cdot872-\frac{1}{2}\times\cdot206}{29\cdot9217}\times\frac{273}{273+12\cdot02}\times\cdot001293=\cdot0012334.$$

The distance from each other of the axes of the two paper cylinders was 20·80 centims., and this distance was maintained in all the experiments which follow, except the last, where it was 20·78 centims.

From these and the previous data we can, by means of equation (2), get

$$K = 1{\cdot}6122;$$

and hence, by interpolation, we can obtain from the table on page 46 of Professor STOKES's paper

$$m = 1{\cdot}1327.$$

Again, substituting this value of m in equation (3), we obtain as the value of μ in C.G.S. units, at the temperature of 12°·02 C,

·00018294.

Experiment II.

Two hollow cylinders, made of drawn brass tubing, and closed at both ends, were used instead of the paper cylinders. As measured by a gauge reading to $\frac{1}{100}$th of a millimetre, the mean diameter of one cylinder was 0·96446 centim., and of the other 0·96279 centim. These values were obtained by gauging each cylinder in ten different places, equidistant from each other, and in the calculations each cylinder was assumed to have a mean diameter of 0·96363 centim. The length of one cylinder was 60·92 centims., and of the other 60·85 centims., whilst the mean of these numbers, *i.e.*, 60·885 centims., was assumed to be the length of each cylinder. The mass of each cylinder was 91·900 grammes, and when the cylinders were on the bar V V the moment of inertia of the whole vibrator, in centimetre-gramme units, was 36702. The value of the vibration-period was 7·0590 seconds. The temperature of the air was 14°·63 C., and the barometric height 29·707 inches. The uncorrected logarithmic decrement due to the resistance of the air on the cylinders was ·0012338, and the corrected logarithmic decrement was ·0012546. From these data was deduced a value of μ, at the temperature of 14°·63 C., of

·00017718.

Experiment III.

Everything else was arranged in the same manner as in Experiment I., but, instead of the annealed copper wire, an annealed silver wire, 97 centims. in length and 0·100863 centim. in diameter, was used. The paper cylinders employed in Experiment I. were used here, and when these cylinders were on the vibration-period was 3·0198 seconds. The temperature of the air was 11°·69 C., and the barometric height 30·207 inches. The uncorrected logarithmic decrement due to the resistance of the air against the cylinders was ·0016871, and the corrected logarithmic decrement ·0016905. The value of μ at the temperature 11°·69 C. was calculated to be

·00018143.

Experiment IV.

Acting on the advice of Professor STOKES, I modified Experiment III. as follows:—The logarithmic decrement was determined with the paper cylinders already used, and also with another pair of the same diameter, and made in the same manner, but having a length of 7·700 centims., the vibration-period being by the usual device maintained very nearly the same in both cases. The difference between the two logarithmic decrements, ·0024564 and ·0009933, will therefore equal the logarithmic decrement due to the resistance of the air on cylinders having each a length of (60·875–7·700) centims., *i.e.*, 53·175 centims. When the longer paper cylinders were on the bar the vibration-period was 2·9994 seconds. The temperature of the air and the barometric height were 10°·64 C. and 30·057 inches respectively. The uncorrected logarithmic decrement was ·0014631, and the corrected logarithmic decrement ·0014638. The value of μ at the temperature of 10°·64 C., deduced from the above data, was

·00017955.

Experiment V.

The previous experiments had given such closely according values of μ that, though my investigations on the internal friction of metals only required that the formulæ for cylinders should give consistent results, I felt that it would be of interest to ascertain whether the use of spheres would be attended with the same satisfactory agreement. The main difficulty to be encountered with spheres is that the mass of a properly constructed spherical shell makes it rather unsuitable for experiments on the viscosity of gases. After thinking over various plans of obtaining hollow spherical shells of sufficiently accurate make, and not feeling satisfied that I should be able to get, without much difficulty, what I wanted, I decided on using solid spheres made of fairly light wood. These spheres were specially turned for me, with instructions to make each as exactly as possible $2\frac{1}{2}$ inches in diameter. The turner executed his commission very fairly, for, on gauging each sphere at ten different places with calipers reading to $\frac{1}{1000}$th of an inch, I found that none of the readings differed from the mean by so much as ·3 per cent., and that the mean diameters of the two spheres were 2·5103 inches and 2·5007 inches respectively. In the calculations each sphere was reckoned as having a diameter of 2·5055 inches or 6·364 centims. The masses of the two spheres were not quite so equal as I could have wished, the apparent mass of one in air being 64·823 grammes, and of the other 63·761 grammes. No appreciable error will, however, be introduced by considering the apparent mass of each in air to be 64·292 grammes. The correction for the mass of air displaced by each sphere amounted to 0·168 gramme, so that in the calculations the mass of each sphere was taken as 64·460 grammes.

The spheres were attached to the suspenders S, K, in the same manner as the cylinders, but the disks were now dispensed with. The moment of inertia of the whole vibrator when the spheres were on was 30,927 in centimetre-gramme units, the vibration-period was 2·8791 seconds, and the temperature of the air and the barometric pressure were 9°·97 C. and 29·607 inches respectively. The uncorrected logarithmic decrement due to the friction of the air on the spheres was ·0003462, and the corrected logarithmic decrement was ·0003483.

In deducing the value of μ from the above data by the aid of equations (4), (5), and (6), I assumed, in finding $2k\mathrm{M}'$, a value for μ equal to the mean of that got from the other experiments; this step is admissible, because $2k\mathrm{M}'$ is very small compared with I.* Having determined the value of k' by means of equation (4), I substituted it in equation (6), and thus obtained a quadratic equation for finding μ. The quadratic may, however, be converted into a simple equation by making use of the same value of μ as above in calculating the term $\frac{1}{a}\sqrt{\frac{2\mu\tau}{\pi\rho}}$, which was thus found to be 0·16085. The last number is not small compared with unity, and, had the final result proved to be as much as 10 per cent. greater or less than the mean of those got from the other experiments, the above conversion of the quadratic into the simple equation would not have been admissible. It will be seen eventually, however, that the conversion is legitimate, and the value of μ at a temperature of 9°·97 C. as determined from the simple equation is

·00019334.

Mathematical Formulæ required for the Effect of the Rotation of the Spheres or Cylinders about their own Axes.†

Professor G. G. Stokes has been good enough to furnish me with the following formulæ for the corrections not yet made for the effect of the rotation of the spheres or cylinders about their own axes:—

Let λ_a be the logarithmic decrement due to the rotation, then for the spheres

$$\lambda_a = \frac{2\mu\mathrm{M}'\tau}{\mathrm{I}\rho}\,\frac{\nu a+3+\frac{3}{\nu a}+\frac{3}{2(\nu a)^2}}{1+\frac{1}{\nu a}+\frac{1}{2(\nu a)^2}}\log_{10}e, \quad \ldots\ldots\ldots \quad (7)$$

where I is the moment of inertia of the whole system, τ is the time of a vibration

* In fact, is quite neglectable in the case before us.

† What follows was added Sept. 16, 1886.

from rest to rest, M′ is the mass of fluid displaced by each sphere, a is the radius of the sphere, and

$$\nu = \sqrt{\frac{\pi\rho}{2\mu\tau}}.$$

In the case of the cylinders, which were hollow, we have to take into account the effect of the air both inside and outside. For the air outside we may take

$$\lambda_a = \frac{2M'\mu\tau}{I\rho}\log_{10} e\mathrm{P}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where P is the real part of the imaginary expression

$$ma\,\frac{1+\frac{3.5}{1(8ma)}+\frac{1.3.5.7}{1.2(8ma)^2}-\frac{1^2.3.5.7.9}{1.2.3(8ma)^3}+\frac{1^2.3^2.5.7.9.11}{1.2.3.4(8ma)^4}-\ldots}{1+\frac{1.3}{1(8ma)}-\frac{1^2.3.5}{1.2(8ma)^2}+\frac{1^2.3^2.5.7}{1.2.3(8ma)^3}-\ldots},$$

where

$$m = \sqrt{\frac{\pi\rho}{\mu\tau}}(\cos 45^\circ + \sqrt{-1}\sin 45^\circ).$$

On expanding P in descending powers of $\sqrt{\frac{\pi\rho}{\mu\tau}}a$, we get

$$\mathrm{P} = \frac{f}{\sqrt{2}}+1.5+\frac{0{\cdot}375}{\sqrt{2}f}-\frac{0{\cdot}4922}{\sqrt{2}f^3}-\ldots, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where

$$f = \sqrt{\frac{\pi\rho}{\mu\tau}}.a.$$

This series may be used with advantage in all the experiments relating to the cylinders to estimate approximately the effect of the air outside, but, unless the value of f is decidedly larger, the value of λ_a is best found from the formula

$$\lambda_a = \frac{4M'\mu\tau}{I\rho}\log_{10} e.\frac{k^2+k'^2-1}{(k-1)^2+k'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where k, k', are the quantities tabulated at p. 46 of Professor Stokes's paper.

The corrections, as calculated from both formulæ, were found to agree satisfactorily.

For the air inside we may use, for such values of $\sqrt{\frac{\pi\rho}{\mu\tau}}a$ as we have here, the formula

$$\lambda_a = \frac{2M'\mu}{I\rho}(-\mathrm{Q})\log_{10} e, \quad . \quad . \quad . \quad . \quad . \quad (11)$$

where M′ is the mass of air inside, and Q is the real part of the imaginary expression

$$-\frac{\frac{2}{2.4}(ma)^2+\frac{4}{2.4^2.6}(ma)^4+\frac{6}{2.4^2.6^2.8}+\dots}{1+\frac{1}{2.4}(ma)^2+\frac{1}{2.4^2.6}(ma)^4+\frac{1}{2.4^2.6^2.8}(ma)^6+\dots},$$

which is of the form

$$\frac{A+\sqrt{-1}B}{C+\sqrt{-1}D}$$

and of which the real part is

$$\frac{AC+BD}{C^2+D^2}.$$

In the following table will be found the corrections necessary to be made for the rotations of the spheres and cylinders about their axes :—

Cylinders.

Number of experiment.	Logarithmic decrement uncorrected for rotation about axes.	Effect of air outside. *	Effect of air inside. *	Corrected logarithmic decrement.
I.	·0027411	·0000313	·0000011	·0027087
II.	·0012546	·0000030	·0000000	·0012516
III.	·0016905	·0000173	·0000019	·0016713
IV.	·0014638	·0000150	·0000017	·0014471
		Spheres.		
V.	·0003483	·0000159	..	·0003324

There is still a further slight correction to make, inasmuch as the mercury of the barometer was not at 0° C. when the aneroid was compared with the mercury barometer, whereas the density of the air was calculated on the assumption of the mercury being at 0° C. The correction is very slight, but the closeness of agreement of the different experiments justifies us in making it. It will be sufficient for this purpose to multiply each value of μ as determined from the above table by $(1+\cdot00018t)$, where t is the temperature at which the aneroid was compared with the mercury barometer. Applying all the corrections, the final results are as follows :—

* In making these corrections, an approximate value of μ was used.

TABLE II.—Cylinders.

Number of experiment.	Length in centims.	Diameter in centims.	Distance between the centres in centims.	Vibration-period in seconds.	Temperature in degrees Centigrade.	Coefficient of viscosity of air in C.G.S. units.
I.	60·875	2·5636	20·80	6·8373	12·02	·00017900
II.	60·885	0·9636	20·80	7·0590	14·63	·00017680
III.	60·875	2·5636	20·80	3·0198	11·69	·00017767
IV.	53·175	2·5636	20·80	2·9994	10·64	·00017581
Spheres.						
V.	..	6·364	20·78	2·8811	9·97	·00017626

Taking the means of the numbers in the sixth and seventh columns, we find that the value of μ at a temperature of 11°·79 C. is

·00017711.

The Effect of the Presence of Aqueous Vapour on the Viscosity of Air.

The above experiments extended over a period of some months, during which the air was in various conditions with respect to being saturated with aqueous vapour, so that for a rough approximation we may assume that the mean value for μ just given will apply to air half saturated with vapour at a temperature of 12° C., and it would appear that the presence of the small quantity of aqueous vapour which this implies would not affect the value of μ to an extent equal to that of the probable error in experimenting. From the careful investigations of Mr. CROOKES* we learn that at a temperature of 15° C., and under pressures of from 760 to 350 millims., the presence of aqueous vapour has little or no influence on the logarithmic decrement. By the aid of Professor STOKES's note,† I have estimated that at 15° C., and under a pressure of 760 millims., the air when *saturated* with aqueous vapour would be *more* viscous than perfectly dry air‡ to the extent of only ·2 per cent. It is not until the air is under a less pressure than 350 millims. that the aqueous vapour begins to show appreciable effect, but when the rarefaction is great the moist air becomes considerably *less* viscous than dry air.

According to MAXWELL§ damp air over water at a temperature of 21°·11 C., and under a pressure of 101 millims., is *less* viscous than dry air by about $\frac{1}{60}$th part.

* 'Phil. Trans.,' Part II., 1881, p. 427.

† See p. 440 of the above paper.

‡ Mr. CROOKES adopted great precautions to render the air dry.

§ 'Phil. Trans.,' vol. 156, 1866.

On the whole it would seem that the aqueous vapour in the air used in my experiments would hardly influence the value of μ to the extent of ·1 per cent.

The presence of carbon dioxide in the air would still less affect the result, as not only is the viscosity of carbon dioxide not very remote from that of air, but the amount of the gas present is also very minute.

Comparison of the Results of Recent Investigations of the Coefficient of Viscosity of Air.

In the beginning of this memoir I pointed out the very large discrepancies which existed between the results of different experimenters, but, since I entered on my task, not only have I acquired fresh information respecting what had already been done, but also quite recently fresh investigations have been made. Table III. contains the required information.

TABLE III.

Authority.	Coefficient of viscosity of air at 0° C.	Method.
O. E. MEYER*	·0001875	Oscillating plates.
,,	·0001727	Transpiration.
PULUJ*	·0001798	,,
SCHNEEBELI†	·0001707	,,
OBERMAYER†	·0001705	,,

In order to reduce my own observations to 0° C., I have made use of the investigations of Professor SILAS W. HOLMAN on the effect of temperature on the viscosity of air.‡ According to the exceedingly careful and elaborate observations of this experimenter, the coefficient of viscosity of dry air is not proportional to the absolute temperature, but

$$\mu_t = \mu_0(1+0{\cdot}002751t-0{\cdot}00000034t^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where t is the temperature in degrees Centigrade, and μ_t, μ_0, are the coefficients of viscosity at $t°$ C. and 0° C. respectively.

My own observations were made with too small ranges of temperature to show the relation between the value of μ and the temperature, but the above formula expresses more nearly this relation as deduced from my experiments than the formula

$$\mu_t = \mu_0(1+0{\cdot}00366t).$$

* 'Phil. Mag.,' vol. 21, 1886, p. 220.
† 'Archives Sci. Phys. Nat.,' vol. 14, 1885.
‡ 'Phil. Mag.,' vol. 21, 1886.

Adopting, therefore, formula (12), we have the following equation for determining the value of μ at any temperature :—

$$\mu_t = \cdot00017155(1 + \cdot002751t - \cdot00000034t^2). \quad . \quad . \quad . \quad . \quad . \quad (13)$$

The differences between the observed and calculated values of μ_t for the five different sets of experiments are given below :—

Experiment.	Observed value of μ_t.	Calculated value of μ_t.	Difference.
I.	·00017900	·00017760	+·00000140
II.	·00017680	·00017850	−·00000170
III.	·00017767	·00017704	+·00000063
IV.	·00017581	·00017653	−·00000072
V.	·00017626	·00017622	+·00000004

The probable error is about ·2 per cent., and, considering the manner in which the five sets of experiments varied as regards their conditions, it would seem that, even when all allowance has been made for aqueous vapour, &c., the number ·00017155 must represent the value of μ_0 for *dry* air within at least $\frac{1}{2}$ per cent. Now, this number agrees fairly with the values of μ_0 obtained by other observers with the transpiration method ; it is, however, more than 9 per cent. less than that obtained by MEYER with oscillating plates, and by MAXWELL. The mathematical difficulties attending Professor MEYER's method of oscillating plates have been already mentioned, but the method of Professor MAXWELL does not seem open to these objections, and indeed appeared to me to be so good that I for some time attempted, though in vain, to account for the difference between MAXWELL's result and my own. Professor G. G. STOKES has, however, kindly interested himself in the matter, and has shown in the accompanying note the possibility of MAXWELL's result being too high. I may perhaps be allowed to add that, if we only take the first two of the five sets of MAXWELL's experiments, in which two the distances of the fixed from the oscillating plates are so great as to render any error such as suggested by Professor STOKES very small, we obtain a value for the coefficient which is nearly identical with that obtained by myself.

ADDENDUM.

Note on the preceding Paper, by Professor G. G. STOKES, *P.R.S.*

(Received January 14, 1886.)

The consistency of Mr. TOMLINSON's different determinations of the coefficient of viscosity of air, notwithstanding the great variation in the circumstances of the experiments, and the consistency with one another of the numbers got by a different process by MAXWELL, led me to endeavour to make out the real cause of the difference, and I think the main part, at any rate, of it can be explained by a very natural supposition.

The fact that Mr. TOMLINSON worked with air in its ordinary state, whereas MAXWELL's air was dry, even if it tends in the right direction, would evidently not go nearly far enough. But it occurred to me that the effect of any error of level in the movable disks employed by MAXWELL must have been much greater than might at first sight appear. For suppose a very small error δ to exist, and suppose the fixed disks adjusted to be parallel to the movable ones in the position of equilibrium of the latter. Then the two systems must be, very nearly indeed, parallel throughout the motion, since the angle of oscillation of the movable disks to one side or other of the position of equilibrium is very small. If 2α be the whole amplitude, the greatest error of parallelism will be of the order $\delta\alpha$, and it would naturally appear at first sight that the effect of so small an error of parallelism must be insignificant for any such error of level as we can reasonably suppose to have existed. But a little consideration will show that this need not be the case when the distance between the fixed and movable disks is very small compared with the diameter of the latter. For suppose the disk to have been rotated through a small angle ρ round a vertical axis; the rotation ρ may be decomposed into a rotation $\rho\cos\delta$ round the axis of figure, and a rotation $\rho\sin\delta$ round a horizontal axis in the plane of the disk. As regards the former, the motion takes place as supposed in the investigation. But as regards the latter the disk oscillates about a horizontal axis in its own plane. Now, when the disks are very near one another this oscillation entails a squeezing thinner of the stratum of air opposite to one half of the disk, and a widening of the stratum opposite the other half, the two halves being alternately squeezed thinner and widened; and, since for such slow motions the air is practically incompressible, this transfer of air cannot be effected without a motion of the air along the surface of the disk far larger than what would be produced by an equal rotation about the axis of figure. Accordingly a very slight error of horizontality in the movable disk might produce a sensible error in the result, though an error of direction of similar amount in the orientation of the fixed disk would be quite insignificant in its influence on the final result.

This conclusion is fully borne out by the result of mathematical calculation founded on the equations of motion of a viscous uncompressed fluid. The calculation becomes

very simple if we treat the distance between the disks as very small compared with the radius, neglect the special actions about the edge, and further neglect the inertia of the air, as we safely may, since it was small in MAXWELL'S experiments, especially those in which the disks were at a small distance apart, and therefore the influence of viscosity the greatest; or those again in which the air was rarefied.

Let the plane of a movable disk in its position of equilibrium be taken for the plane of x, y, the axis of figure for the axis of z, and the intersection of a horizontal plane with the plane of the disk for the axis of y; and let the opposed fixed plane be parallel to the plane of x, y, and at a distance h from it. Let a be the radius of the disk.

First, as regards motion round the axis of figure. Let ω be the angular velocity of the disk. Then, according to the simplifications adopted, the motion of the fluid will be a motion of simple shearing, such that the velocity at a point whose semi-polar coordinates are r, θ, z, will be $\omega r(h-z)/h$ in a direction perpendicular to the radius vector. It will suffice to write down the moment of the force which this calls into play, which is

$$\frac{\pi\mu a^4\omega}{2h}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Next, for motion round the axis of y. Let ω' be the angular velocity; u, v, w, the components of the velocity; U, V, the mean values of u, v, from 0 to h. Consider the prism of fluid standing on the base $dx\,dy$, and extending between the planes. As the volume of the prism is diminished at the base by $\omega' x\,dx\,dy\,dt$ in the time dt, the excess of the volume of the fluid which flows out across the face $h dy$, whose abscissa is $x+dx$, over that which flows in across the face $h dy$, whose abscissa is x, *plus* the similar difference for the pair of faces $h dx$, must equal $\omega' x\,dx\,dy\,dt$. This leads to the equation

$$h\frac{d\mathrm{U}}{dx}+h\frac{d\mathrm{V}}{dy}=\omega' x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But, for motion between two close parallel planes, the velocity parallel to the plane, and its components in two fixed directions in that plane, vary as $z(h-z)$, and therefore

$$u=\frac{6z(h-z)}{h^3}\mathrm{U}, \qquad v=\frac{6z(h-z)}{h^3}\mathrm{V}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The first equation of motion is

$$\frac{dp}{dx}=\mu\left(\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Now, on account of the smallness of h, the space-variations of the components u, v,

of the velocity are much greater for z than for x or y. Hence in (3), and the corresponding equation for v, the first two terms in the right-hand members may be omitted, giving, by (2),

$$\frac{dp}{dx} = -\frac{12\mu}{h^2}\mathrm{U}, \qquad \frac{dp}{dy} = -\frac{12\mu}{h^2}\mathrm{V},$$

and then, from (1),

$$\frac{d^2p}{dx^2}+\frac{d^2p}{dy^2} = -\frac{12\mu\omega'}{h^3}x,$$

or, in polar coordinates,

$$\frac{d^2p}{dr^2}+\frac{1}{r}\frac{dp}{dr}+\frac{1}{r^2}\frac{d^2p}{d\theta^2} = -\frac{12\mu\omega'}{h^3}r\cos\theta; \quad \ldots\ldots\ldots \quad (4)$$

and if we take, as we may, p to mean the excess of pressure over the pressure in equilibrium, we have the conditions that p shall vanish when $r=a$, and that p shall not become infinite at the centre.

The equation (4) and the conditions at the mouth and centre may be satisfied by taking

$$p = f(r)\cos\theta,$$

which gives, from (4),

$$f''(r)+\frac{1}{r}f'(r)-\frac{1}{r^2}f(r) = -\frac{12\mu\omega'}{h^3}r.$$

The integral of this equation is

$$f(r) = -\frac{3\mu\omega'}{2h^3}r^3+\mathrm{A}r+\frac{\mathrm{B}}{r},$$

where A, B, are arbitrary constants. The conditions at the centre and mouth give

$$\mathrm{B} = 0, \qquad \mathrm{A} = \frac{3\mu\omega'a^2}{2h^3},$$

whence

$$p = \frac{3\mu\omega'}{2h^3}(a^2r-r^3)\cos\theta.$$

The moment of this pressure about the axis of y is $\iint p.r\cos\theta.rdrd\theta$, or

$$\frac{\pi\mu\omega'a^6}{8h^3}. \quad \ldots\ldots\ldots \quad (\mathrm{B})$$

The moments (B) and (A) are as $a^2\omega'$ to $4h^2\omega$, and the works of these moments in the time dt are as $a^2\omega'^2$ to $4h^2\omega^2$. If this ratio be denoted by n to e, and ω, ω', are the components of an angular velocity round an axis in the plane of xz, inclined at an angle δ to the axis of z,

$$\tan^2\delta = \frac{4h^2}{a^2}n.$$

In MAXWELL'S experiments a was 5·28 inches, and when the fixed and movable disks were closest h was 0·18475. If we suppose the whole loss of energy 8 per cent. greater than that due to rotation round the axis of figure, to which it was deemed to be due, we have $n=0·08$, giving $\delta=1°\ 8'$. Now, no special adjustment was made to secure the strict horizontality of the movable disks, or at least none is mentioned; the final adjustment is stated to have been that of the fixed disks, which were presumably adjusted to be parallel to the movable ones, and at the desired distance. Hence such small errors of level as that just mentioned may very well have occurred.

Second Note.—*On the Effect of the Rotations of the Cylinders or Spheres round their own Axes in increasing the Logarithmic Decrement of the Arc of Vibration.*—By the same.

(Received October 22, 1886.)

In Art. 9 of my paper on Pendulums I pointed out that in the case of a ball pendulum the resistance due to the rotation of the sphere round its axis need not be regarded, on account of the large ratio which the distance of the centre from the axis of suspension bears to the radius of the sphere. In Mr. TOMLINSON'S experiments the corresponding ratio is not near so great, and its squared reciprocal is not small enough to allow us to neglect the correction altogether, especially in the case of the spheres, the radius of which is much larger than that of the cylinders. In both cases the problem admits of solution.

In both cases the motion of the suspended body may be regarded as compounded of a motion of translation, in which the centre oscillates in an arc of a circle, and a motion of rotation about its axis of figure, supposed fixed; and, the motion being small, the effects of the two may be considered separately. It is the latter with which we have at present to deal. As regards the motion of translation, the spheres or cylinders were sufficiently far apart to allow us to regard each as out of the influence of the other, and accordingly as oscillating in an infinite mass of fluid; and this is still more nearly true as regards the motion of rotation. The problem, then, is reduced to this: a sphere or cylinder performs small oscillations of rotation about its axis of figure, which is vertical and regarded as fixed, in an infinite mass of viscous fluid; it is required to determine the motion, and thereby to find the effect of the fluid in damping the motion of the system of which the suspended body forms a part.

In the case of the sphere the problem of determining the motion of the fluid is identical with that solved by Professor VON HELMHOLTZ in a paper published in the 40th volume of the 'Sitzungsberichte' of the Vienna Academy, p. 607, and reprinted in the first volume of his collected works, p. 172, with the exception that the arbitrary constants which occur in the integral of the fundamental ordinary differential equation are differently determined, since the condition that the motion shall not

become infinite at the centre is replaced by the condition that it shall not be infinite at an infinite distance.

In the present case the motion is necessarily symmetrical about the axis, so that it is alike all round any circle that has the axis for its axis; it is, moreover, tangential to the circle. Let the fluid be referred to polar coordinates r, θ, ϖ; r being the distance from the centre, θ the inclination of the radius vector to the axis, and ϖ as usual. Then, taking ρ, μ, to denote the density and coefficient of viscosity, and observing that $v=q\cos\varpi$, where q is the velocity, we easily get from the second equation of motion, by putting, as we may, $\varpi=0$ after differentiation,

$$\frac{d^2q}{dr^2}+\frac{2}{r}\frac{dq}{dr}+\frac{1}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dq}{d\theta}\right)-\frac{q}{r^2\sin^2\theta}-\frac{\rho}{\mu}\frac{dq}{dt}=0; \quad . \quad . \quad . \quad (1)$$

and we have the condition at the surface—

$$q=\omega a\sin\theta \quad \text{when} \quad r=a, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where ω is the angular velocity of the sphere, and a its radius.

The motion with which we have to deal is periodic, subject to a secular diminution. The latter being actually very slow, it will suffice, in calculating the force of the air on the sphere, to take the motion as periodic, and expressed, so far as the time is concerned, by the sine or cosine of nt. It will be more convenient, however, to use the symbolical expression e^{int}, where $i=\sqrt{(-1)}$. The general equation (1) and the equation of condition (2) can both be satisfied by taking q to be expressed, so far as θ is concerned, by $\sin\theta$. Assuming, then—

$$q=e^{int}\sin\theta f(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

and writing

$$\frac{i\rho n}{\mu}=\frac{i\pi\rho}{\mu\tau}=m^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

we have

$$f''(r)+\frac{2}{r}f'(r)-\frac{2}{r^2}f(r)-m^2f(r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Taking $+m$ for that root of the imaginary m^2 which has its real part positive, we have for the integral of (5), subject to the condition of not becoming infinite at an infinite distance—

$$f(r)=Ae^{-mr}\left(\frac{1}{r}+\frac{1}{mr^2}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Omitting the pressure in equilibrium, we shall have for the force of the fluid on an element of the sphere a tangential pressure (say T, referred to a unit of surface)

acting perpendicularly to the plane passing through the axis and the element, the expression for which, reckoned positive when it acts in the direction of ϖ positive, is

$$\mathrm{T} = \mu\left(\frac{dq}{dr} - \frac{q}{r}\right)_{r=a} = \mu e^{int}\sin\theta\left(f'(a) - \frac{f(a)}{a}\right);$$

and the moment of the force taken all over the sphere is

$$\int_0^\pi \mathrm{T}\, 2\pi a^2 \sin\theta . a\sin\theta\, d\theta = \tfrac{8}{3}\pi\mu a^3 e^{int}\left(f'(a) - \frac{f(a)}{a}\right)$$
$$= 2\mathrm{M}'\mu' e^{int}\left(f'(a) - \frac{f(a)}{a}\right)$$

if $\mu' = \mu/\rho$, and M' is the mass of the fluid displaced by the sphere.

Now we have, by (2), (3),

$$\omega a = e^{int} f(a),$$

whence the expression for the moment becomes

$$2\mathrm{M}'\mu'\left(\frac{af'(a)}{f(a)} - 1\right)\omega.$$

To get the whole moment, the above must be doubled, as there are two spheres. If ϑ be the angular distance of the vibrating system from the position of equilibrium, we may write $d\vartheta/dt$ for ω; and if the mixed imaginary within parentheses, with sign changed, be denoted by $\mathrm{P}+i\mathrm{Q}$, the real part, P, will be that which affects the arc of vibration, the imaginary part falling upon the time, which we do not want. The Napierian logarithmic decrement in one vibration will be got by dividing half the real part of the expression for the moment of the forces by the moment of inertia, or, say, MK^2. It will therefore be $2\mathrm{M}'\mu'\mathrm{P}/\mathrm{MK}^2$.

Now we get, from (6),

$$1 - \frac{af'(a)}{f(a)} = \frac{ma + 3 + \frac{3}{ma}}{1 + \frac{1}{ma}};$$

and, taking the real part of this, we get finally, after reduction,

$$\text{Nap. log. dec.} = \frac{2\mu'\mathrm{M}'}{\mathrm{MK}^2} \frac{\nu a + 3 + \frac{3}{\nu a} + \frac{3}{2\nu^3 a^3}}{1 + \frac{1}{\nu a} + \frac{1}{2\nu^3 a^3}} \tau, \quad \ldots\ldots \quad (7)$$

where $\nu = \sqrt{\left(\frac{\pi}{2\mu'\tau}\right)}$.

In the case of the cylinder the motion is in two dimensions, and is most conveniently referred to polar coordinates r, θ, the origin being in the axis. The radius of the cylinder will be denoted by a, the outer or inner radius according as we are dealing with the air outside or inside.

The mode of proceeding is precisely analogous to that in the case of the sphere, and, q being the whole velocity, we have

$$q = e^{int} f_1(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

where

$$f_1''(r) + \frac{1}{r} f_1'(r) - \frac{1}{r^2} f_1(r) - m^2 f_1(r) = 0; \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and the condition at the surface gives

$$e^{int} f_1(a) = \omega a. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

If T be the tangential pressure on the cylinder,

$$\mathrm{T} = \pm\mu\left(\frac{dq}{dr} - \frac{q}{r}\right)_{r=a}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

the sign being + or − according as we are dealing with the air outside or inside. The moment of this pressure on a length, l, of the cylinder is

$$\pm 2\pi\mu a^2 l e^{int}\left(f_1'(a) - \frac{1}{a} f_1(a)\right) = \pm 2\mathrm{M}'\mu'\left(\frac{a f_1'(a)}{f_1(a)} - 1\right)\omega. \quad . \quad . \quad . \quad . \quad (12)$$

The equation (9) cannot be integrated in finite terms. Nevertheless, in the case of the air outside, the expression (12) for the moment may be obtained in a finite form in terms of two functions, k, k', which I had occasion to tabulate for the purpose of finding the resistance of a viscous fluid to a pendulum of the form of a cylindrical rod.

Putting, as in my former paper,

$$f_1(r) = f_0'(r), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

(f_1, f_0, are the functions there denoted by F_2, F_3), we have

$$f_0''(r) + \frac{1}{r} f_0'(r) - m^2 f_0(r) = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Now in both problems (that of my former paper and that of the present note) the function $f_0(r)$ satisfies the same differential equation (14) and the same condition of

vanishing at infinity. Hence the function $f_0(r)$ is the same in the two cases, save as to the value of the arbitrary constant, which is a factor of the whole, and which disappears from the expression (12) as well as from those of k and k'.

The definition of k and k' is given by equation (99) of my former paper, viz. :—

$$1-\frac{4f_0'(a)}{m^3af_0(a)}=k-ik'. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Now, by (13), (14), and (15),

$$1-\frac{af_1'(a)}{f_1(a)}=1-\frac{af_0''(a)}{f_0'(a)}=2-\frac{m^2af_0(a)}{f_0'(a)}=2\frac{k+1-ik'}{k-1-ik'}=2\frac{k^2-1+k'^2+2ik'}{(k-1)^2+k'^2};$$

whence we get, as before, for the part of the logarithmic decrement due to the external air, in consequence of the rotations of the two cylinders round their own axes, M′ denoting the mass of air which would be displaced by one if solid and of radius a,

$$\text{Nap. log. dec.}=\frac{4\mathrm{M}'\mu'\tau}{\mathrm{MK}^2}\cdot\frac{k^2-1+k'^2}{(k-1)^2+k'^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

In the Table given in Art. 37 of my paper, m denotes half the modulus of ma, or

$$\frac{a}{2}\sqrt{\frac{\pi}{\mu'\tau}}.$$

This Table is not available for calculating the effect of the internal air, for which we must have recourse to the differential equation (9). The integral of this equation, expressed in ascending series, subject to the condition of not becoming infinite at the origin, is

$$f_1(r)=\mathrm{A}\left\{r+\frac{m^2r^3}{2.4}+\frac{m^4r^5}{2.4^2.6}+\frac{m^6r^7}{2.4^2.6^2.7}+\ldots\right\},$$

which gives

$$\frac{af_1'(a)}{f_1(a)}-1=\frac{\frac{m^2a^2}{4}+\frac{m^4a^4}{2.4.6}+\frac{m^6a^6}{2.4^2.6.8}+\ldots}{1+\frac{m^2a^2}{2.4}+\frac{m^4a^4}{2.4^2.6}+\frac{m^6a^6}{2.4^2.6^2.8}+\ldots} \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

Let the numerator of this fraction be denoted by $\mathrm{E}+i\mathrm{F}$, and the denominator by $\mathrm{G}+i\mathrm{H}$, where E, F, G, H, are real; then the real part will be $\mathrm{EG}+\mathrm{FH}$ divided by $\mathrm{G}^2+\mathrm{H}^2$, and we shall have for the correction due to the internal air

$$\text{Nap. log. dec.}=\frac{2\mathrm{M}'\mu'}{\mathrm{MK}^2}\cdot\frac{\mathrm{EG}+\mathrm{FH}}{\mathrm{G}^2+\mathrm{H}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

When the modulus of ma is small, it is rather more convenient to expand (17) according to ascending powers of ma. This may be done by actual division, or more conveniently by assuming a series with indeterminate coefficients, and using the non-linear differential equation of the first order in z obtained from (9) by putting $f_1'(r)=zf_1(r)$. Carried as far as to a^{12}, the development is

$$\frac{m^2a^2}{4}-\frac{m^4a^4}{96}+\frac{m^6a^6}{1536}-\frac{m^8a^8}{23040}+\frac{13m^{10}a^{10}}{4423680}-\frac{11m^{12}a^{12}}{55050240};$$

and, denoting the modulus of ma by f, and taking the real part, we have

$$\text{Nap. log. dec.}=\frac{2M'\mu'}{MK^2}\left\{\frac{f^4}{96}-\frac{f^8}{23040}+\frac{11f^{12}}{55050240}-\ldots\right\} \quad . \quad . \quad . \quad . \quad (19)$$

This series must not be used when f is at all large, as the convergence is too slow, and, as appears by a theorem due to CAUCHY, it becomes actually divergent when $f=3{\cdot}340^*$ nearly, whereas the series in (17) are always convergent, and when f has the above value converge rapidly.

When f is decidedly large the series in (17), though ultimately convergent, begin by diverging, so that the calculation is troublesome, and moreover my Table giving k and k' is not carried beyond $f=8$, as the calculation by a different method then becomes very easy. In this case we should employ the integral of (9), which is of the form e^{-mr} or e^{mr} multiplied by a descending series. The former exponential only will come in when we are treating of the external air, and the latter only when of the internal.

For the external air the integral is of the form

$$f_1(r)=\mathrm{B}e^{-mr}r^{-1}\left\{1+\frac{1.3}{1.(8mr)}-\frac{1^2.3.5}{1.2(8mr)^2}+\frac{1^2.3^2.5.7}{1.2.3(8mr)^3}-\ldots\right\}, \quad . \quad . \quad (20)$$

the signs being alternately + and −, and the new factors in the numerator being two less and two greater than the last factor in the term before. We get from (12), (20), and the expression for the logarithmic decrement in terms of T and the moment of inertia,

* The square root of the smallest real root of the equation

$$1-\frac{x}{2.4}+\frac{x^2}{2.4^2.6}-\ldots=0.$$

The series would have become divergent still earlier if the equation just written had had an imaginary root with a modulus smaller than 3·340 . . .².

$$\text{Nap. log. dec.} = \frac{2M'\mu'\tau}{MK^2} \times \text{real part of } ma \frac{1+\frac{3.5}{1.8ma}+\frac{1.3.5.7}{1.2(8ma)^2}-\frac{1^2.3.5.7.9}{1.2.3(8ma)^3}+\dots}{1+\frac{1.3}{1.8ma}-\frac{1^2.3.5}{1.2(8ma)^2}+\frac{1^2.3^2.5.7}{1.2.3(8ma)^3}-\dots} \quad . \quad (21)$$

Instead of the latter part of (21), in which, however, the law of either series is manifest, we may use its development according to descending powers of a, which is

$$ma+\frac{3}{2}+\frac{3}{8ma}-\frac{24}{(8ma)^2}+\frac{252}{(8ma)^3}-\frac{3456}{(8ma)^4}+\frac{60768}{(8ma)^5}-\frac{1327104}{(8ma)^6}+\dots \quad . \quad . \quad (22)$$

The expression for the correction for the internal air will be got from the above by changing the sign of ma and of the whole, or, in other words, by changing the signs of the 2nd, 4th, 6th . . . terms in the series in (21) or (22). It will be remembered that ma is $f(\cos 45°+i \sin 45°)$.

Appendix.

(Received November 15th, 1886.)

In the previous experiments the *main* loss of energy arising from the friction of the air may be characterised as being due to the fact that the air is *pushed*. A small portion, however, of the loss is occasioned by the rotation of the cylinders or spheres about their own axes, and in this case the air may be said to be *dragged*. Professor G. G. Stokes has, in the preceding note, deduced formulæ by means of which this last portion of the whole loss of energy can be calculated, and it seemed of interest to determine whether the coefficient of viscosity of air would prove to be the same as before, when the air was *entirely dragged*. This will occur when only one sphere or one cylinder is used, whose axis is made to coincide with the axis of rotation. Accordingly I followed out a suggestion of Professor Stokes in the manner detailed in the following experiments.

Experiment VI.

A paper cylinder was made by wrapping drawing-paper several times round a metal cylinder, which had been turned true throughout its whole length, the different layers being pasted together. When dry, the paper cylinder was removed from its metal core, and its external diameter very carefully gauged by calipers reading to $\frac{1}{1000}$th of an inch at six different places equidistant from each other. It was then gauged at the same distances from the ends, but in directions at right angles to the first. The following were the two sets of gauges :—

Set I. Diameter in inches.	Set II. Diameter in inches.
6·026	6·073
6·083	6·010
6·106	6·051
6·106	6·020
6·090	6·030
6·010	6·006
Mean 6·0701	6·0323

The circumference of the cylinder was next measured by a steel tape at five different equidistant places :—

Circumference in centims.
48·64
48·66
48·60
48·56
48·35
Mean 48·562

Allowing for the thickness of the steel tape, the circumference is 48·485.

From the measurements made with the calipers and tape, the mean diameter of the cylinder was 15·370 and 15·433 centims. respectively, and the total mean 15·4015 centims.

It will be observed that the external diameter is nearly, but not quite, uniform throughout; this no doubt arises from the fact that the paper was not quite uniform in thickness. Inside, as far as could be judged by inserting a straight edge, the bore of the cylinder was perfectly uniform throughout.

The inside diameter was determined by the calipers at the top and bottom, at eight different places in all. It was also determined by gauging the thickness of the walls of the cylinder at the top and bottom by means of a wire gauge, and subtracting twice the thickness from the external diameter as measured by the tape. The internal diameter, measured in the two different ways mentioned above, was exactly the same for both, namely, 14·872 centims. The mean of the internal and external diameters is 15·1395 centims., and the mean radius 7·5698 centims.

The length of the paper cylinder was 60·80 centims., and the mass, allowing for the air displaced, was 543·6 grammes.

The wire was inserted into a hole bored in the centre of one end of a vertical brass rod 2 millims. thick and 15 centims. long, and there soldered: the other extremity of

the rod was soldered into the centre of a horizontal, hollow, brass tube, of length 17·85 centims., of diameter 1·25 centim., and of mass 29·20 grammes.

From the hollow brass bar the paper cylinder was suspended; two holes, whose centres were $2\frac{1}{2}$ centims. from the top, being cut in the walls of the paper cylinder for this purpose.

Great care was taken in arranging the cylinder, so that the axis of rotation might coincide in direction as accurately as possible with the axis of the suspended system. As the paper cylinder did not quite hang truly, it was made to do so by placing small strips of tinfoil, as riders, on the top of the cylinder, and these strips were carefully padded down by hand to the walls of the cylinder. The usual previous precautions having been taken, the logarithmic decrement was determined from seven sets of observations, each involving 100 vibrations, as follows:—

Number of observation.	Logarithmic decrement.
1	·0026307
2	·0025856
3	·0025837
4	·0025810
5	·0025700
6	·0025849
7	·0025550

These observations were consecutive, and the mean of them is ·0025844.

The paper cylinder was now removed, and in its place was substituted a much shorter cylinder, made partly of paper and partly of tinfoil, and having nearly the same mass and mean radius. The dimensions of this cylinder were as carefully measured as those of the longer cylinder, with both steel tape and calipers. The mean of the inside and outside radius was 7·5132 centims., and its real length was 12·80 centims. Since the radius is, however, not quite the same as that of the longer cylinder, we must assume its length to be

$$12{\cdot}80 \times \left(\frac{7{\cdot}5132}{7{\cdot}5698}\right)^3 \text{ centims.},$$

or 12·52 centims. if we are to use it for the purpose mentioned below.

The same pieces of tinfoil as had been used with the long cylinder were used here, and for the same purpose. The logarithmic decrement was then determined by six sets of experiments, each involving three times the number of vibrations employed with the longer cylinder.

Number of observation.	Logarithmic decrement.
1	·0009162
2	·0009015
3	·0008743
4	·0008871
5	·0009019
6	·0008993

These, like the others, are consecutive observations, and the mean of them is ·0008967.

Applying the corrections, mentioned in the paper, for small differences in the vibration-periods, temperature, &c., when the two cylinders were used, we have for the logarithmic decrement due to a cylinder (60·80−12·32) centims. or 48·28 centims. in length the value

·0017029.

It follows, from Professor STOKES's formulæ, that the logarithmic decrement arising from the friction of the air against the inner and outer walls taken together will be

$$\frac{M\mu\tau}{I\rho}\log_{10}e\,(\sqrt{2}f+\sqrt{2}\times 0{\cdot}375f^{-1}-\sqrt{2}\times 0{\cdot}4922f^{-3}+\&c.),$$

f being equal to $\sqrt{\frac{\pi\rho}{\mu\tau}}.a$,

where a is the mean radius of the cylinder, τ the vibration-period, μ the coefficient of viscosity, ρ the density of the air, M the mass of air which would be contained in a cylinder of the same length, and having an internal radius equal to a, and I the moment of inertia.

The values of I and τ were 36966 centimetre-grammes and 3·6038 seconds respectively. The corrected height of the barometer was 29·354 inches, and the temperature 12°·225 C. The value of ρ was calculated, as usual, on the supposition that the air is half saturated with moisture.

The terms $0{\cdot}375f^{-1}$ and $0{\cdot}4922f^{-3}$ are so small that we may calculate them by using an approximate value of μ, and the series converges so rapidly that it is quite unnecessary to include any more terms in it.*

The value of μ, determined from the data given above, was found to be

·00017580.

* Indeed, the third term might have been dispensed with in this case, but not in the next experiment.

Experiment VII.

The copper wire used in the last experiment was about 4½ feet in length and 0·1 centim. in diameter. This was now changed for one of the same length, but of 0·063 centim. diameter, so that the vibration-period became 8·930 seconds. The rest of the arrangements were the same as in Experiment VI. The corrected logarithmic decrement was ·0027040, and the value of μ deduced as above was found to be ·00017902 at a temperature of 13°·100 C.

The mean of the two last experiments is ·00017741 at a temperature of 12°·663 C. This result agrees so well with the mean of those deduced from the previous experiments that it is unnecessary to make any alteration in the formula already given for finding the viscosity at any temperature.

I have entered more into the details of these last experiments, as I think the present method can be more advantageously employed than any of the others. Indeed, by spending sufficient time over the experiments, whereby the errors likely to arise from the somewhat unstable nature of the internal friction of the metal may be more perfectly eliminated, it seems likely that very considerable accuracy can be attained by it.

[Note added Dec. 8th, 1886.—A much greater number of observations were afterwards made with the same cylinders and wires, and resulted as follows:—With the wire used in Experiment VI. the value of μ obtained was ·00017708 at a temperature of 12°·225 C., and with the finer wire of Experiment VII. the value was ·00017783 at a temperature of 13°·075 C. The mean of these values is ·00017746 at 12°·650 C., as compared with ·00017711 at 11°·79 C., the mean of the other five sets of experiments. If we allow for the difference of temperature by using the previously given formula, the agreement between these two means is perfect.]

In conclusion, my warmest thanks are due to Professor Stokes for his valuable suggestions and advice throughout the investigation. To myself the experimental verification of Professor Stokes's formulæ has been a source of great pleasure.

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XX. *The Influence of Stress and Strain on the Physical Properties of Matter.*

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Communicated by Professor W. GRYLLS ADAMS, *M.A., F.R.S.*

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[PLATE 43.]

PART I.—ELASTICITY (continued).

THE INTERNAL FRICTION OF METALS.

Origin and Purpose of the Investigation.

I HAVE already had the honour of presenting to the Royal Society* an account of researches carried on with a view of determining the effects of stress and strain on the elasticity and electrical resistance of metals. Side by side with these researches were conducted others on magnetic induction and thermo-electricity, and a very considerable amount of experimental detail was collected with reference to the two last physical properties of matter. The results of the last-mentioned investigations have, however, now remained unpublished for several years, for it seemed desirable that, previously to publication, certain phenomena should, if this were possible, have light thrown upon them by further experiments on elasticity. More especially was I anxious to examine into the causes of the loss of energy experienced by a wire when vibrating torsionally, for the interesting memoirs of G. WIEDEMANN† and D. E. HUGHES led me to expect that my doing so would cause some insight to be gained regarding the above-mentioned phenomena. The results of these labours, which have now occupied almost the whole of my spare time for the last three years, I offer to the Society in the hope that they may prove as interesting to others as they are to myself.

Researches of THOMSON *and* WIEDEMANN.

Under the title of "The Elasticity and Viscosity of Metals," Sir W. THOMSON published a memoir,‡ the first portion of which deals with the loss of energy of a wire

* 'Phil. Trans.,' Part 1, 1883.
† 'Wiedemann's Annalen,' vol. 6, 1879.
‡ 'Roy. Soc. Proc.,' vol. 14, 1865, p. 289.

when vibrating under the influence of torsional elasticity. It is pointed out (*a*) that, though no change of volume or shape can be produced without dissipation of energy, because of the accompanying change of temperature, estimates founded on the thermodynamic theory of elastic solids suffice to prove that the loss of energy due to this cause is small in comparison with the whole loss of energy which has been observed in many cases of vibration. (*b*) That, as a result of experiments in which a spring was vibrated alternately in air of ordinary pressure and in the exhausted receiver of an air-pump, there is an internal resistance to its motions immensely greater than the resistance of the air. Hence it is concluded that with *solids as with liquids* there exists an internal resistance to change of shape *depending upon the rapidity of the change*. The results of THOMSON's experiments are briefly as follows:—

(*a*.) The loss of energy in a vibration through one range was greater, the greater the velocity;* but the difference between the losses at low and high speeds was much less than it would have been, had the resistance been approximately as the rapidity of the change of shape.

(*b*.) When the weight of the vibrator was increased, whilst its moment of inertia was maintained constant, the viscosity was always at first much increased; but then, day after day, it gradually diminished and became as small in amount as it had been with the lighter weight.

(*c*.) A wire which was kept vibrating nearly all day, from day to day, after several days showed very much more molecular friction than another kept quiescent except during each experiment.†

The investigation was continued with much smaller degrees of maximum angular distortion, to discover, if possible, the law of the molecular friction, and, so far as the irregularities depending upon previous conditions of the elastic substance allowed any simple law to be indicated, it was proved that, as with fluids, the diminution of range per equal numbers of oscillations bore a constant ratio to the diminishing range. But, on the contrary, as with the larger angular distortions, the relation between the law of subsidence in two sets of oscillations having different periods, with the same elastic body in the same circumstances, was not that which would occur if the molecular resistance were simply proportional to the velocity of the change of shape in the different cases. If the molecular friction followed this simple law, the proportionate diminution of range per period would be inversely as the periods.‡ This proportion was not found to hold good, but the loss of energy was, in fact, as it would be if the result were wholly or partially due to the "elastische Nachwirkung"—elastic after-working—as the Germans call it.

* I shall be able to show that this is not the case if we eliminate the effect of the resistance of the air.

† This so-called "fatigue of elasticity" does not occur if the oscillations are kept well within the limits of elasticity.—H. T.

‡ There is a slip in § 36 of 'Encycl. Brit.,' 9th edit., Art. "Elasticity," in which article Sir WILLIAM THOMSON refers to his previously-mentioned memoir; the words "directly as the square roots of the periods" should be "inversely as the periods."

GUSTAV WIEDEMANN, at the end of a very interesting paper on "Torsion,"* discusses the question of the loss of energy experienced by a wire vibrating torsionally. He had by previous statical experiments proved—

(*a*.) That permanent torsion occurs even after the slightest temporary torsion.

(*b*.) That when a wire is repeatedly twisted by a given torsional stress, first one way and then the other, the distance between the permanent positions of equilibrium on each side gradually diminishes to a minimum; the period during which this diminution is taking place is called "the accommodation period."

(*c*.) That when the wire has been "accommodated" by stresses $\pm W_\infty$ repeatedly acting in opposite directions, till the temporary† and permanent torsions $\pm T_\infty$ and $\pm P_\infty$ respectively are constant, if the wire be again twisted by increasing stress in the direction of the last torsion, $T_p - P_p$ increases nearly in proportion to W_p, where T_p, P_p, and W_p represent any temporary torsion, permanent torsion, and torsional stress respectively between 0 and T_∞, P_∞, and W_∞.

WIEDEMANN, from these and other results, feels himself justified in concluding that "the hypothesis according to which the diminution of the amplitude is to be referred to an internal friction which for the entire course of the oscillations is a function of the velocity can no longer be maintained." Neither does he consider that the elastic after-action will alone account for what is observed. The chief cause of the diminution of the amplitude lies rather, according to him, in the *shifting of the position of permanent torsion of the molecules at the end of each oscillation*; what takes place at the alternate oscillations may be well described in his own words as follows:—"In the absolute first position of rest of the wire, before any torsion in its molecules, we will draw axes parallel with the axis of the wire. Let the wire be next 'accommodated' by frequent vibrations to and fro, and at last be temporarily twisted in the positive direction, describing an angle of $+a$, while the molecules may be rotated so that the lower ends of their axes, looked at from the axis of the wire, describe an angle, $+\alpha$ say, to the left. If the wire be slowly brought back into the permanent torsion-position $+b$, the axes of the molecules will retain a portion $+\beta$ of their rotation to the left. If the wire now receive an impulse in the positive direction, which again sends it to $+a$, it will, according to the laws of perfect elasticity, swing back to $+b$. If it now swing beyond this position further to the right, and if the molecules in their rotation had no friction to overcome, it would arrive at the position $-a$, while the axes of the molecules would be rotated to $-\alpha$. Again, with perfect elasticity the wire would go back to the position $-b$, in which the molecules would be rotated $-\beta$, and so forth. The to-and-fro motions of the wire between $\pm a$ and $\pm b$ are perfectly elastic; therefore the performances of work in the swingings backward and forward

* 'Wiedemann's Annalen,' vol. 6, 1879, pp. 485-520; 'Phil. Mag.,' Jan. and Feb., 1880.

† WIEDEMANN employs the term "temporary torsion" so as to include not only the strain which disappears on the removal of the stress, but also the permanent torsion. I, on the contrary, will always denote the torsion which disappears on the removal of the torsional stress as the temporary torsion.

must within these limits completely compensate one another. In fact, however, there results a diminution of the amplitude of oscillation; hence the loss of energy therein can only correspond to the work which is expended in the alteration of the positions of equilibrium, or the rotation of the molecules from $+\beta$ to $-\beta$ which determine it. The diminution of amplitude may therefore be taken as a measure of this work. Since the shifting of the permanent torsion-position is, within certain narrow limits, proportional to the temporary torsion, and therefore the rotation of the molecules therewith is likewise approximately so, the amplitudes must, within those limits, diminish according to the law of a geometrical series."

Again, Professor WIEDEMANN agrees with Sir W. THOMSON that, whilst an increase of load at first increases the loss of energy, this loss, under repeated oscillations, becomes less and less, and finally approaches in magnitude that obtained with the lighter load. WIEDEMANN'S explanation of the matter is that, just as in his own statical experiments the torsional displacements were found to be initially greater with a heavier load than with a lighter one, so also must the decrements attendant upon oscillations be greater in the former case, but, like those, gradually approach the magnitude which is observed with the lighter loading.

Briefly expressed, then, the views of Sir W. THOMSON and Professor WIEDEMANN are as follows :—The former, whilst acknowledging that his experiments do not show that the proportionate loss of energy of a wire when vibrating torsionally is inversely as the period of vibration, which should be the case if the internal friction of a solid resemble that of a fluid, still employs the term *viscosity* with reference to metals, and appears to think that the internal friction of a metal partakes of the nature of the viscosity of a fluid ; the latter is of opinion that the loss of energy is mainly due to a shifting from side to side of the positions of stable equilibrium of the molecules caused by the torsional vibrations of the wire.

I now proceed to give an account of my own researches relating to the internal friction of metals.

Description of Apparatus and Mode of Experimenting.

The wire under examination was firmly clamped at its upper extremity to a stout brass block, which was itself clamped to the end of a strong iron bracket projecting from the outside of the wall at the back of my house, the height of the bracket above the ground being 685·5 centims. A wooden box A A, nearly 600 centims. in length and 12 centims. square inside, protected the vertically suspended wire from currents of air, and passed through the roof of a small wooden house. The box was secured at its upper extremity to the iron bracket, and rested with its base B B upon another box S S (fig. 1), which measured about 40 centims. in length and breadth and about 86 centims. in height. The box S S had in front a glass window W W and a door, and at each of the two sides a door.* All the doors were lined with caoutchouc, so

* These doors are not shown in the figure.

as to fit air-tight, and were only opened for the purpose of adjustment. The top of the long box A A was also closed by a metal cap, and every precaution taken to ensure that the vibrations of the wire should be undisturbed by currents of air. The wire near its lower extremity passed through the centre of an aperture O made in the top of the box S S. The bar V V of the vibrator was clamped to the lower extremity of the wire, and to it were suspended two cylinders of equal mass and dimensions; by sliding the cylinders backwards or forwards on the bar, the moment of inertia of the whole vibrator could be altered without changing its mass. The bar V V was graduated in millimetres, and the cylinders could be clamped at any required equal distances from the axis of the wire; the position of the cylinders could readily be estimated to the tenth of a millimetre. The mode of attaching the cylinders is sufficiently shown in the figure, and it will be noticed that each cylinder is capable of being divided into two equal parts, each part having a mass of 337·1 grms., a length of 30·195 centims., and a diameter of 1·29864 centim. These cylinders, either complete or in halves, were used for nearly all the wires, except those of tin and lead, with which last lighter cylinders and bars were employed. The bar V V consisted in some experiments of a piece of thin, drawn, hollow brass tubing, of which the length was 30·70 centims. and the diameter 1·420 centims. Two brass caps provided with screws, about 8 centims. in length and 2 millims. in diameter, fit one into each end of this hollow bar, and can be easily removed from or placed in it. This bar will be designated as the bar B_2. In other experiments V V consisted of a thin hollow brass tube, closed at both ends, of length 30·46 centims., and, except at the centre, of mean diameter 0·94980 centim. The central portion of the bar had soldered on to it a solid concentric brass ring 1·55 centims. in length and of the same diameter as the bar B_1, so that either bar might be fitted into the brass block K, which was itself clamped to the wire. The bar last described will be designated as the bar B_1; both B_1 and B_2 can just pass through a circular aperture in the block K and be clamped at their centre to K. When the cylinders suspended to V V are used complete they will be designated as the cylinders C_2, and when they are used in halves as the cylinders C_1.

The torsional vibrations of the wire were observed by the usual mirror-and-scale arrangement, which is sufficiently shown in fig. 1, where M is the light mirror reflecting an illuminated circle of light crossed by a vertical, fine, dark line on to a scale placed at the distance of 1 metre from the mirror. The scale was bent into an arc of a circle 1 metre in radius, and was supported on brickwork, which last, together with the box S S, rested on the flagstones which formed the floor of the small wooden house previously mentioned. So perfectly firm and free from disturbing causes of whatever sort was the whole arrangement that the spot of light, when at rest, would remain perfectly so for any required length of time.*

* This, of course, shows that the arrangements made for securing freedom from currents of air answered their purpose satisfactorily.

Three thermometers, one near the top of the long box, one in the middle of the latter, and one in the box S S, served to determine the mean temperature of the wire; these thermometers will be designated as the thermometers T_1, T_2, and T_3 respectively. The thermometer T_3 was divided so as to read to one-tenth of a degree Centigrade, and had been tested recently at Kew; the thermometers T_1 and T_2 were carefully compared by myself with T_3 degree by degree, and were graduated in degrees Centigrade.

A delicate aneroid barometer, placed on a level with the middle of the box S S, served to determine the pressure of the air in the box; the aneroid had been recently compared with a standard mercury barometer, and was graduated to one-hundredth of an inch. Readings of the three thermometers T_1, T_2, and T_3, and of the aneroid, were taken at regular intervals during the experiments.

When all the adjustments had been completed, the wire, whether in the annealed or hard-drawn condition, was allowed to rest for a period of time varying, with the nature of the metal, from one day to several weeks, until there was no shifting of the position of torsional equilibrium and no appreciable "running down" of the wire either before or after vibration. It was necessary to adopt this precaution inasmuch as, though great care had been used in adjusting the wires to avoid imparting permanent or rather sub-permanent torsion, such was frequently found to exist, and manifested itself by a gradual shifting with time of the position of torsional equilibrium. Now, when a wire is either possessed of sub-permanent torsion or is "running down," or even on the point of doing so, the loss of energy which occurs during torsional vibration is appreciably greater than when there are no such things. Even when there may be no apparent permanent torsion, or "running down," it is necessary to allow the wire to rest for some time, and to cause it to undergo frequent vibrations before the internal friction assumes constancy.

When it was necessary to excite vibrations a side-door of the box S S was opened, and slight impulses were given by means of a worsted thread to the bar V V until the amplitude of the oscillations was of the required magnitude; the door was then shut, and another door* at the bottom of the long box A A was opened, so that any accidental pendulous oscillations might be checked: finally this door was closed, and when the amplitude of the oscillations had become reduced to the requisite extent the observer commenced to write down the extreme limits of several consecutive vibrations, as shown by the numbers on the scale. At intervals the time of transit of the light across the middle point of the scale was noted. When the amplitudes decreased rapidly the observations were continued throughout the experiment; but when the decrement was small the observer frequently left the room till the amplitude was sufficiently reduced, when he again noted down the extreme limits of the same number of consecutive vibrations as before, and so on. In all cases the experi-

* Not shown in the figure.

ments were repeated again and again until the logarithmic decrement became sensibly constant in the different experiments.*

In reducing the observations, the following plan was adopted in those cases where the decrement of the amplitude was comparatively slight. Suppose that a_1, b_1; a_2, b_2; a_3, b_3, and a_4 are seven consecutive readings,† the six corresponding arcs from rest to rest will be a_1+b_1, b_1+a_2, a_2+b_2, b_2+a_3, a_3+b_3, b_3+a_4. The means of a_1+b_1, b_3+a_4; b_1+a_2, a_3+b_3; a_2+b_2, b_2+a_3, were written down, and if these agreed well with each other, which was almost invariably the case, the logarithm of the total mean was taken. Now, say that n single vibrations have taken place between the end of this and the beginning of the next seven consecutive readings, the difference of the logarithms of the first and second total means will, when divided by $n+6$, give the mean logarithmic decrement for a single vibration. When the decrements of amplitude were comparatively large, the following mode of reduction was used :— Suppose a_1, b_1; a_2, b_2; a_3, b_3, and a_4 to be, as before, seven consecutive readings, so that the arcs from rest to rest are a_1+b_1, b_1+a_2, a_2+b_2, b_2+a_3, a_3+b_3, and b_3+a_4: the difference of the logarithms of a_1+b_1 and b_2+a_3; of b_1+a_2 and a_3+b_3; of a_2+b_2 and b_3+a_4, were put down, and, if they agreed well, the mean of these differences, divided by three, was taken as the mean logarithmic decrement of a single vibration. The following experiments will furnish illustrations of the methods pursued in the two cases, and will also give a fair idea of the accuracy to be attained.

* This is very necessary, as sometimes, from slight fluctuations of temperature, the internal friction is very variable. I have not unfrequently had to go on experimenting for a couple of hours or more before the logarithmic decrement became constant.

† In the earlier experiments seven readings were taken; in the later ones eleven readings.

Experiment I.

A WIRE of pure copper, hard-drawn, 602·5 centims. in length and ·009383 square centim. in section, allowed to rest for 24 hours after adjustment. Time of a single vibration, 13·930 seconds.

Number of arc.	Observed arc in scale-divisions.	Pairs of arcs between which means are taken.	Mean of pairs.	Total mean.	Probable error of total mean.	Logarithmic decrement for one vibration.	Departure of logarithmic decrement for one vibration from the mean.
1	281·9						
2	280·3	1—6	279·60				
3	280·4						
4	279·8	2—5	279·10	279·60	·06		
5	277·9						
6	277·3	3—4	280·10				
71	233·0						
72	232·5	71—76	231·75				
73	232·0						
74	231·0	72—75	231·70	231·65	·04	·001167	·000000
75	230·9						
76	230·5	73—74	231·50				
141	193·8						
142	192·3	141—146	192·30				
143	191·8						
144	191·8	142—145	192·00	192·03	·09	·001164	−·000003
145	191·7						
146	190·8	143—144	191·80				
211	159·6						
212	159·6	211—216	159·15				
213	159·6						
214	158·7	212—215	159·10	159·14	·01	·001166	−·000001
215	158·6						
216	158·7	213—214	159·15				
421	91·0						
422	90·5	421—426	90·50				
423	90·3						
424	90·2	422—425	90·35	90·36	·06	·001170	+·000003
425	90·2						
426	90·0	423—424	90·25				
Mean	..	..	..	..	..	·001167	..

Experiment II.

A TIN wire, as pure as could be obtained by the ordinary process of distillation, 607·3 centims. in length and ·04972 centim. in diameter, tested about two months after suspension, having been in the meantime vibrated almost every day. Time of a single vibration, 3·898 seconds.

Number of arc.	Observed arc in scale-divisions.	Pairs between which the differences of logarithms are taken.	Difference of logarithms.	Logarithmic decrement for three vibrations.*
1	170·2			
		1—4	·06926	
2	160·2			
3	150·1	2—5	·07396	·07190
4	145·1			
5	135·0	3—6	·07246	
6	127·0			
7	123·1	7—10	·05550	
8	116·0			
9	112·1	8—11	·06016	·06392
10	108·0			
11	101·0	9—12	·07611	
12	94·1			
13	90·0	13—16	·06779	
14	85·1			
15	81·0	14—17	·06613	·06578
16	77·0			
17	73·0	15—18	·06343	
18	70·0			
19	68·0	19—22	·06907	
20	67·0			
21	64·0	20—23	·09375	·08160
22	58·0			
23	54·0	21—24	·08197	
24	53·0			
25	51·0	25—28	·04481	
26	49·0			
27	47·0	26—29	·04675	·05387
28	46·0			
29	44·0	27—30	·07004	
30	40·0			
31	38·0	31—34	·06127	
32	37·0			
33	35·0	32—35	·07684	·06835
34	33·0			
35	31·0	33—36	·06695	
36	30·0			
Mean	...	..	..	·06757

* The great variations which occur in the numbers in this column I afterwards found to be entirely due to the fact that in this case the period of torsional vibration was nearly synchronous with the period of pendulous vibration. (See note on pp. 809, 810.)

Remarks on Experiments I. and II.

It is worth while to pause here and to consider what we can glean by a careful examination of these two experiments, because they are typical of most of the others, whose number now can be reckoned by hundreds. I had in my mind rather a strong conviction that there was an intimate relationship between the loss of energy from internal molecular friction and the electrical resistance of a metal, and have already observed to that effect.* Accordingly, regarding this apparent relationship as a matter of considerable importance, I spared no pains in attempting to get accurate values for the coefficients of viscosity of metals, such as have been obtained by MAXWELL and others in the case of fluids. All the wires mentioned in this paper have been examined over and over again; nearly all have been set up a second time, and most of them a third time, and, whilst in no case has a wire been under examination for less than a week, in some instances the experiments have extended over several weeks. As the work progressed, the conviction became more and more forced upon me that I was dealing with something very distinctly different from what is met with in fluid friction. In Experiment I. we see apparently that one of the laws of fluid friction is *exactly* carried out, for according to the seventh column, in which is given the logarithmic decrement for one vibration, the diminution of range per equal numbers of oscillations bears a constant ratio to the diminishing range, the departures from the mean value of the logarithmic decrements being in no case greater than ·3 per cent. A closer scrutiny, however, shows us that this is only the case because, in calculating each of the logarithmic decrements, the mean of a great number of oscillations is taken.† If we look into each set of six arcs, we may notice that in some instances two consecutive arcs will show no diminution whatever, whilst in others the proportionate diminution is much more than the average. These are not the results of errors of observation, because the differences are appreciably greater than any which could be accounted for in this way, and when we turn to Experiment II. the phenomenon is more marked; we have most indubitable evidence of a continued rise and fall of the logarithmic decrement. This rise and fall would probably escape notice, were we able to take the mean of as many oscillations as were taken with copper; moreover, if we commence a fresh experiment altogether, we find a total mean logarithmic decrement agreeing, within the limits of errors of observation, with the previous one.

[Note added Sept. 18, 1886.—I have since discovered the cause of this rise and fall: it is due to the fact that it is impossible to ensure that the axis of rotation always passes *exactly* through the centre of mass. Of course, if the wire were *perfectly* flexible, and we could *entirely* avoid pendulous vibrations when we started

* 'Phil. Trans.,' Part 1, 1883, p. 168.

† It will be observed that in Experiment I. the mean logarithmic decrement for 70 oscillations is taken in all the series except the last, where we have the mean for 210 oscillations given.

the torsional vibrations, this would not be the case; but, as a matter of fact, even if we could *start* the wire without any pendulous motion, the axis of the wire never *exactly* passes through the centre of mass, and, as a consequence, when the wire vibrates torsionally, so-called "centrifugal force" causes torsional motion to be partially converted into pendulous motion, or *vice versâ*.

When, as in Experiment I., the period of pendulous vibration does not approximately synchronise with the period of torsional vibration, the phenomenon is not very pronounced; but when, as in Experiment II., the two periods approach to synchronism, it is very apparent. Indeed, when the two periods approach sufficiently to synchronism, and the damping is not great, the phenomenon may become so marked that, after a time, almost the whole of the torsional motion becomes converted into pendulous motion, and presently reconverted again, so that the torsional vibrations, having first diminished to nearly zero, begin to increase again until nearly the initial amplitude has been reached, when once more the torsional motion begins to be converted into pendulous motion, and so on.* An interesting fact in connexion with this phenomenon is that, if we allow the wire to rest, with oscillations at intervals, the rise and fall becomes much less pronounced, in consequence, probably, of the wire gradually allowing its axis to pass more nearly through the centre of mass, but we can easily reproduce it in all its intensity by jarring the wire,† or even sometimes by change of temperature.

Unless there is nearly synchronism between the pendulous and torsional vibrations, there is little or no *appreciable* effect on the value of the logarithmic decrement, and, if we reckon from maximum to maximum, I have found that the logarithmic decrement is *sensibly constant, whatever the amplitude*, provided the latter does not exceed a certain limit. Similarly there will be little error introduced in our estimation of the internal friction of the metal from the mean logarithmic decrement deduced from a number of trials, which number was in my experiments very considerable.

When, however, the synchronism between the two periods is sufficient, there may be a very serious interference with the *mean* logarithmic decrement, and even a sensible interference with the torsional vibration-period; and I hold it as a matter of some considerable importance that, in all experiments connected with the damping, or even, when great accuracy is required, with the vibration-period of torsionally vibrating bodies, this point should be well looked to.‡]

* I had the pleasure of showing this phenomenon to the Physical Society, June 26, 1886.

† A very slight shock sometimes will effect this.

‡ In the case of a magnet suspended with its axis horizontal, the axis of rotation will *of necessity* not pass through the centre of mass anywhere except at the magnetic equator.

Formula for determining the Loss of Energy experienced by a Metal Wire vibrating under the Influence of Torsional Elasticity and encountering a Resistance proportional to the Velocity.

The diminution of amplitude of a wire vibrating as in the present experiments is due to three causes:—

1. Energy imparted to the support to which the upper extremity of the wire is attached, or to the parts of the vibrator which may be capable of independent motion.*
2. Energy imparted to the air surrounding the wire and vibrator.
3. Internal molecular friction.

Of these causes, 1 may be dismissed at once as not affecting the results within the limits of errors of observation, and a means of eliminating the effect produced by the resistance of the air will be shown presently. I propose, therefore, in dealing with the problem before us, to confine myself to the consideration of Cause 3, on the assumption that the internal friction varies as the first power of the velocity.

Let F be the moment of the couple necessary to twist the wire through one radian; $2k$ the resistance due to internal friction, when the wire is twisting or untwisting with unit angular velocity; τ the period of vibration from rest to rest; and M the moment of inertia: then it can readily be shown that very approximately

$$\tau = \pi \sqrt{\frac{M}{F}\left(1+\frac{k^2}{2FM}\right)}, \quad \ldots \ldots \ldots \quad (1)$$

and that the proportionate diminution of amplitude is

$$\frac{k\pi^2}{\tau F}. \quad \ldots \ldots \ldots \quad (2)$$

From (2) it follows that the logarithmic decrement should be independent of the amplitude of vibration, and should vary inversely as the period of vibration.

Discussion of THOMSON'S *Views concerning the internal Friction of Metals.*

According to the mathematical formula just given, the logarithmic decrement should, with the same vibration-period, be a constant for all amplitudes if the friction between the molecules of a metal resemble fluid friction. This we have seen is the case.† When I turned, however, to examine the effect of changing the moment of inertia of

* If, for example, the cylinders attached to the bar of the vibrator be capable of a motion independent of the bar, as they would be if suspended by fine wires or threads.

† Except, of course, in such cases as that of Experiment II., where the synchronism between torsional and pendulous vibration-periods causes the logarithmic decrement to vary; or, rather, prevents there being any such thing as a logarithmic decrement.

the vibrator without at the same time altering the mass in order to prove the second law deduced above, I failed entirely, and, though I adopted numerous changes of the mass of the vibrator, of the amplitude of vibration, of the vibration-period, and of the nature of the metal, in no case was I able to get a step further towards proving the above-mentioned law. On the contrary, when I had eliminated the effect of the resistance of the air in the manner now to be described, I found the logarithmic decrement nearly independent of the vibration-period in some cases and increasing with it in others.

Mode of eliminating the Effect of the Resistance of the Air.

In a paper quite recently presented to the Royal Society, I have shown that the following formulæ for finding the logarithmic decrement due to the resistance of the air may be deduced from Prof. G. G. Stokes's valuable memoir "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums":—*

For a cylinder vibrating horizontally

$$\log_{10}\text{dec.} = \frac{\pi^3 k' \rho a^4 l^2}{24 I} \log_{10} e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ρ is the density of the air, a the radius of the cylinder, l the length of the cylinder, I the moment of inertia of the vibrator, and k' is a constant, provided the vibration-period, the diameter of the cylinder, and the nature of the fluid remain unchanged.

For a pair of cylinders of equal mass and dimensions, suspended vertically at equal distances from the axis of the wire,

$$\log_{10}\text{dec.} = \frac{\pi^3 \rho \beta^3 l d^2 k'}{16 I} \log_{10} e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where β is the diameter and l the length of each cylinder, and d is the distance between the two axes of the cylinders.

We further have from Prof. Stokes's paper the formula

$$\mathfrak{m} = \frac{\beta}{4} \sqrt{\frac{\pi \rho}{\tau . \mu}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where τ is the vibration-period and μ is the coefficient of viscosity of air. When $\mathfrak{m}$ has been determined, k' can also be determined by the aid of the Table given on p. 46 of Prof. Stokes's paper.

By several very carefully executed series of experiments with vibrating cylinders

* (1850) 'Camb. Phil. Soc. Trans.,' vol. 9 No. X.

and spheres I have proved, in the paper just alluded to, that the coefficient of viscosity of ordinary air is given by the formula

$$\mu = \cdot00017155(1+0\cdot002751t-\cdot00000034t^2),^* \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where t is the temperature of the air in degrees Centigrade and the units C.G.S. units. From these four formulæ we can readily determine the logarithmic decrement due to the resistance of the air on the horizontal bar V V and on the two vertically suspended cylinders. The mode of proceeding is as follows:—First, from equation (6) is found the value of μ, which enables m to be obtained from equation (5). Then we obtain by interpolation, from the Table given on p. 46 of Prof. Stokes's paper, the value of k', and, by substituting this last in equations (3) or (4), according as we are determining the effect of the resistance of the air on the bar V V or on the two vertical cylinders, the logarithmic decrement required. The vertical cylinders are connected to the bar V V by the two suspenders x, x, the cylindrical portions of which are affected by the air to an extent which must be calculated independently of the long cylinders, which they connect with V V.† The effect of the resistance of the air on the other portions of the suspenders was determined experimentally by using the bar B_1 for V V, and observing the logarithmic decrement, first when the bar was vibrating with the suspenders attached, and afterwards with the suspenders removed, but with brass cylinders introduced into the hollow bar V V, of equal mass with the suspenders, and so adjusted that the moment of inertia remained unaltered.‡ By this means it was ascertained that the portions of the suspenders which were not cylindrical were equivalent to a certain length of the long vertical cylinders. This equivalent length would vary so slightly with the vibration-period that we may without sensible error consider it a constant; accordingly, in order to allow for this portion of the suspenders, the length of each of the long vertical cylinders was estimated at so much more.

* For the coefficients of t and t^2 in this formula I have taken as my authority Holman ('Phil. Mag.,' vol. 21, 1886, p. 199), as not only do Prof. Holman's investigations seem to have been very carefully conducted, and to agree as regards the effect of change of temperature on the coefficient of viscosity with those of Meyer, Puluj, and others, but also they are more in accordance with my own in this respect than are those of Maxwell, who found the coefficient to vary as the absolute temperature.

Meyer and Puluj have obtained values for the absolute coefficient of viscosity of air which differ from my own, but I prefer to use my own results, as these were obtained by experimenting with cylinders and bars of somewhat similar dimensions to those used here, and also under similar conditions to those holding here. I believe the equations given above to be capable of expressing the logarithmic decrement due to the resistance of the air with certainly a less error than 1 per cent.

† Because k' varies for cylinders of different diameters.

‡ For this purpose the caps were removed from the ends of B_1, and the brass cylinders, by means of companion-screws cut along their axes, were adjusted at the required distance on the screws attached to the caps; the caps were then replaced.

Equation (4) does not take into account the effect of the air due to the rotation of the cylinders about their axes; the correction required for this is, however, very slight.* When necessary, this correction was made by means of the following formula:†

$$\lambda_a = \frac{2M'\mu\tau}{I\rho} P \log_{10} e, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where λ_a is the natural logarithmic decrement due to the rotation of *both* cylinders about their axes, M′ is the mass of fluid displaced by *each* cylinder, and

$$P = \frac{f}{\sqrt{2}} + 1{\cdot}5 + \frac{0{\cdot}375}{\sqrt{2}f} - \frac{0{\cdot}4922}{\sqrt{2}f^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

f being put, by way of abbreviation, for $\sqrt{\frac{\pi\rho}{\mu\tau}}.a$, where a is the radius of the cylinder.

The Extent to which the Internal Friction of Metals depends upon the Vibration-Period.

We are now in a position to examine how far the loss of energy, resulting from internal friction, depends upon the vibration-period of the wire. I propose, therefore, to give, in illustration, the results of four sets of experiments on piano-steel, on copper, on zinc, and on tin.

Experiment III.

UNANNEALED Piano-Steel 0·0824 centim. in diameter and 602 centims. in length. Bar B_1 and Cylinders C_2.

Vibration-period in seconds.	Observed logarithmic decrement for one vibration.	Logarithmic decrement due to the resistance of the air.‡	Logarithmic decrement due to internal friction.	Temperature in degrees Centigrade.
3·9337	·0010497	·0006950	·0003547	10·00
5·5210	·0009563	·0006459	·0003104	10·64
8·1750	·0009633	·0005918	·0003715	10·12
12·5130	·0010710	·0007077	·0003633	9·70
16·9720	·0012052	·0008502	·0003550	11·38
Mean	..	..	·0003510	10·37

* Would not in any case exceed ¼ per cent. of the whole.

† Kindly furnished by Prof. G. G. STOKES.

‡ It will be observed that the numbers in this column at first decrease with the time of swing and then increase. This arises from the fact that the time of swing, as well as the dimensions of the vibrator, come into the equations used for determining the logarithmic decrement due to the resistance of the air.

Experiment IV.

ANNEALED Copper Wire ·1622 centim. in diameter and 602 centims. in length. Bar B_3 and Cylinders C_3.

Vibration-period in seconds.	Observed logarithmic decrement for one vibration.	Logarithmic decrement due to the resistance of the air.	Logarithmic decrement due to internal molecular friction.	Temperature in degrees Centigrade.
1·7180	·0006597	·0003833	·0002764	17·13
2·2984	·0005971	·0003644	·0002327	15·20
3·1190	·0006136	·0003481	·0002655	15·17
4·7230	·0006552	·0003776	·0002776	14·60
6·3010	·0006835	·0004222	·0002613	17·90
7·9335	·0007347	·0004752	·0002595	16·20
Mean	..	..	·0002622	16·03

Experiment V.

UNANNEALED Zinc Wire ·07913 centim. in diameter and 602 centims. in length. Bar B_1 and Cylinders C_1 used.

Vibration-period in seconds.	Observed logarithmic decrement for one vibration.	Logarithmic decrement due to the resistance of the air.	Logarithmic decrement due to internal molecular friction.	Temperature in degrees Centigrade.*
5·875	·02116	·00113	·02003	19·86
7·222	·02243	·00106	·02137	19·86
15·313	·02632	·00094	·02538	19·86

Experiment VI.

A wire of tin, unannealed, 607·3 centims. in length and ·09944 centim. in diameter. The vibrator consisted of a solid cylindrical bar 23·3 centims. in length, 0·320 centim. in diameter, and 16·282 grms. in mass; suspended from the bar were two brass cylinders, each 1·127 centim. in diameter, 1·85 centim. in height, and 16·006 grms. in mass. The total mass of the vibrator was 49 grms.

* The temperatures varied from 18°·34 C. to 21°·37 C., but data were at hand for correcting the value of the logarithmic decrement when the temperature differed from the average, 19°·86 C. A similar remark applies to the temperatures 10°·25 C. and 3°·40 C. of the next experiment.

Vibration-period in seconds.	Observed logarithmic decrement for one vibration.	Logarithmic decrement due to the resistance of the air.	Logarithmic decrement due to internal molecular friction.	Temperature in degrees Centigrade.
1·579	·020236	·000913	·019323	10·25
2·362	·022346	·000831	·021515	10·25
3·898	·024343	·000912	·023431	10·25
The brass cylinders replaced by lead cylinders, so that the total mass of the vibrator was 168 grms.				
1·913	·016784	·000833	·017951	3·40
4·366	·023664	·000535	·023129	3·40
7·989	·024885	·000616	·024239	3·40
1·913	·020533	·000833	·019700	10·25
4·366	·026035	·000535	·025500	10·25
7·989	·027316	·000616	·026700	10·25

Remarks on Experiments III.–VI. inclusive.

Let us now see what we can gather from the data given in these experiments. With tin the logarithmic decrement decidedly increases with the vibration-period, but not in the same proportion as the latter, and may be expressed by the following formula:—

$$\lambda_\tau = a + b\tau + c\tau^2,$$

where τ is the vibration-period and λ_τ the logarithmic decrement, whilst a, b, and c are constants, the last being a negative quantity. Thus, when the mass of the vibrator in Experiment VI. was 49 grms., and the temperature 10°·25 C., the formula became

$$\lambda_\tau = ·012407 + ·005437\tau - ·0006693\tau^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

whilst for the same temperature, and with the mass of the vibrator equal to 168 grms., it was

$$\lambda_\tau = ·012381 + ·004466\tau - ·0003347\tau^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

The logarithmic decrement would thus seem to be capable of being divided into two parts, the one being independent of the load and of the vibration-period, and the other dependent upon the load and the vibration-period.

When the mass of the vibrator was 168 grms., and the temperature 3°·40 C., the formula became

$$\lambda_\tau = ·011470 + ·0039473\tau - ·0002925\tau^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Now, if we allow for the difference of temperature, the value of a given in (11) will be found to accord almost exactly with the values of a given in (9) and (10).

The effect of change of temperature is very considerable, for a rise of 6°·85 C. caused an increase of 10·05 per cent. in the internal friction, and proved to be nearly the same for the three vibration-periods: 1·913 second, 4·366 seconds, and 7·989 seconds.

With zinc the logarithmic decrement also increases with the vibration-period, but not to the same extent as with tin, the formula in this case becoming

$$\lambda_\tau = \cdot 01019 + \cdot 0022274\,\tau - \cdot 0000941\tau^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

With the metals possessing less internal friction, steel and annealed copper, the variation of the logarithmic decrement with the period of vibration is so slight as to be masked by errors due to small fluctuations of temperature* and other causes; and, had not a great many other observations than those here recorded proved that the internal friction does slightly increase with the vibration-period, it would have almost appeared not to do so. If, however, we compare in Experiments III. and IV. the *mean* value of the first half of the logarithmic decrements with that of the second half, we may trace indications of the truth of the above statement. To sum up, then, these and many other experiments of a like kind have nearly satisfied me that, with *all* metals, the logarithmic decrement does increase with the period of vibration, but that, the less internal friction the metal possesses, the more nearly is the former independent of the latter.

Effect of Fluctuations of Temperature on the Internal Friction.

Perhaps one of the most remarkable features presented by a torsionally vibrating wire is the ease with which the molecules are disturbed by changes of temperature from those positions of stable equilibrium into which they will, after a period of time, whose length depends upon the nature of the metal, naturally creep. It is truly astonishing to observe the profound change in the internal molecular friction of such an elastic metal as unannealed piano-steel† which is wrought by a sudden rise or fall of one or two degrees of temperature. I have explained that, though the wires were shielded by a long box, this box was itself exposed, for the greater part of its length, to the external air. Now, the first wire examined was one of unannealed piano-steel, and some hundreds of trials were made with it during a period of three months, and

* See what follows shortly.

† I do not know that unannealed piano-steel is more remarkable than other metals in the respect mentioned.—H. T.

at various hours of the day. Sometimes the proportionate diminution of amplitude would seem quite as constant for different ranges of amplitude as could be expected, but at others there would be variations far exceeding anything which could be attributed to errors of observation. For some weeks I was quite at a loss to account for these variations, but at length was successful. Luckily, I had determined to make frequent observations of all three thermometers, and, whilst examining the contents of my note-book, it occurred to me that the greatest variations took place on those days and hours at which the variations of temperature were greatest. I also noticed that *the logarithmic decrement became larger, whether the change of temperature was of the nature of a rise or a fall.* At first this explanation seemed almost incredible, for the variations of temperature would sometimes be only 1° C., or even less, and yet at the same time the value of the logarithmic decrement would be increased 30, 40, or even 100 per cent. My suspicions having been thus roused, I found it comparatively easy to verify them, and among several other experiments, made with the same object, tried the following:—

Experiment VII.

The same piano-steel wire as in Experiment III. The sun was shining rather faintly, and at intervals on that side of the house on which the wire was suspended, but before the end of the experiment passed over to the other side.

Civil time.	Temperature of the wire in degrees Centigrade.	Alteration of temperature in degrees per hour.	Time of 100 vibrations in seconds.	Logarithmic decrement due to internal friction.
h. m. s.			m. s.	
10 10 42 A.M.	10·33	..	..	..
10 40 57	9·33	1·98−	30+13·05	·000636
11 11 10	10·00	1·32+	+14·35	·000635
11 41 24	11·00	1·98+	+12·80	·000753
12 18 39 P.M.	10·23	..	..	vibrator restarted
12 48 52	10·43	0·396+	+12·90	·000428
1 49 15	10·23	0·198−	+11·08	·000356
2 49 37	10·10	0·129−	+10·93	·000501
4 50 21	9·17	0·461−	+10·61	·000702

From 7 P.M. to 9 P.M., when the experiment terminated, the logarithmic decrement and the temperature became nearly constant, the value of the former being ·000356.

The changes of temperature, as shown in the second column, are very small, but we see what a large effect they have on the internal friction, which in one case became twice as great as when under a fairly steady temperature. The effects of fluctuations of temperature on the internal friction resemble those of mechanical jarrings, such as would be produced when the mass of the vibrator is altered, or even when the

cylinders are shifted along the bar without any alteration of mass;* they also, as I shall show in a future paper, can be imitated by means of magnetic stress.

Attention should also be directed to the fourth column of the experiment, where it will be seen that not only is the loss of energy increased by change of temperature, but so also is the vibration-period; the latter is, however, so much less altered than the former that it requires careful observation to detect any such similarity between them.

In consequence of the above-mentioned effect of fluctuations of temperature on the internal friction, it was necessary to choose a day, or time of day, when the temperature was fairly steady, and all the results which will be recorded in the following portion of this memoir were obtained when this was the case, for with all the metals which I have examined the phenomenon is more or less pronounced.

Since it is impossible, with the arrangements which I have described, to obtain a *perfectly* steady temperature, it seems probable that most of the values of the logarithmic decrements previously recorded are slightly in excess of what they would have been, could we have secured greater uniformity in this respect.

The question next arises: will any change of temperature, *however small*, introduce increase of internal friction? I think not; for not only in some instances have I obtained values for the logarithmic decrements, which were as constant as the limits of errors of observation would allow, and yet where during the experiments the temperature did change several times, and in an irregular manner, though by small amounts, but I shall in another paper bring forward instances where the molecules were slightly twisted backwards and forwards by small magnetic stress without appreciably affecting the value of the logarithmic decrement.

The Effect of Loading on the Internal Friction.

We have seen that according to both Sir WILLIAM THOMSON and Professor G. WIEDEMANN, when the load on the wire is increased, there is at first an increase in the internal friction, but this increase diminishes with time, so that the logarithmic decrement tends to become the same as with the lighter load. Neither of the above observers has, however, attempted to ascertain whether or not the internal friction is *entirely* independent of the load, provided sufficient time be allowed, and sufficient account be taken of the effect of the resistance of the air in diminishing the amplitude of vibration. The following arrangements were therefore made for the purpose of settling this point by experiment:—

The brass block, into which was clamped the bar V V of the vibrator (fig. 2),

* This is one of the reasons why the logarithmic decrement is at first increased when the mass of the vibrator is decreased or increased, even though the change of mass produce no perceptible permanent alteration whatever in the length of the wire, and why it is prudent to allow the wire to oscillate for some time after shifting the cylinders, as in Experiments III., IV., V.

terminated at its lower extremity S in a ring, through which a pin *pp* could just pass. The pin *pp* served to connect the brass box A with the bar of the vibrator by passing through two rings *r*, *r*, attached to the top of A, and placed one on each side of the ring at S. In this way the box A was well secured to V V, and without the least chance of any shifting taking place, as the wire vibrated torsionally. Two smaller boxes B, B, were suspended from V V, and could be clamped, as shown in the figure, to any part of it. All three boxes were provided with hinged doors * of the same height as the boxes, so as to admit of the introduction of two sets of cylinders C and L, the one set of brass and the other of lead; these doors, after the cylinders had been arranged in the boxes, were kept shut by means of springs and clasps. The mass of each of the brass cylinders C was 100 grms., and that of each of the lead cylinders L 2000 grms. The masses of each of the cylinders in the two sets were adjusted by myself, so as to agree with some standard masses within 1 part in 10,000. Each of the cylinders L had been cast in the same mould, and the dimensions had been carefully measured by means of calipers reading to $\frac{1}{1000}$th inch. The moment of inertia of each cylinder was determined from the dimensions and mass, and afterwards by certain indirect methods, which last, when suitable precautions were taken, gave results in good accordance with those which had been obtained by the direct method. Moreover, the moments of inertia of all the lead cylinders were found to be very nearly equal to each other. The mode of proceeding was as follows:—Eight of the lead cylinders L were placed in the box A, and on the top of these was put a wooden cylinder W, fitting rather tightly in A. On the top of W, and fitting into a circular cavity let into the centre of it, were placed sixteen of the brass cylinders C, arranged in sets of four each, each set having its four constituents bound together by an elastic band *d*. The wooden cylinder W served not only to prevent any rocking of the lead cylinders as the wire vibrated, but also to enable the experimenter to readily adjust the brass cylinders C in their proper positions in the box A.

When all the necessary adjustments had been carefully completed, the whole was allowed to remain for five days at rest, except that the wire was caused to vibrate several hundred times on each day; finally, the logarithmic decrement and the vibration-period were determined from a very large number of oscillations. As soon as these had been determined in a sufficiently satisfactory manner, two of the lead cylinders L were removed from A, and four of the little brass cylinders C were transferred from A to the boxes B, B, two being placed in each box. The boxes B, B, had been at the outset so placed on V V that the above transference would exactly make up for the loss of the moment of inertia which would follow from the removal of the two lead cylinders. After this second adjustment and a rest of twenty-four hours, followed by a large number of preliminary vibrations, the logarithmic decrement and the vibration-period were again determined. Again, two more lead cylinders were

* Not shown in the figure.

removed from A and four more transferred from A to B,* and so on, until at length all the lead cylinders had been removed from A and all the brass cylinders transferred from A to the boxes B, B. The next experiments show the results obtained by treating steel and copper wires in this manner.

Experiment VIII.

Unannealed piano steel, 0·0824 centim. in diameter and 602 centims. in length. Mean vibration-period, 21·2825 seconds. The temperature ranged from 6° C. to 7° C.

Load on the wire in kilos.	Logarithmic decrement due to internal friction.
4·635	·0003633
8·635	·0003381
12·635	·0003479
16·635	·0003658
20·635	·0003547

Experiment IX.

Annealed copper wire, 0·1622 centim. in diameter and 602 centims. in length. Mean vibration-period, 7·340 seconds. The temperature ranged from 13° C. to 20° C.

Load on the wire in kilos.	Logarithmic decrement due to internal friction.
4·635	·0002653
8·635	·0002239
12·635	·0002610
16·635	·0002616
20·635	·0002991

The variations, which occur in the values of the logarithmic decrements for different loads, are to be attributed to differences in the rate at which the temperature varied on different days. Had a greater number of observations been made with each of the loads on *different* days, the mean values of the logarithmic decrements would have varied still less. We may say, then, that for comparatively large loads the internal friction is independent of the load, whether the wire be annealed or hard-drawn.

We will now turn to consider the effect of comparatively light loading on the internal friction.

* The figure shows the arrangements of the cylinders after this last operation.

Experiment X.

The piano-steel wire used in Experiment VII. was, after the removal of the boxes, tested with suspended cylinders of different masses. The temperature ranged from 10° C. to 15° C., and the vibration-period from 3·934 seconds to 4·608 seconds.

Total mass of the vibrator in grammes.	Logarithmic decrement due to internal friction.
164	·0003602
838	·0003763
1512	·0003547

Experiment XI.

The annealed copper wire of Experiment IX., after having been tested in the manner described in this experiment, was tried with much lighter loads. The temperature ranged from 17°·9 C. to 18° C., and the vibration-period from 6·301 seconds to 7·363 seconds.

Total mass of the vibrator in grammes.	Logarithmic decrement due to internal friction.
1570	·0002756
4635	·0002653

Experiments X. and XI. teach us that both for hard-drawn and annealed wires the internal friction is as independent of the load for light loads as we have seen it to be for heavy ones. We must, however, in such experiments as these be careful to begin with the heaviest load first, as loading tends more or less to produce a diminution of the internal friction, which is partly permanent and partly sub-permanent, provided no sensible permanent elongation results from the loading. Care should also be taken, in changing from the heavier load to the lighter one, to avoid shocks. If these precautions be not taken, the logarithmic decrement will probably be found less for the heavier load than the lighter one.

The Effect of Loading on the Torsional Elasticity.

It is, I believe, generally assumed that the torsional elasticity of a wire is independent of the load, but apparently no experimental evidence has been brought forward in support of this assumption, at any rate as far as large stresses are

concerned.* We might, perhaps, be inclined to infer that, because the internal friction is independent of the load, so also will be the torsional elasticity, but my own experience had taught me that it was by no means safe to do so. It is true that Experiments VIII. and IX. had shown that the torsional elasticity must be nearly independent of the load when the latter is at all large, inasmuch as the period of vibration was nearly the same for all the loads. There were, however, slight differences in the vibration-period for the different loads, which I felt inclined to attribute to the fact that the boxes might not always hang quite vertically. The following experiments seem to show conclusively that for loads ranging from moderate amounts to great ones the period of vibration is independent of the load.

Experiment XII.

A pan made of brass was suspended by two hooks, H, H (fig. 3); the bottom of the pan was circular, and the two suspenders H A, H A, made of drawn brass, were soldered into A. The lead cylinders before-mentioned were in the first instance placed on the pan, so that their centres were in the same line as the axis of the wire. The brass cylinders, C_2, were placed near the ends of the bar V V, at equal distances from the wire, and had in consequence of their position and mass a very considerable moment of inertia as compared with the rest of the vibrator. After the arrangements had been completed the apparatus was allowed to remain suspended for several days, and was during this period frequently vibrated. The vibration-period was then determined, and, two of the lead cylinders having been removed, a further rest of 24 hours was given,† when again the vibration-period was determined; and so on, until all the lead cylinders had been removed. In this experiment the same piano-steel wire as before was used. The temperature ranged from 9°·30 C. to 13°·40 C.

Vibration-period in seconds.	Load on the wire in kilos.	Couple in dyne centims. required to twist the wire through one radian.
16·658	1·208	5713
18·106	5·208	5738
19·889	9·208	5758
21·331	13·208	5766
22·720	17·208	5754
24·003	21·208	5756

* G. Wiedemann has shown in his paper already quoted that for moderate loads the torsional elasticity is independent of the load.

† Rest should always be given either after putting on or taking off load, as, though the torsional elasticity is not so much affected by shocks as the internal friction, it is affected.

Experiment XIII.

The copper wire used in Experiment IX. was treated in the same manner as the steel in the last experiment, except that now the cylinders C_3 were dispensed with. The temperature throughout remained nearly constant at 20° C.

Vibration period in seconds.	Load on the wire in kilos.	Couple in dyne centims. required to twist the wire through one radian.
1·891	0·534	46539
3·327	4·534	46841
4·312	8·534	46818
5·105	12·534	46708
5·801	16·534	46817

Remarks on Experiments XII. and XIII.

Experiment XII. shows that for the unannealed piano-steel the couple required to twist the wire through one radian is quite independent of the load for loads ranging from 4 to 20 kilos. For smaller loads than 4 kilos. the value of the couple is slightly less. With the annealed copper, the couple required to twist the wire through one radian is independent of the load for all the loads.

Having thus endeavoured to prove that the torsional elasticity does not depend upon the load when the latter is comparatively large, both for hard-drawn and annealed wires, I will consider the effect of light loading. As regards annealed wire, Experiment XIII. furnishes evidence that the elasticity is very nearly independent of the load, even for light loads, for the lightest load in this experiment does not much exceed $\frac{1}{2}$ kilo.* Care must, however, be taken, with annealed wire, to begin with the heaviest load first, as loading, if not carried to the extent of causing perceptible permanent elongation, invariably produces an increase of elasticity, which is partly permanent and partly sub-permanent, and this last must not be disturbed by shocks in unloading if we are to strictly compare the values of the torsional elasticity with different loads.†

As regards the unannealed piano-steel, the case is otherwise, for if the loads be small the torsional elasticity may vary very perceptibly with them.

* This would produce only a very small stress per square centimetre as compared with the stress used with the unannealed piano-steel.

† See similar remarks respecting the internal friction. With the heavier loads still remaining on the wire, the sub-permanent elasticity is not so readily shaken out as with the lighter ones. I have since thoroughly satisfied myself that the torsional elasticity is independent of the load, even for light loads, in the case of annealed wires, provided sufficient time be allowed after changing each load.

Experiment XIV.

The same piano-steel wire as before. The temperature ranged from 9°·30 C. to 12°·38 C.

Vibration-period in seconds.	Total load on the wire in grammes.	Couple in dyne centims. required to twist the wire through one radian.
4·303	164	5496
16·658	1208	5713
18·106	5208	5738

It will be noticed that there is a comparatively rapid rise in the torsional elasticity as the load increases from 164 grammes to 1208 grammes, and a much less proportionate increase as the load rises still further. Now we find this phenomenon to be closely associated with the permanent torsion, which is, to a greater or less extent, always imparted during the process of wire-drawing. Permanent torsion produces decrease of torsional elasticity, but, as I have already shown,* unannealed piano-steel, if permanently twisted by hand, begins to temporarily untwist† when loaded, the amount of untwist being, for small loads, very rapid, and consequently we might perhaps expect the elasticity to be increased by loading, rapidly at first, and afterwards much more slowly as the loads were increased in amount. Similarly piano-steel behaves when permanently twisted by wire-drawing; this is apparent from the next experiment.‡

Experiment XV.

The unannealed piano-steel used in the previous experiments, and which had been, during a period of several weeks, frequently loaded and unloaded with weights reaching to upwards of 20 kilos., was allowed to rest for some time with only the bar of the vibrator§ attached; a pan whose mass was 388·8 grammes was then suspended

* 'Phil. Trans.,' Part I., 1883, p. 21.

† This expression needs a little explanation. By the words "*permanently* twisted" is meant that the torsion will remain in the wire so long as we do not alter the load under the influence of which the wire was when the torsion was originally imparted, or, if we do alter the load, but afterwards restore it to the original amount, the twist originally imparted will be restored.

‡ I have since examined the matter more closely, and find that, with all very hard-drawn wires, there is a slight difference between the torsional elasticity, as inferred from torsional vibrations, when different light loads are used. This is also the case with annealed metals which, after annealing, have suffered considerable permanent torsion. Whether the torsional elasticity increases or diminishes with the load depends apparently upon whether loading causes the wire to temporarily untwist or twist more.

§ The mass of this bar was 164 grammes.

by means of hooks from the bar, and afterwards loaded carefully with weights, which eventually reached 16 kilos.

Load added to the bar in grammes.	Untwist in scale-divisions produced by each additional load.
388·8*	1814·0
1000	453·5
1000	113·8
2000	106·3
2000	78·3
2000	59·3
2000	43·8
2000	45·8
2000	39·0
2000	37·8

The Effect of Permanent Strain on the Internal Friction.

Experiment XVI.

An annealed iron wire, 602 centimetres in length and 0·960 centimetre in diameter, was provided with a vibrator whose mass was 3606 grammes; a rest of 24 hours was now allowed, and after this period the wire was tested, the results of the experiments showing a logarithmic decrement of ·001131. Next the wire was permanently elongated to the extent of 3·16 per cent., and, having been shortened to its original length, was again tested about ten minutes afterwards; the logarithmic decrement was now found to be ·001422. After a rest of 24 hours, the internal friction had so far diminished that the logarithmic decrement had become ·001214, i.e., nearly 8 per cent. greater than it was before the permanent extension had taken place. The vibration-period before the permanent strain was 4·309 seconds, and after the extension 4·505 seconds; therefore, allowing for the diminution of section, resulting from the strain, the effect of the latter was to cause a decrease of the torsional rigidity of 2·8 per cent.

We must observe, with reference to such an experiment as the above, that, in order that permanent extension may produce increase of internal friction, the wire must be loaded, when tested before the extension, to a degree which, though not sufficient to cause sensible permanent elongation, is yet enough to secure a certain amount of permanent strain among the molecules; if the wire be only tested with a light vibrator, permanent extension may possibly produce a diminution of internal friction, as the following experiment shows.

* The load produced by the pan itself.

Experiment XVII.

An annealed copper wire 602 centims. in length and 0·960 centim. in diameter was provided with a vibrator whose mass was 430 grammes, and after resting for several days was tested; the logarithmic decrement proving to be ·000297. The wire was now permanently lengthened 2·67 per cent., and ten minutes after it had been shortened to its original length and re-suspended was again tested, with the following results:—

Logarithmic decrement due to internal friction.	Remarks.
·000717	Mean value for the first 100 vibrations
·000496	Mean value for the next 200 vibrations
After a rest of two days.	
·000262	Mean value for 400 vibrations

Thus we see that, though the permanent extension increased considerably the internal friction, when the wire was vibrated shortly after the strain had been produced, the effect was rapidly lessened by rest, so that after two days the logarithmic decrement became slightly less than it was before the extension. The time of vibration was, before the extension, 5·125 seconds, and after the extension 5·275 seconds. If we allow for the diminution of section, caused by the strain, there results from extension a *diminution** of the *torsional rigidity* of about 0·7 per cent.

Experiment XVIII.

The same wire was now twisted through twenty complete revolutions, of which six came out, leaving a permanent torsion of fourteen revolutions; ten minutes afterwards the wire, which had been kept vibrating, was again examined, with the following results:—

Time of vibration in seconds.	Logarithmic decrement due to internal friction.	Remarks.
5·475	·001175	Mean value for the first 100 vibrations
5·440	·001051	Mean value for the next 200 vibrations
After a rest of 24 hours.		
5·425	·000461	Mean value for 400 vibrations

* Observe that diminution of torsional rigidity is not always attended by increase of internal friction.

It would appear that permanent torsion has much more effect than permanent extension in increasing the internal friction. The torsional rigidity of the metal is also much more diminished by permanent torsion than by permanent extension, but the effect on the rigidity is nothing like so great as that on the internal friction, which last amounts to nearly 76 per cent.

Having shown that when sensible permanent deformation is produced, either by traction or torsion, there is an increase of internal friction, I will now consider the effect of stress, which is not sufficient to produce sensible molar permanent deformation, but which causes molecular deformation.

Experiment XIX.

An annealed iron wire 601 centims. in length and 0·960 centim. in diameter. Attached to the bar of the vibrator was a pan, which, together with the bar, weighed rather more than 2 kilos. The wire was, after adjustment, allowed to remain for five days, and was in the meantime frequently vibrated. After this period weights were put into the pan, and after each addition of load, followed by a rest of about ten minutes, the logarithmic decrement was determined; before each determination the wire was suffered to make about 100 vibrations.

Load in kilos.	Logarithmic decrement due to internal friction.	Remarks.
2	·003241	
4	·002726	
6	·002231	
8	·001406	
2	·001492	
4	·001246	
6	·001606	
8	·001346	
10	·002377	
12	·004177	
14	·005563	
14	·002441	After a rest of 30 minutes.
14	·002366	Started again.
14	·005606	After having been vibrated for about ten minutes through an angle of 60°.*
14	·001268	After a rest of 20 minutes.

An additional load of 1 kilo. was now put on and caused a slight permanent extension, which, after a rest of two days, was ascertained to be 2·6 centims. The wire was then reduced to its original length, and, 14 kilos. being the load, was allowed to vibrate for ten minutes. After this the logarithmic decrement was again determined.

* Notice the "fatigue of elasticity" (see p. 832).

Load in kilos.	Logarithmic decrement due to internal friction.	Remarks.
14	·001017	After a rest of 24 hours and a very large number of previous oscillations.
10	·001089	Ditto.
6	·001109	Ditto.
2	·001307	Ditto.

The above experiment suffices to show the great extent to which the internal friction can be diminished by the permanent molecular strain resulting from loads which are not sufficient to cause any appreciable permanent extension. For we see that even with a load of 15 kilos. the permanent extension was only about 0·4 per cent., whereas a load of half this was sufficient to cause a permanent molecular strain which reduced the friction to less than one-half of the original amount.

The Effect of Permanent Strain on the Torsional Rigidity.

I have already pointed out in Experiments XVI., XVII., and XVIII. that when very sensible permanent molar deformation is produced, either by traction or torsion, the torsional rigidity is decreased,* so that we need only now consider the effect of stress which, without producing sensible molar deformation, suffices to cause molecular permanent strain.

Experiment XX.

An annealed copper wire 602 centims. in length and 0·1622 centim. in diameter. The mass of the vibrator attached to the wire was 1554 grammes, and the determinations of the logarithmic decrements were all made with only the vibrator. When, however, it was necessary to strain the wire, a pan was suspended from the bar of the vibrator, and on this pan weights were placed which were suffered to remain for some time. The pan and weights were then removed, and 24 hours after the removal the testing, preceded by a large number of preliminary vibrations, began. The temperature ranged from 15° C. to 20° C.

Load in kilos. used to produce the permanent strain.	Number of days during which the load acted before removal.	Couple in dyne-centimetres required to twist the wire through one radian.
1·554	0	41477
1·554	8	44373
21·554	6	48158

* Sir W. Thomson had previously shown this to be the case ('Encycl. Brit.,' 9th edit., Art. "Elasticity," § 81).

There was no perceptible permanent elongation produced by the loading, so that the increase of torsional elasticity, which is very apparent, must be due to molecular and not molar strain. Part of the strain is only sub-permanent and can be shaken out; when this is done there is an immediate loss of elasticity, which is regained if the wire be again loaded to the same extent as before, and subjected to the same previous treatment.

Effect of Wire-drawing on the Internal Friction.

We have seen that both permanent extension and permanent torsion are attended with permanent increase of internal friction. Now, in the process of wire-drawing there occurs both permanent extension and torsion, combined with lateral pressure. As a consequence, wire-drawing increases the internal friction in an astonishing manner. I shall have occasion to dwell more fully on the effects of annealing in a memoir which I hope will shortly follow this one, but the following Table will bear witness to the truth of the above statement:—

TABLE.

Name of metal.	Logarithmic decrement due to internal friction in the hard-drawn metal. A.	Logarithmic decrement due to internal friction in the annealed metal. B.	Ratio A : B.
Silver	·001540	·0002970	5·2
Copper	·000526	·0002622	2·0
Platinum	·002239	·0000766	29·3
Platinum-silver . .	·001792	·0001166	15·3
Zinc	·018530	·0032070	5·8

We thus see that annealing the hard-drawn metals reduced the internal friction to an amount which varied from *one-half to one-thirtieth of the original.*

The Logarithmic Decrement due to Internal Friction is independent of the Length and Diameter of the Wire.

Experiment XXI.

The annealed copper wire of Experiment IV. was drawn to a diameter which was nearly one-half of the original and afterwards re-annealed. With the same length of wire in both cases, and with nearly the same preliminary treatment, the logarithmic decrement due to internal friction only was for the thicker wire ·0002622 and for the thinner one ·0002695.* The temperature of the wire was nearly the same in each case,

* Great care must be used in an experiment of this kind to ensure uniformity of temperature during the period of experimenting, otherwise the thinner wire will show a greater internal friction than the thicker one (see Experiment VII.).

and the difference in the two logarithmic decrements is not greater than could be accounted for by slight difference in the steadiness of the temperature.* Two nickel wires, whose diameters were also nearly in the same ratio, namely 2 : 1, gave values for the logarithmic decrements due to internal friction which agreed with each other within the errors of observation.

Experiment XXII.

Two pieces of the same specimen of annealed zinc wire, one 602 centims. in length and the other only 97 centims. in length, were both tried under nearly the same conditions as regards preliminary treatment and time of vibration, &c. The logarithmic decrement due to internal friction was, in the case of the longer wire, ·003207, and, in the case of the shorter one, ·003327. The difference between these two numbers can quite be accounted for by the fact that the temperature of the shorter wire was about 3° C. higher than that of the longer.†

Two pieces of iron wire, cut from the same hank, but one having a length of 96 centims. and the other of 602 centims., gave, when treated in a similar manner, exactly the same logarithmic decrements.

We may, therefore, regard the logarithmic decrement as independent both of the length and of the diameter of the wire, provided care be taken that the two pieces of different lengths which are compared shall have suffered the same preliminary treatment.

"Fatigue of Elasticity."

Sir WILLIAM THOMSON writes,‡ "Experimental exercises performed by students in the physical laboratory of the University of Glasgow, during the session 1864–65, brought to light some very remarkable and interesting results, proving a loss of energy in elastic vibrators incomparably greater than anything which could be due to imperfections in their elasticity, and showing also a very remarkable fatigue of elasticity, according to which a wire, which had been kept vibrating for several hours or days through a certain range, came to rest much more quickly when left to itself than when set in vibration after it had been at rest for several days and then immediately left to itself." This so-called "fatigue of elasticity" seemed to me such a very remarkable phenomenon that, throughout the whole of these investigations, I was constantly on the look-out for traces of it; but, owing no doubt to the very slight maximum deformations employed in these experiments, I could detect nothing of the kind with any metal§ except nickel. The whole of the other metals examined, namely,

* It is necessary to take great care that the temperature should fluctuate very little, as any fluctuation will affect a thin wire more than a thicker one.

† The effect of change of temperature on the internal friction of zinc is rather considerable.

‡ 'Encycl. Brit.,' 9th edit., Art. "Elasticity," § 30.

§ Except in Experiment XIX.

platinum, silver, steel, iron, brass, copper, aluminium, zinc, platinum-silver, and German-silver, with arcs from rest to rest ranging up to 700 scale-divisions, and lead and tin, for arcs ranging up to 200 scale-divisions, showed no sign of the phenomenon. With nickel, however, the case was different, and I met with most marked signs of "fatigue of elasticity," even for such small molecular displacements as attended arcs ranging from 300 to 700 scale-divisions.

Experiment XXIII.

A nickel wire, unannealed, 602 centims., and ·018622 square centim. in section, after having been tested for several days with three different vibrators, was finally tried with the most massive of the three.*

Logarithmic decrement due to internal friction.	Vibration-period in seconds.	Remarks.
·007058	6·010	Immediately after adjustment.
·006623	..	After ten minutes' rest.
·006225	..	After 1½ hours' further rest.
·005964	..	After 2 hours' further rest.
·005585	..	After 3½ hours' further rest.
·005522	5·960	After 16 hours' further rest.
·005644	..	Started again immediately.
·005689	..	Ditto.
·006023	..	An arc of 700 scale-divisions maintained for 2 minutes, and then allowed to subside.
·006427	..	An arc of 700 scale-divisions maintained for 30 minutes, and then allowed to subside.

In all the above trials the initial arc from which the logarithmic decrements were computed was about 600 scale-divisions, and the final arc about 150 scale-divisions. I next give the whole of the observations which were taken in the last trial from the initial to the final arc.

Range of arc.	Logarithmic decrement due to internal friction.
605·2–516·0	·00674
516·0–440·8	·00665
440·8–372·0	·00718
372·0–317·2	·00673
317·2–270·0	·00680
270·0–230·8	·00663
230·8–201·5	·00570
201·5–174·0	·00619
174·0–152·5	·00553
152·5–135·5	·00495

* The total mass of this vibrator was 3666 grammes.

It will be noticed that the logarithmic decrement begins to fall very rapidly as soon as the range of arc has become 230·8—201·5. Immediately, therefore, that the calculations necessary for obtaining the above results had been finished, the vibrator was again started, but now with only an initial arc of about 300 scale-divisions.

Range of arc.	Logarithmic decrement due to internal friction.
299·2–165·8	·004938
165·8– 67·5	·003713
67·5– 34·8	·002687
34·8– 19·0	·002438
19·0– 11·2	·002105

By this time the vibration-period had diminished to 5·900 seconds.

After a rest of five hours the vibrator was started with an initial arc of just over 100 scale-divisions.

Range of arc.	Logarithmic decrement due to internal friction.
100·2–57·8	·002200
57·8–36·8	·001771
36·8–22·0	·002044
Mean	·002005

Remarks on Experiment XXIII.

In the first place, we may direct our attention to the diminution of the vibration-period produced by rest after repeated vibrations; this seems to be almost, if not quite, as pronounced as it is in the case of iron, the vibration-period gradually diminishing from 6·010 seconds to 5·900 seconds, and this, too, in spite of the fact that the same vibrator had been used several times previously with the wire, and had been carefully adjusted at the commencement of the experiment.*

In the second place may be noticed the diminution of the logarithmic decrement which attends the above-mentioned diminution of the vibration-period, and that the former of the two is very much greater than the latter. This last is easily accounted for if we assume with WIEDEMANN that the internal friction is due to the to-and-fro rotation of the molecules about their axes, and not, as is the torsional elasticity of the wire, mainly upon the force necessary to alter the distances separating the centres of the molecules.

Thirdly, the effect of "elastic fatigue" becomes less and less apparent as the ampli-

* It should be understood that the *bar* of the vibrator was not changed in any of the alterations of the mass of the vibrator, but only the cylinders.

tudes through which the logarithmic decrement is calculated diminish, and, moreover, the range through which the logarithmic decrement is a constant becomes wider as the amount of "elastic fatigue" is diminished. We may see here analogies between the increase of the proportionate diminution of amplitude caused by "elastic fatigue" and the increase of the ratio of the temporary molar deformation, whether torsional or longitudinal, to the stress producing it, as the latter increases whenever the wire acted upon has recently suffered permanent longitudinal extension or torsion.*

Lastly, it is desirable to draw attention to the marvellously small deformations which can produce "elastic fatigue" in nickel as compared with those which are attended with corresponding effects in the case of the other metals.

The Effect of Rest and of frequent Oscillations on the Internal Friction.

It has been already observed that rest after strain of any kind invariably diminishes the internal friction, but rest may be considerably aided in this effect by agitating the molecules either by thermal or by mechanical agency. With the part played by thermal agents I shall deal in a later memoir, and now turn to agitations produced by oscillating the wire torsionally.

Experiment XXIV.

The wire used in Experiment XVI., and which had suffered the permanent extension mentioned in this experiment, was allowed to rest for 24 hours with a load of 12 kilos. on, and was then tested with the same load still on.

Logarithmic decrement.†	Range of arc.	Number of oscillations before the logarithmic decrement was determined.
·003050	537·4–500·9	4
·003108	411·5–346·6	40
·002617	346·6–299·9	64
·002875	299·9–255·8	88
·002905	255·8–217·9	112
·002553	217·9–189·2	136
Started again with an initial arc of 600 scale-divisions, and left vibrating until the arc was reduced to 379·0 scale-divisions.		
·002033	379·0–302·7	
·001911	302·7–245·1	

* 'Phil. Trans.,' 1883, Part I., pp. 7, 8. See also WIEDEMANN'S paper *loc. cit.*

† In this experiment the effect of the resistance of the air was not eliminated, as it was impossible to have done so accurately, and the elimination would have been of little importance.

The mean of the two last values of the logarithmic decrement is ·001972, and here it remained very fairly constant, further oscillation not diminishing the internal friction. When, however, rest for one or more days was now given, the internal friction would rise again until once more repeated oscillations would reduce it to ·001972 or thereabout. No doubt the increase of internal friction attending on rest in this case is owing to changes of temperature which occur in the interval, such changes, as we have seen, tending to disturb the molecular arrangements which existed at the end of the frequent oscillations.

Summary.

1. The proportionate diminution of amplitude due to internal molecular friction in a torsionally vibrating wire is independent of the amplitude, provided the deformations produced do not exceed a certain limit. This limit varies with the nature of the metal, and is, for nickel wire, very low.

2. The logarithmic decrement of amplitude increases with the length of the vibration-period, but in a less proportion than the latter, and in a diminishing ratio. The amount of increase of the logarithmic decrement attending on a given increase through a given range of the vibration-period varies with the nature of the metal, and with those metals in which the internal friction is small becomes nearly insensible.

3. It follows from 2 that the internal friction of a solid does not resemble the viscosity of fluids.

4. Permanent molecular strain resulting from loading, not carried to a sufficient extent to produce sensible permanent extension, diminishes the internal friction and increases the torsional elasticity.

5. Considerable permanent longitudinal extension and permanent torsion produce increase of internal friction and diminution of torsional elasticity. The effect of torsion is much greater than that of extension, and the increase of internal friction is much greater than the decrease of torsional elasticity.

6. The internal friction of a metal is independent of the temporary load.

7. Annealing a hard-drawn wire may enormously diminish the internal friction, especially if, after annealing, the wire be kept suspended for some days and frequently vibrated.

8. Rest after suspension, aided by not too large oscillations at intervals, diminishes the internal friction of a wire which has been recently suspended, or which, after suspension, has been subjected to considerable molecular agitations by mechanical or thermal agency.

9. "Fatigue of elasticity" is not felt, provided the deformations produced do not exceed a certain limit. This limit varies considerably with the nature of the metal,

being very low for nickel, so that with this metal it is difficult to avoid "elastic fatigue."

10. Mechanical shocks and rapid fluctuations of temperature beyond certain limits may increase very considerably the internal friction, and, though to a much less extent, diminish the torsional elasticity.

11. The logarithmic decrement is independent both of the length and of the diameter of the wire.

Fig. 1. (§ 6.)

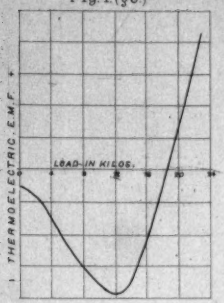


Fig. 2. (§ 7.)

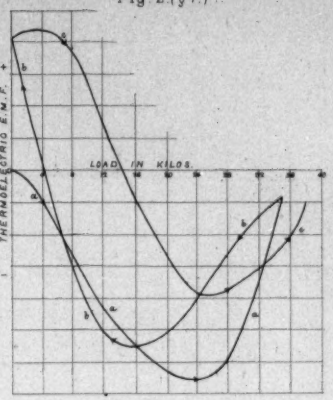


Fig. 3. (§ 8.)

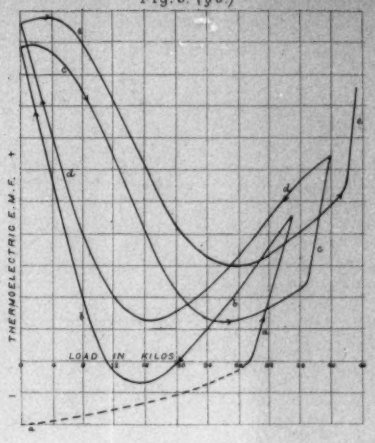
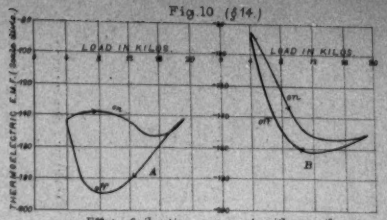


Fig. 10. (§ 14.)



Effects of vibration.

A without vibration.
B with vibration.

Same wire, same cycle of loads.

Fig. 4. (§ 10.)

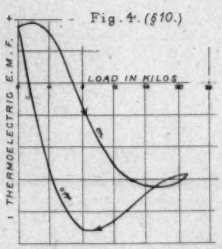


Fig. 5. (§ 12.)

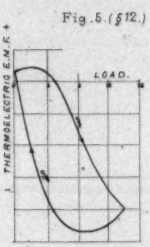
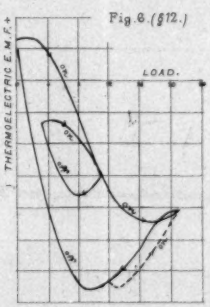


Fig. 6. (§ 12.)



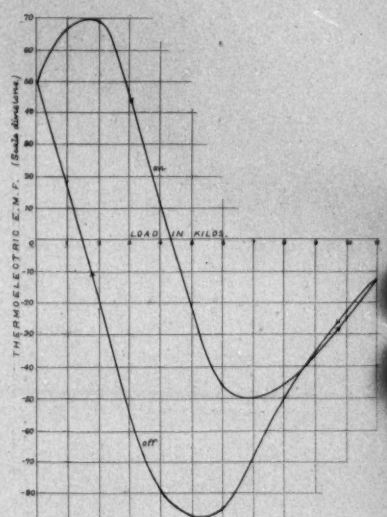


Fig. 12. (§ 22.)

Fig. 7. (§ 12.)

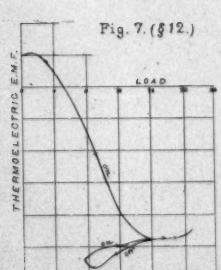


Fig. 8. (§ 12.)

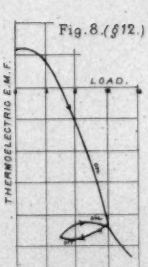
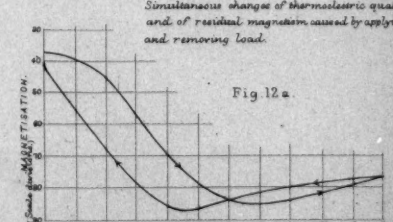


Fig. 9. (§ 12.)

Simultaneous changes of thermoelectric quality and of residual magnetism caused by applying and removing load.

Fig. 12 a.



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EFFECTS OF STRESS ON MAGNETISM AND THERMOELECTRIC QUALITY IN VARIOUS CONSTANT MAGNETISING FIELDS.

Effects of Stress on Thermoelectric Quality shown by Curves Ia, IIa, IIIa, IVa, and Va.
Corresponding effects on Magnetism shown by Curves II, III, IV, and V.

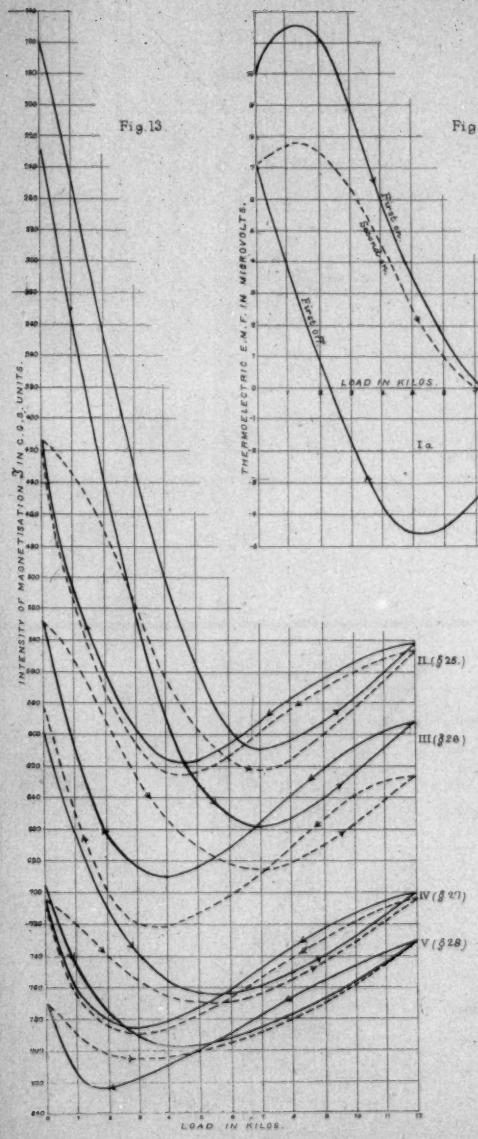

Fig. 13

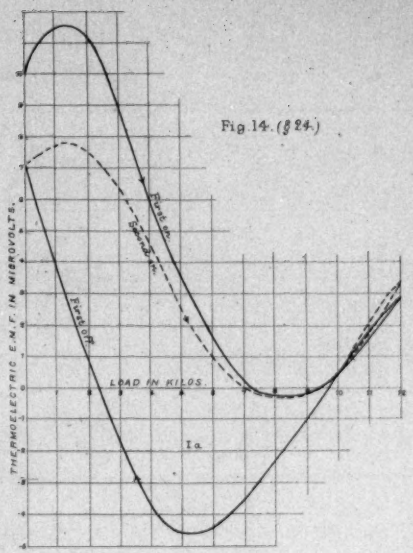

Fig. 14. (§ 24.)

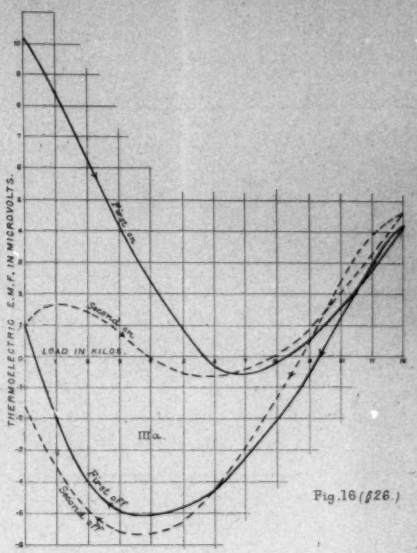

Fig. 16. (§ 26.)

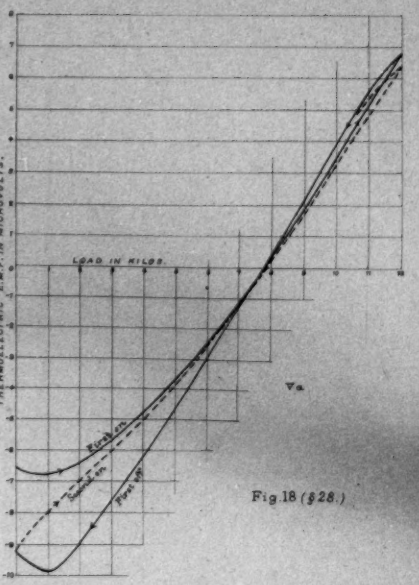

Fig. 18. (§ 28.)

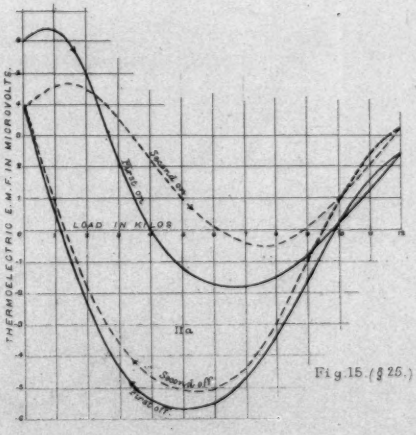

Fig. 15. (§ 25.)

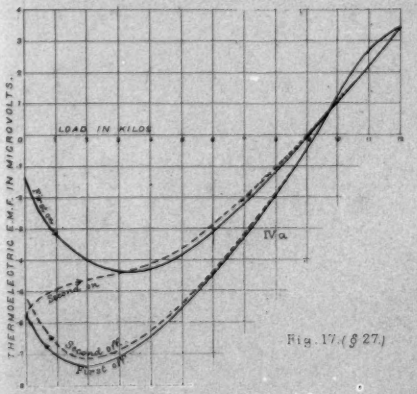

Fig. 17. (§ 27.)

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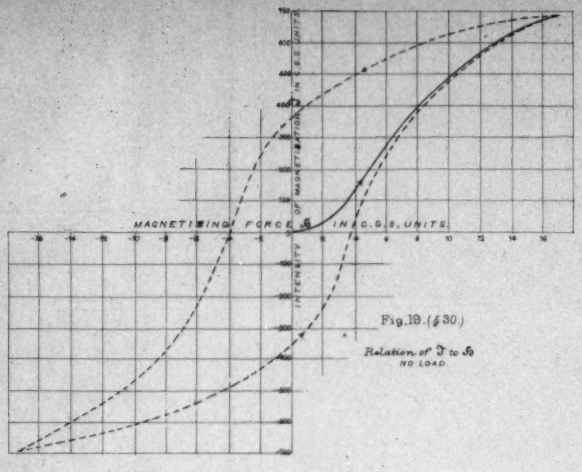

Fig. 19. (§ 30.)
Relation of $\mathfrak{I}$ to $\mathfrak{H}$
NO LOAD.

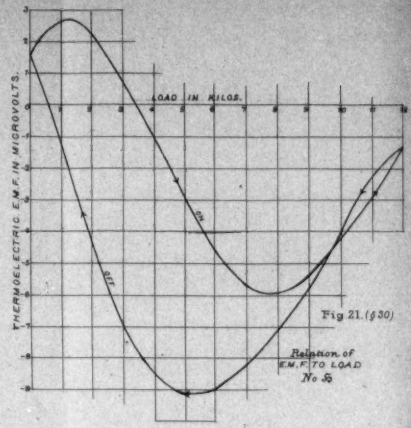

Fig. 21. (§ 30)
Relation of E.M.F. TO LOAD
No 5.

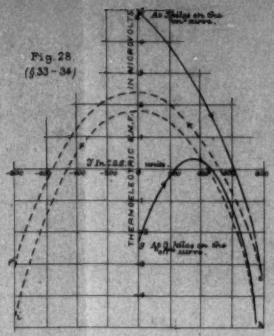

Fig. 28. (§§ 33–34)

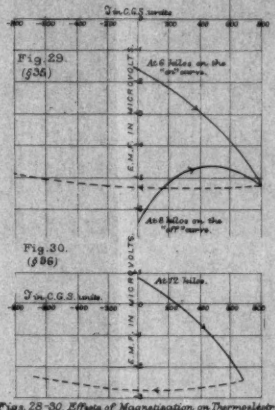

Fig. 29. (§ 35)

Fig. 30. (§ 36)

Figs. 28–30. Effects of Magnetisation on Thermoelectric Quality after applying Stress.

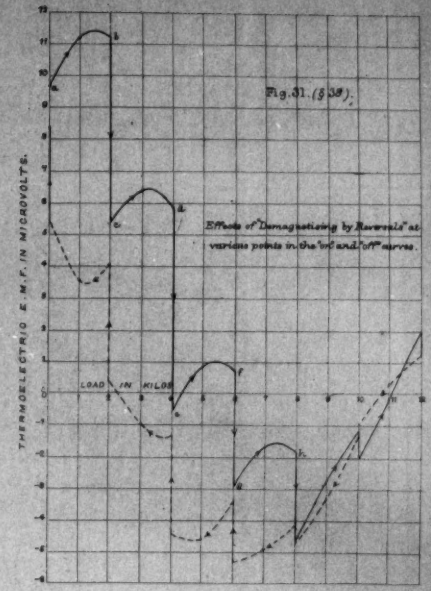
Effects of "Demagnetising by Reversals" at various points in the "on" and "off" curves.
LOAD IN KILOS.
THERMOELECTRIC E.M.F. IN MICROVOLTS.

Fig. 31. (§ 38).

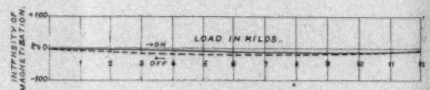

Fig. 22 (§ 30)
Relation of $\mathfrak{I}$ to LOAD.
No 5.

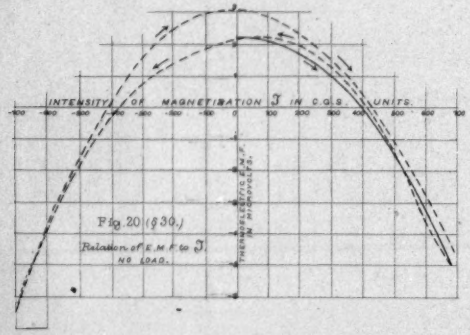

Fig. 20 (§ 30.)
Relation of E.M.F. to $\mathfrak{I}$.
NO LOAD.

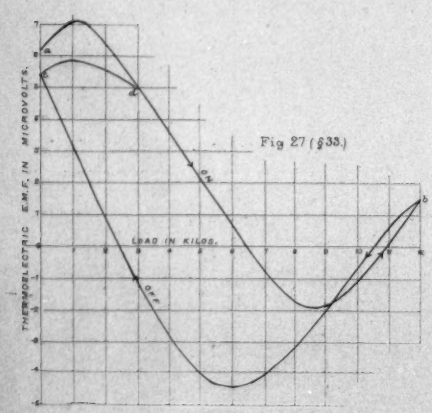

Fig 27 (§ 33.)

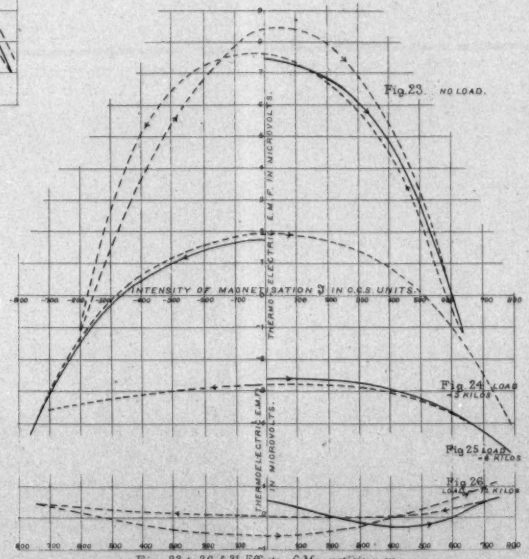
INTENSITY OF MAGNETISATION $\mathfrak{I}$ IN C.G.S. UNITS.
THERMO ELECTRIC E.M.F. IN MICROVOLTS.
THERMOELECTRIC E.M.F. IN MICROVOLTS.

Fig. 23. NO LOAD.
Fig. 24. LOAD = 2 KILOS.
Fig. 25. LOAD = 4 KILOS.
Fig. 26. LOAD = 7 KILOS.

Figs. 23 to 26. § 31. Effects of Magnetising on Thermoelectric Quality under various Constant Loads.

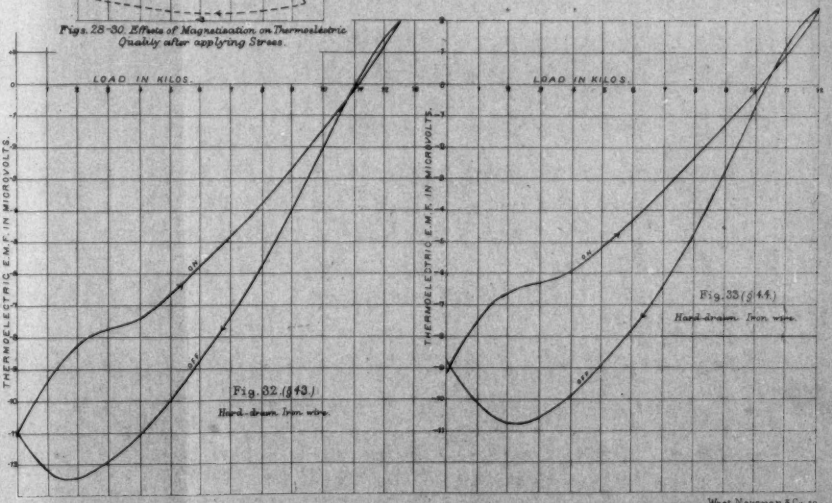

Fig. 32. (§ 43.)
Hard-drawn Iron wire.

Fig. 33. (§ 44.)
Hard-drawn Iron wire.

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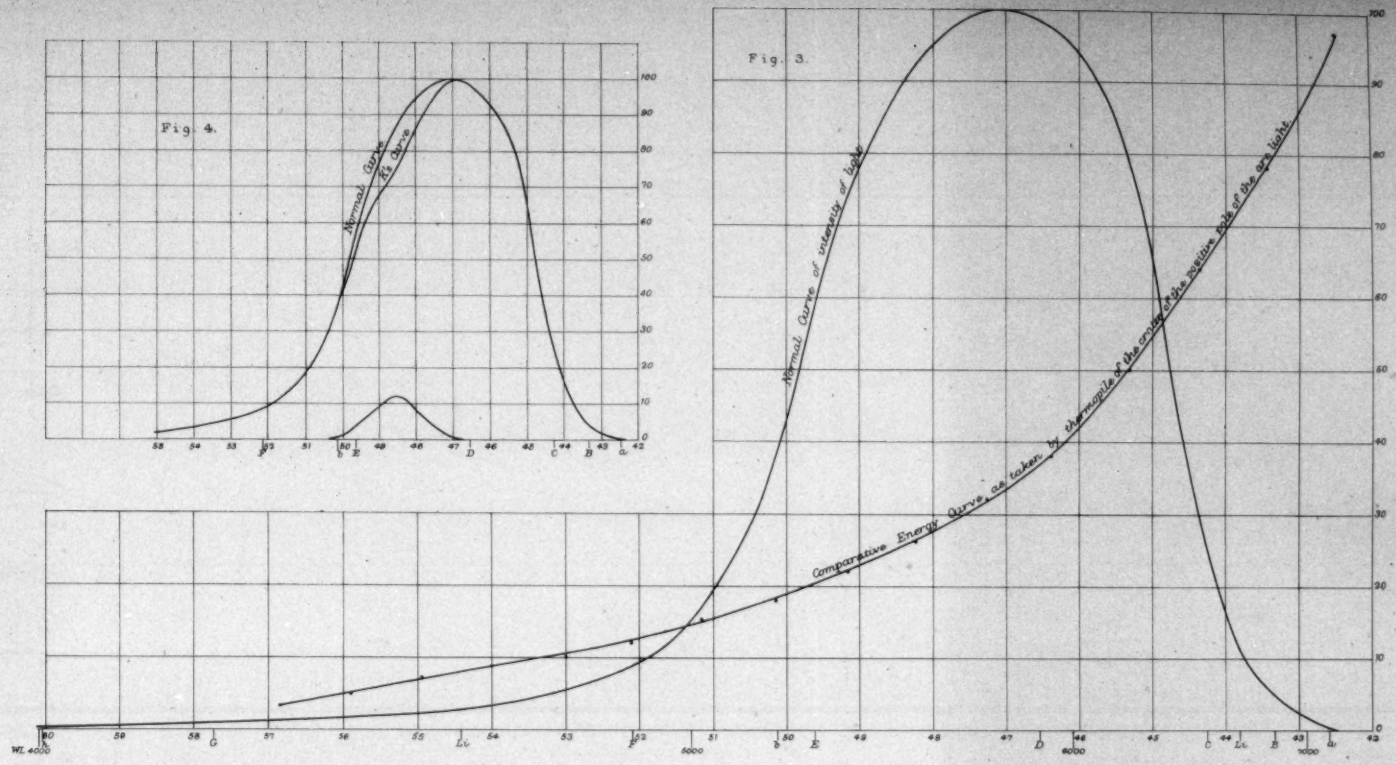
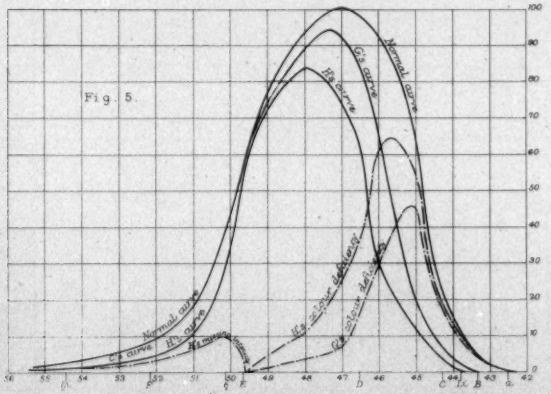
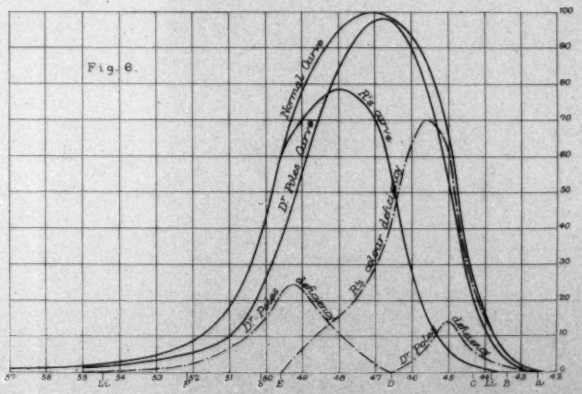


Fig. 3.
Normal curve of intensity of light
Comparative Energy Curve, as taken by thermopile & the centre of the positive pole of the arc light.





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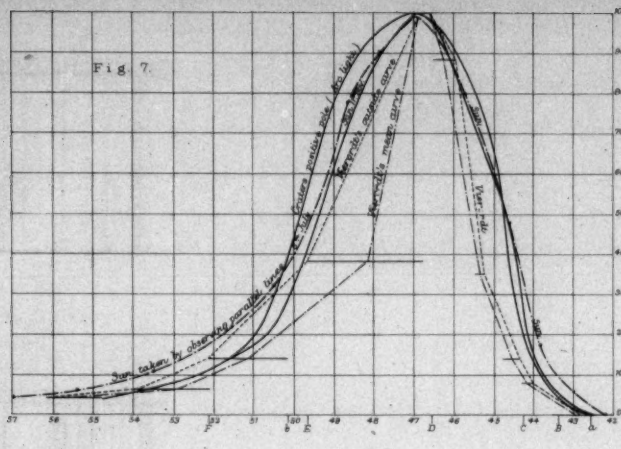
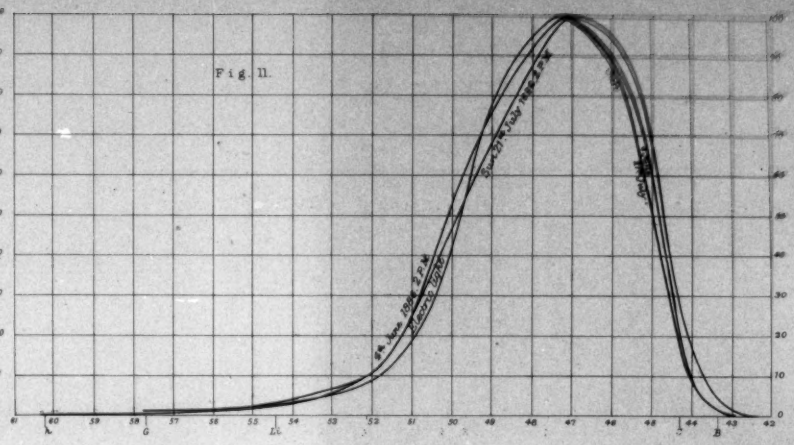
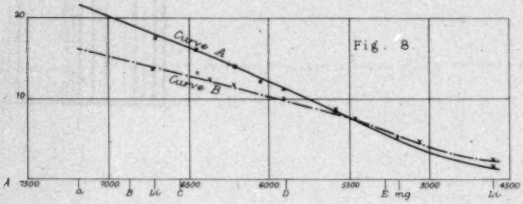






In Curve A the ordinates to be divided by 37·6 to bring to absolute measure

In Curve B the ordinates to be divided by 22·55 to bring to absolute measure.

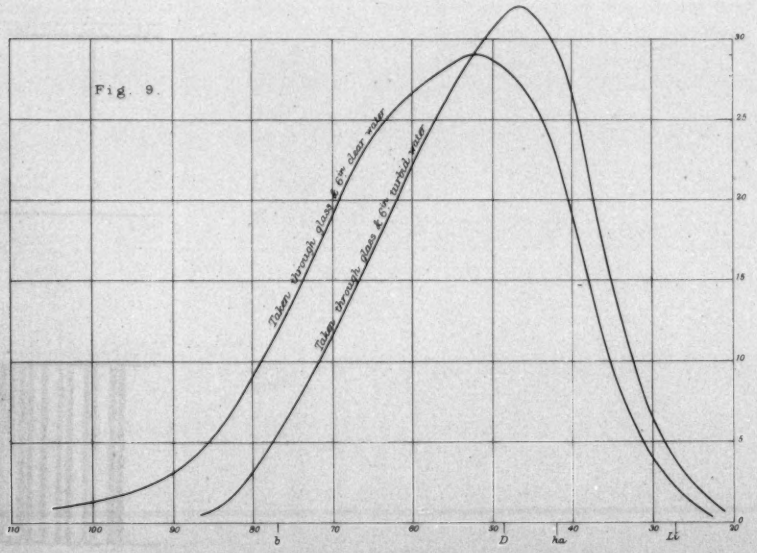
Fig. 10.

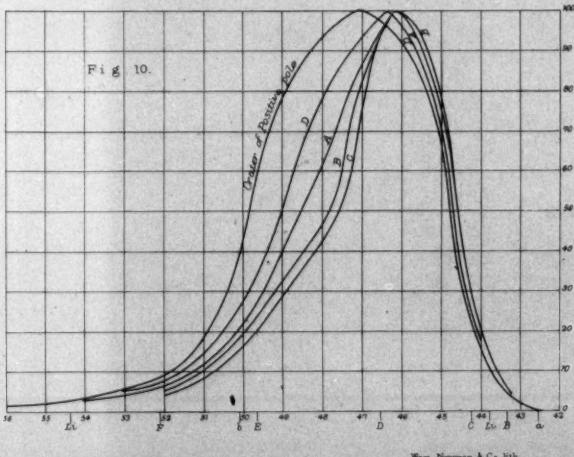
Curve A is Incandescence lamp 102 Watts.

Curve B is Incandescence lamp 143 Watts.

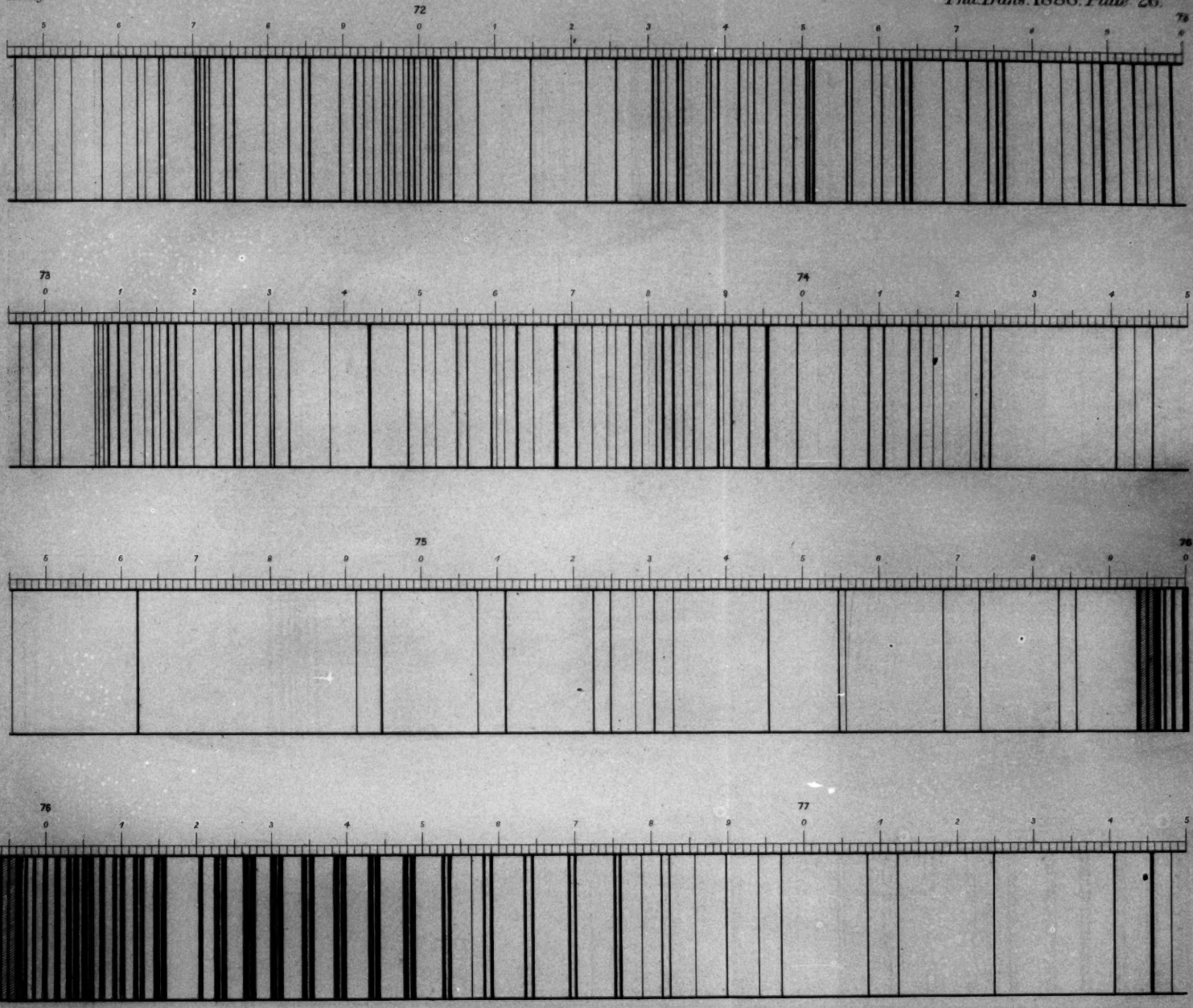
Curve C is Incandescence lamp 266 Watts.

Curve D gas flame.





Wesn. Newman & Co. lith.


72
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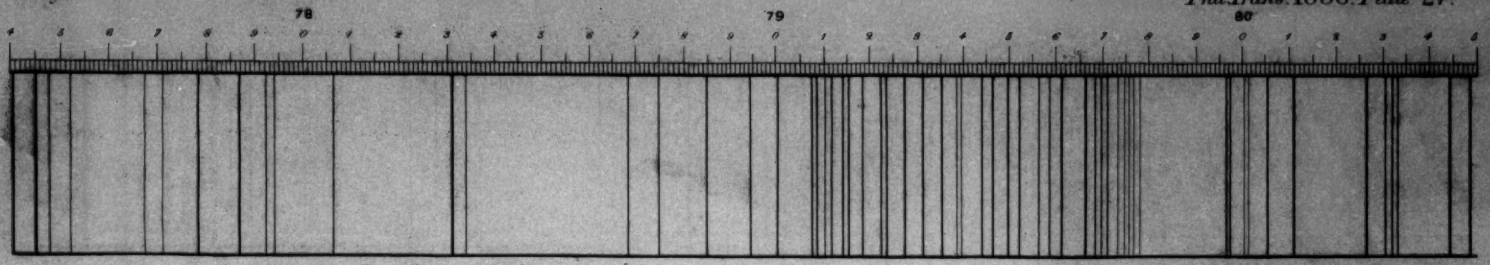
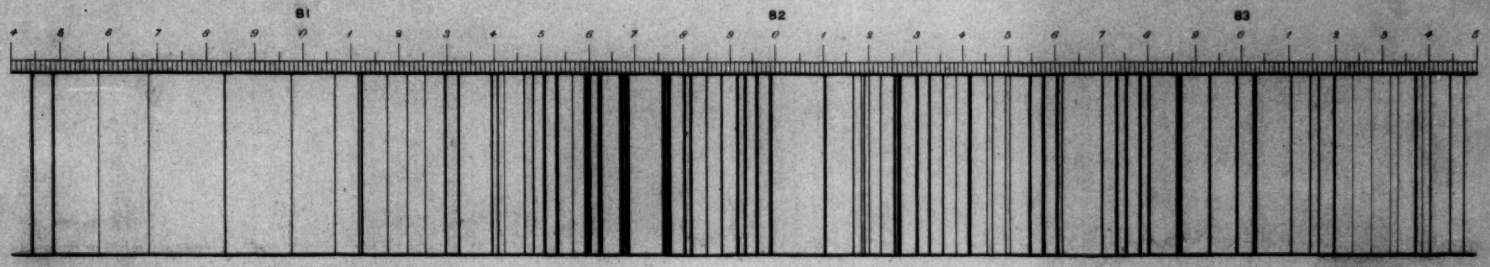
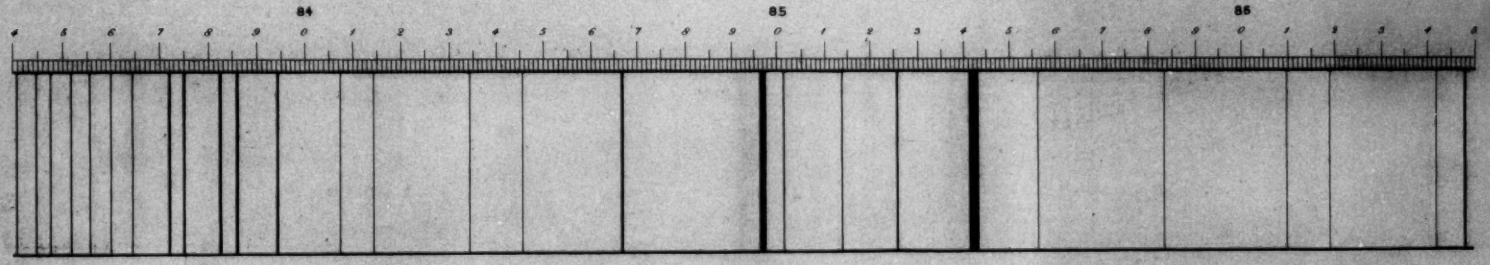
West, Newman & Co. ac

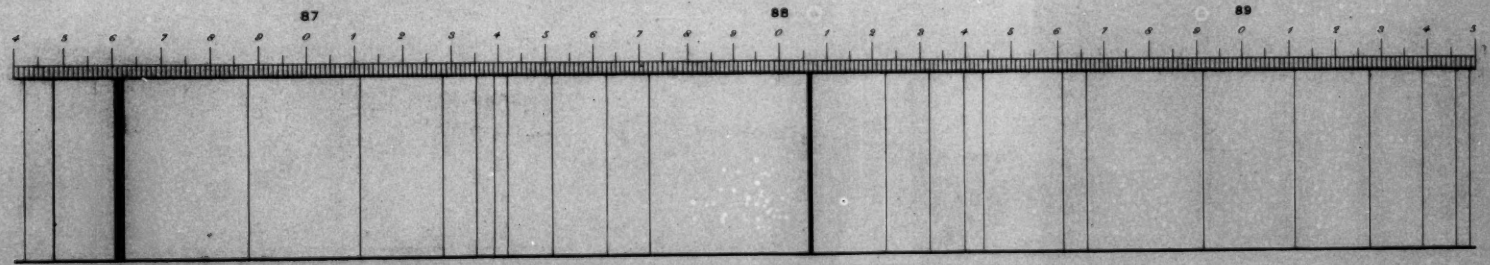




Z



X_I X_{II}



X_{III} X_{IV}

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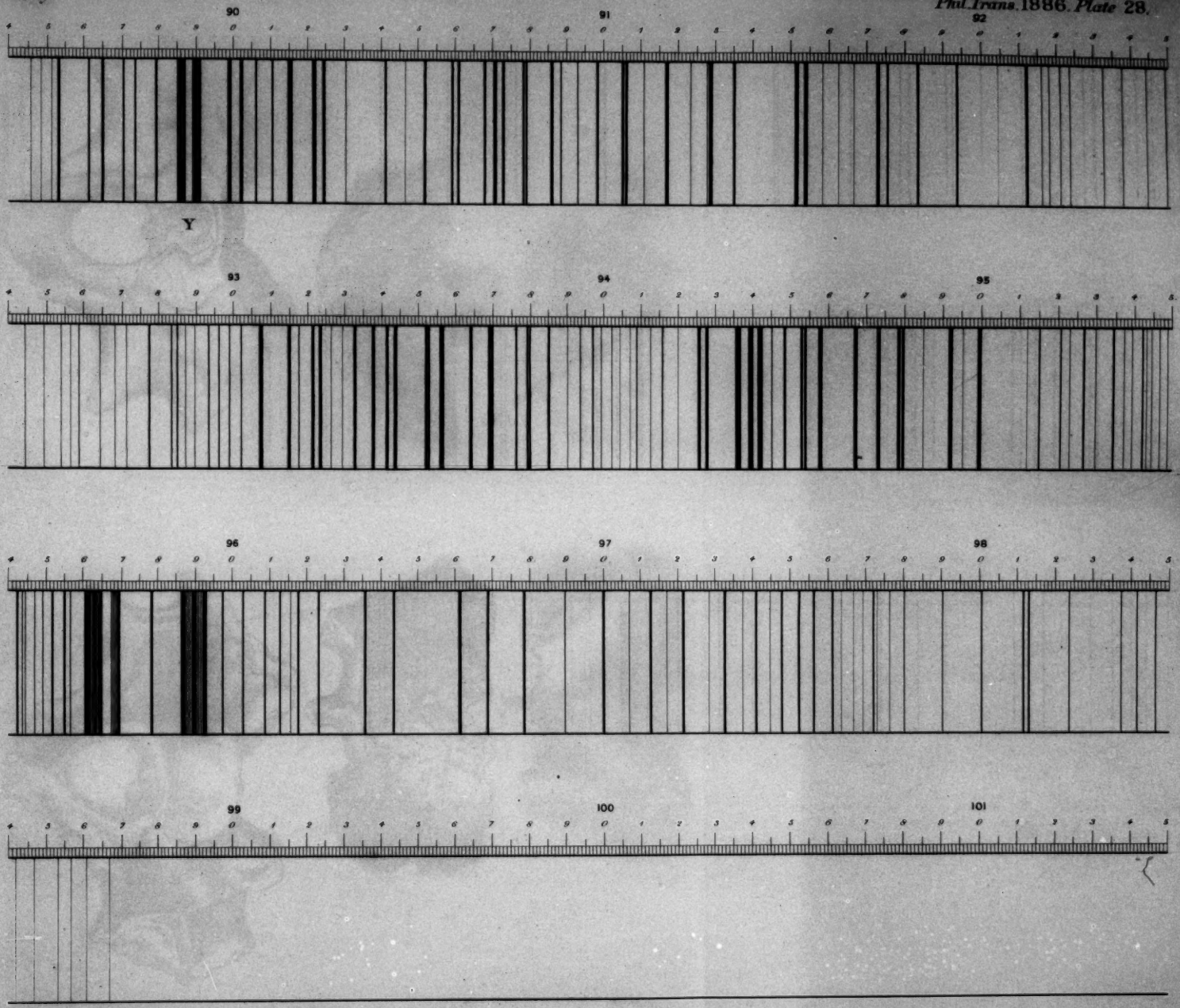


West, Newman & Co. sc.

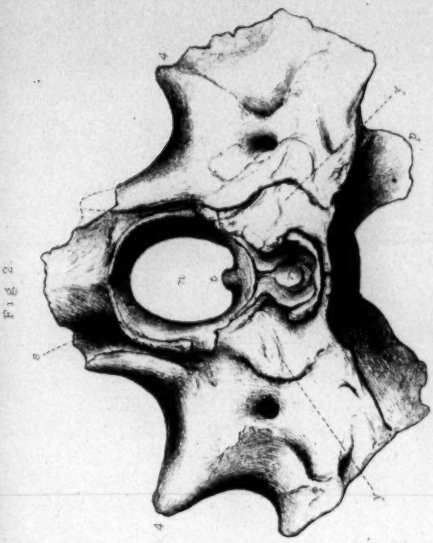


Fig. 2.

Fig. 4.

Fig. 6.

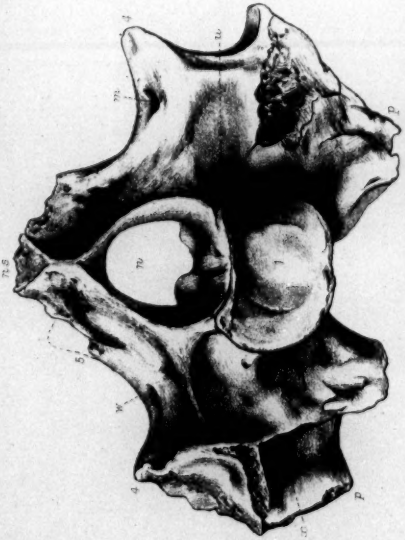


Fig. 1.

Fig. 3.

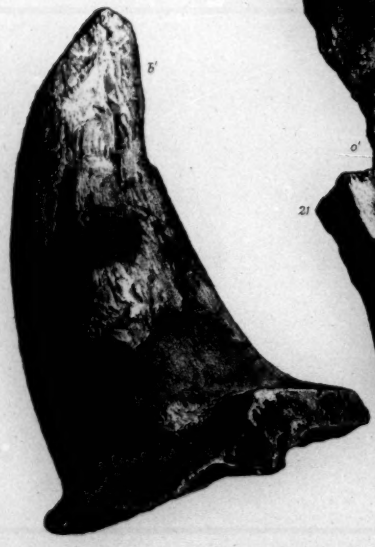
Fig. 5.

W. H. Wesley ad. nat. lith.

West, Newman & Co. imp.

Fig. 1.

Fig. 2.



W. H. Wesley lith.

West, Newman & Co. imp.

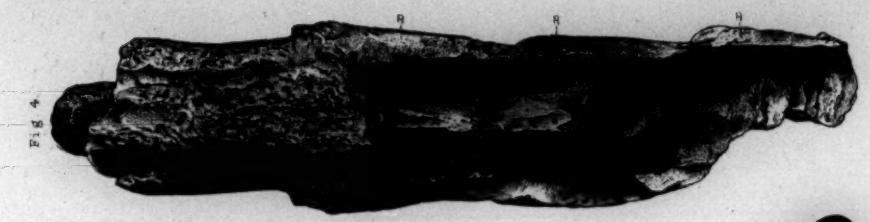


Fig. 4

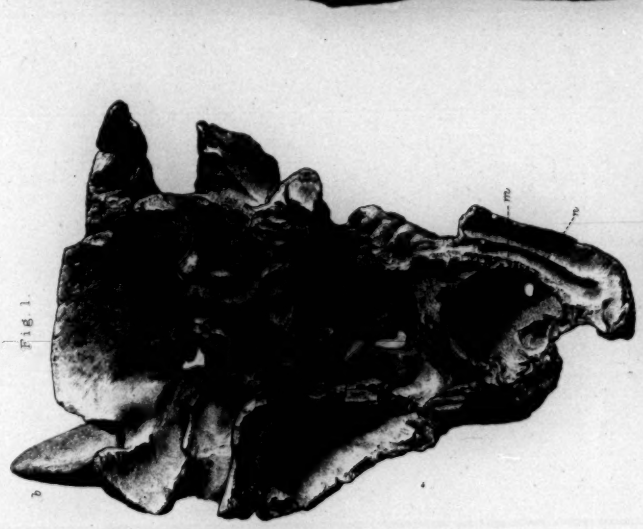


Fig. 1.

Fig. 2.

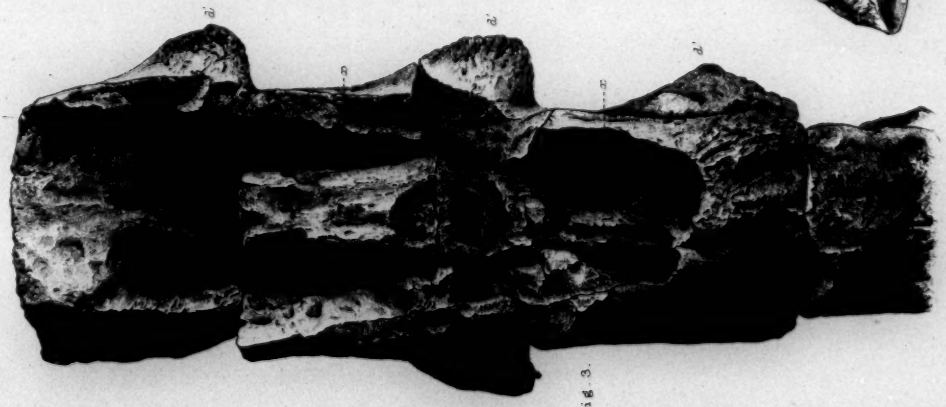


Fig. 3.

W.H.Wesley ad.nat.lith.

West,Newman & Co.imp.

Fig 2.

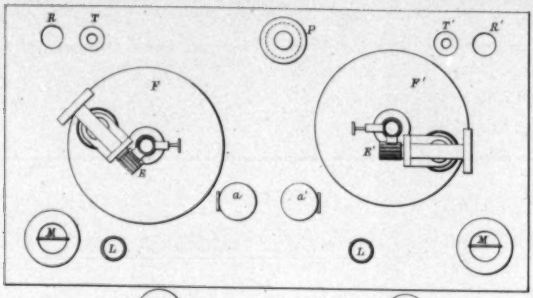
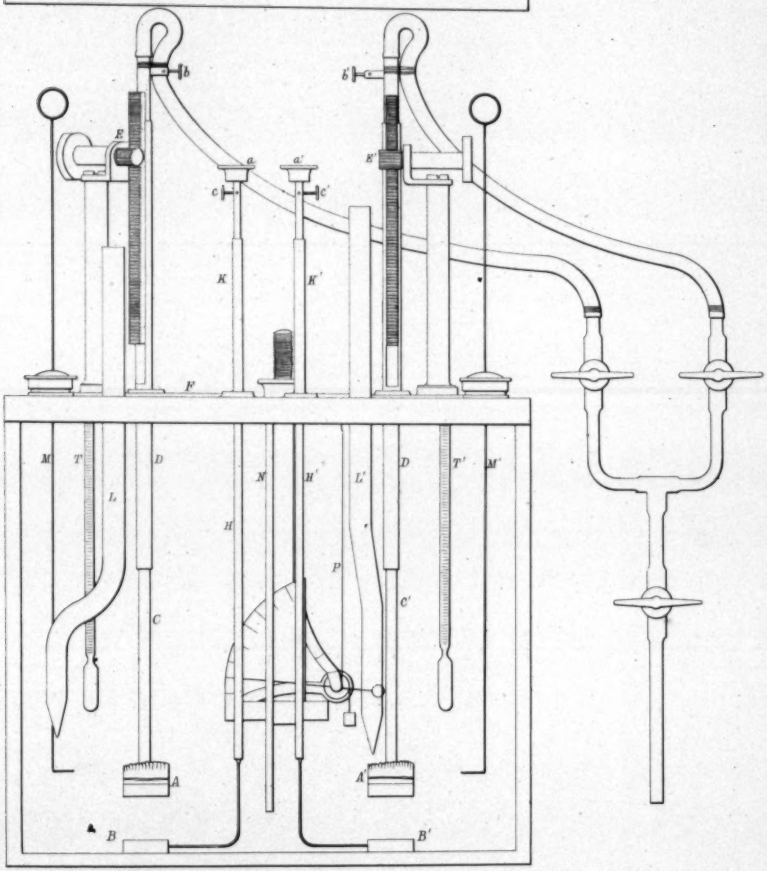
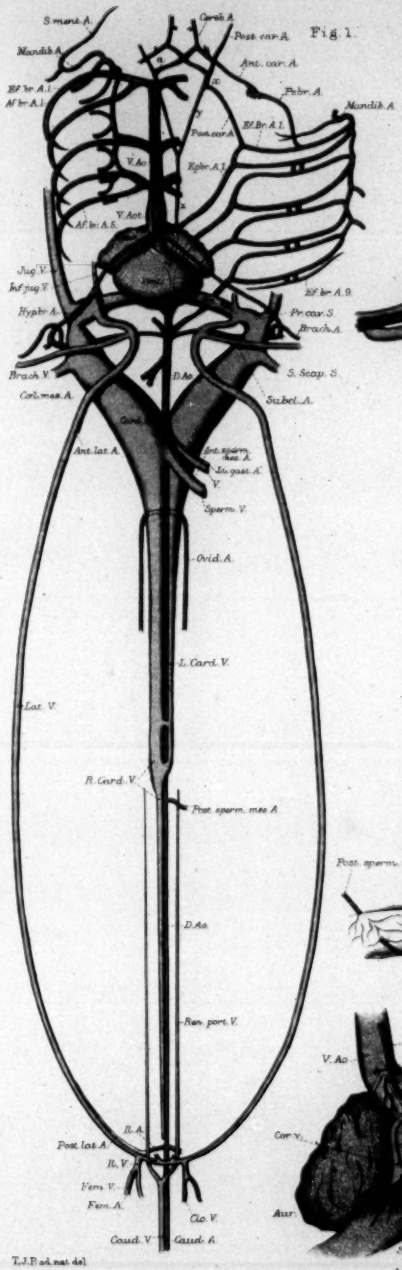


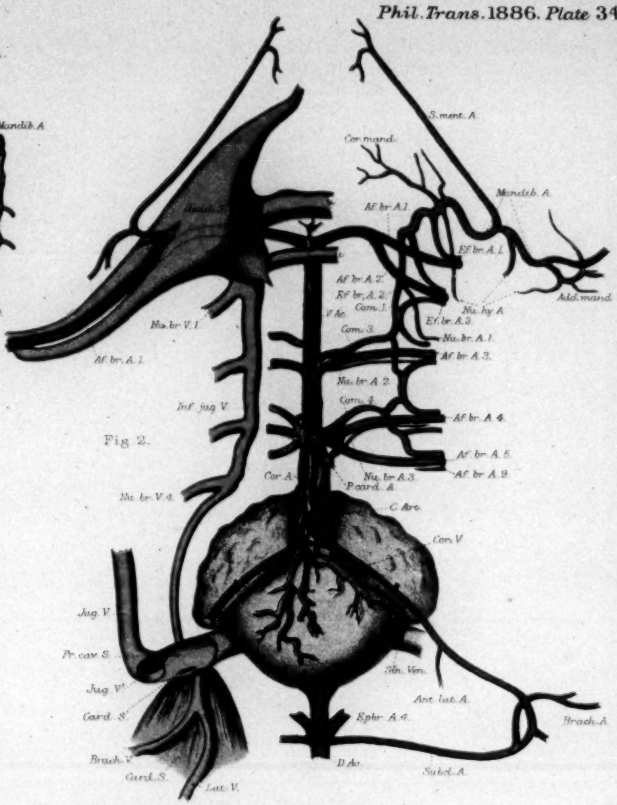
Fig 1.

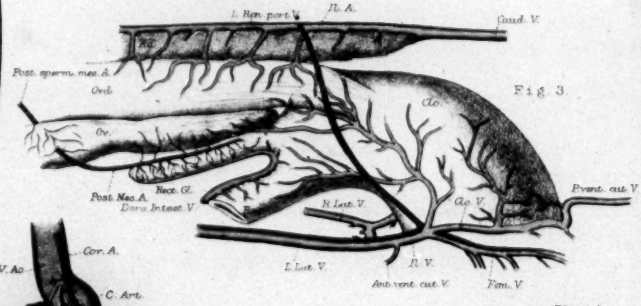


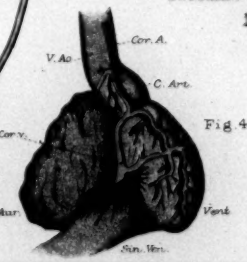
Half Size.

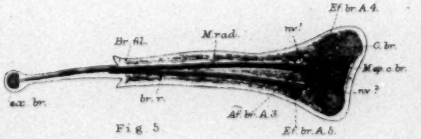
West, Newman & Co. lith.







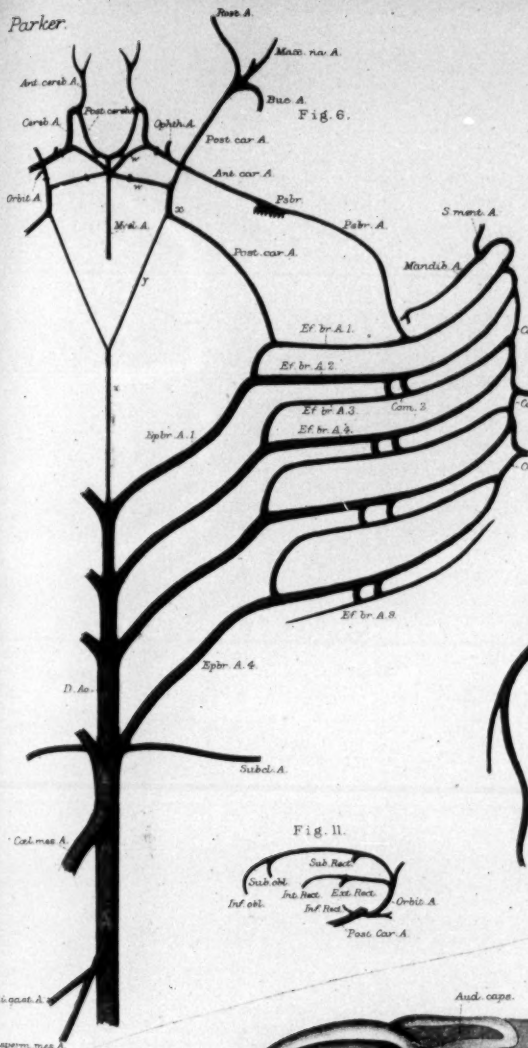


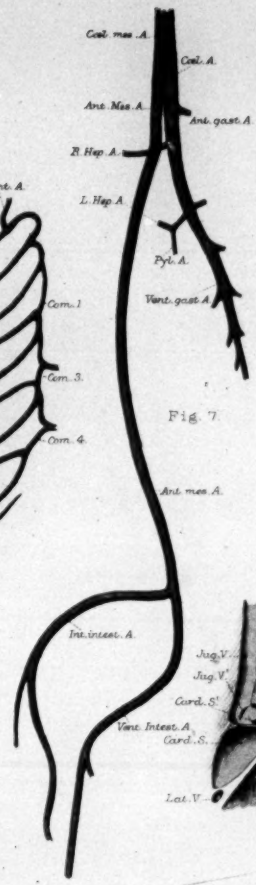


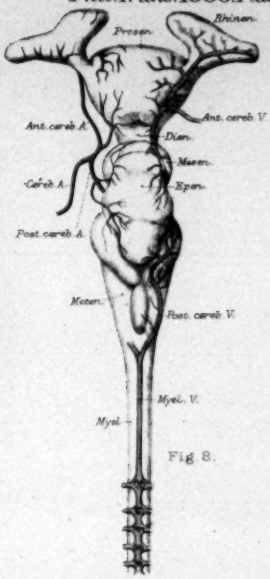
T.J.P. ad. nat. del.

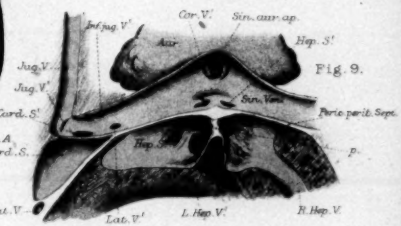
Mustelus antarcticus

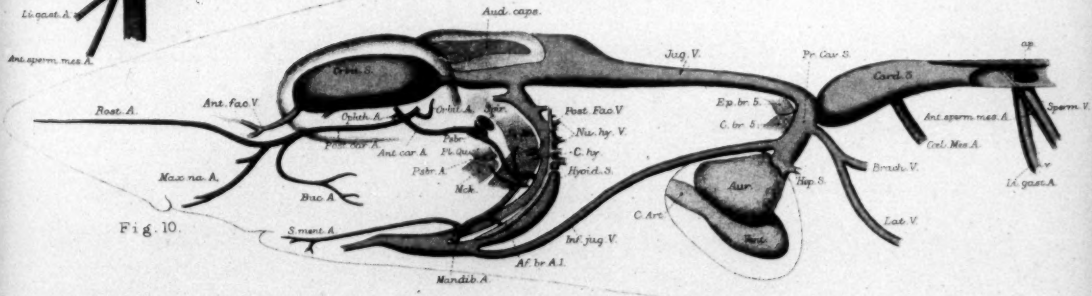
West, Newman & Co. imp.











T. J. P. ad. nat. delt.

Mustelus antarcticus.

West, Newman & Co. imp.

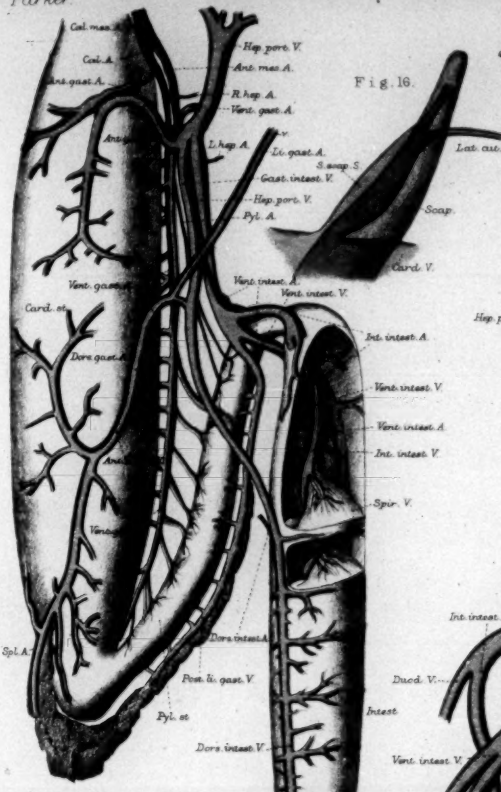
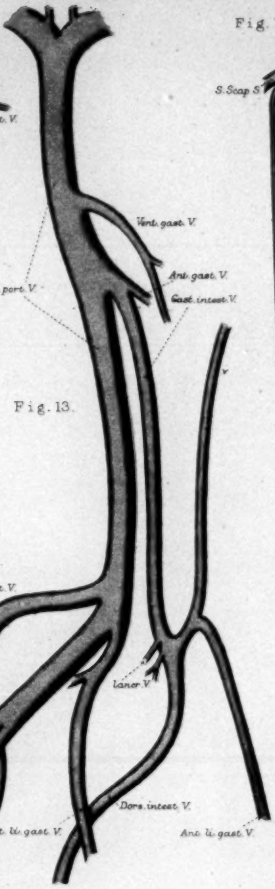


Fig. 12.

Fig. 16.



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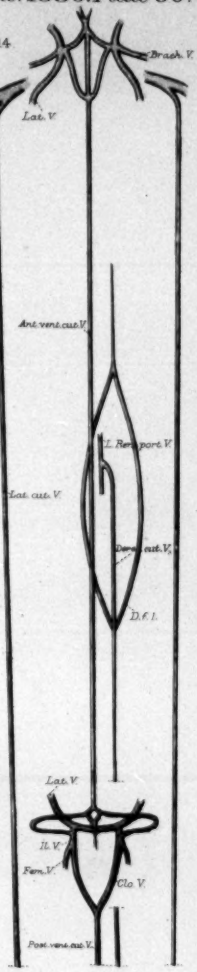

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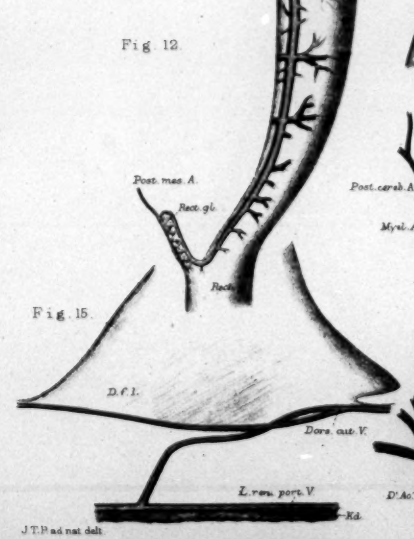

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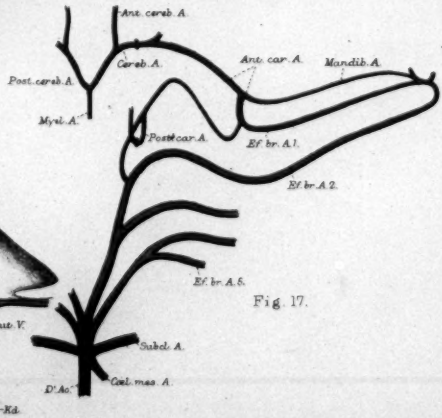

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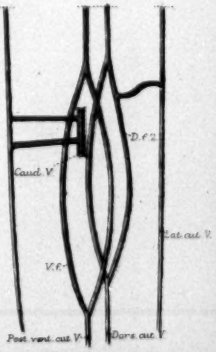
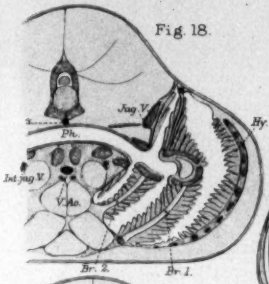
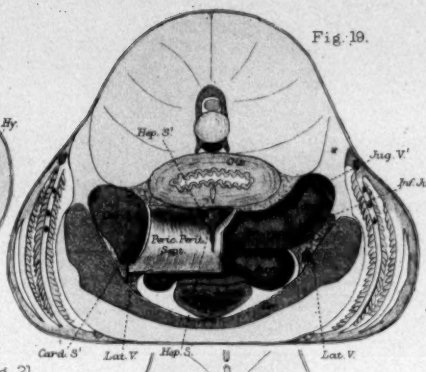


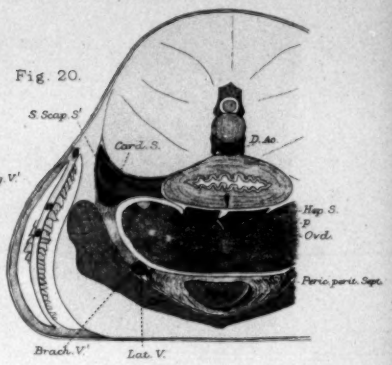
J. T. P. ad. nat. delt.

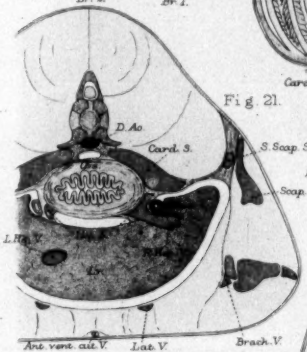
West, Newman & Co. imp.

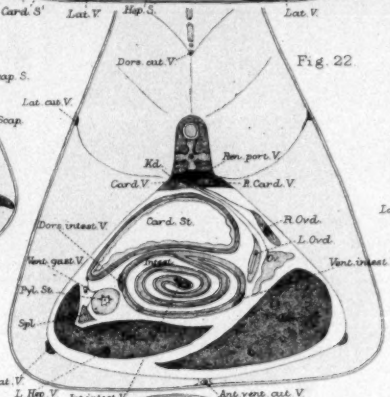
Figs. 12. 16, Mustelus antarcticus: Fig. 17, Callorhynchus antarcticus.

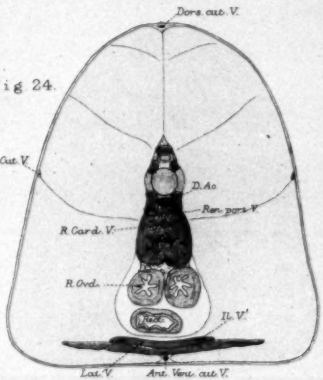


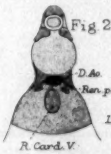


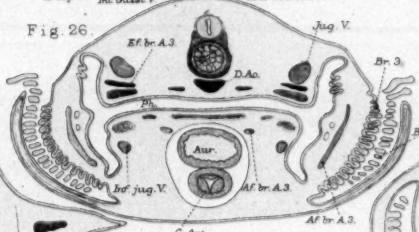


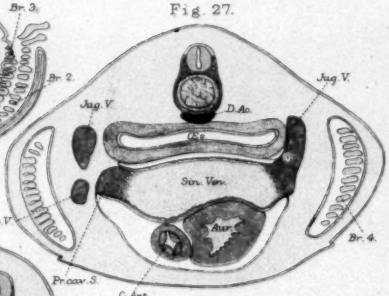


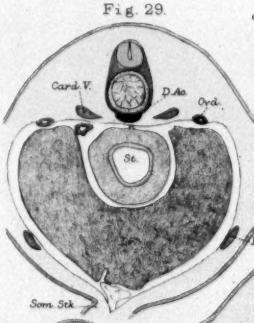


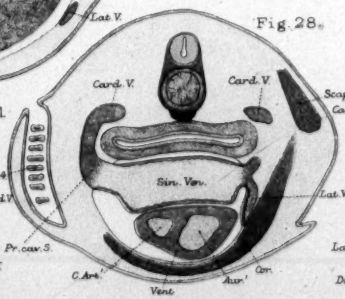


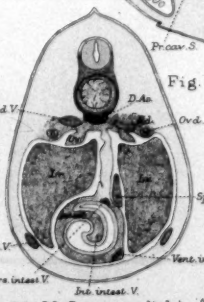


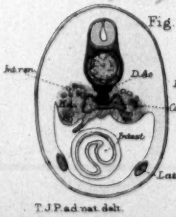


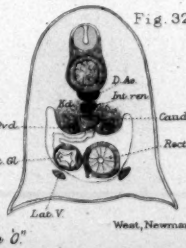












Figs 18–25, Mustelus antarcticus: Figs 26–32, Scymnus lichia, "Stage b."

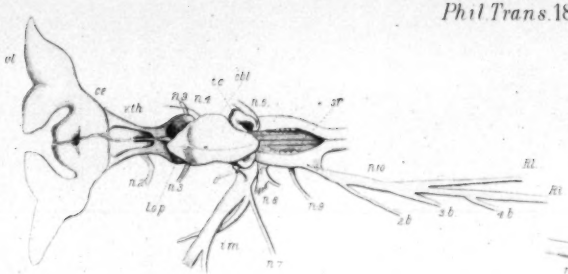
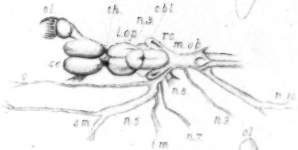


Fig 1



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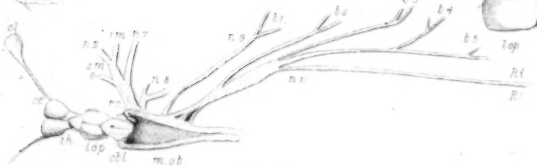
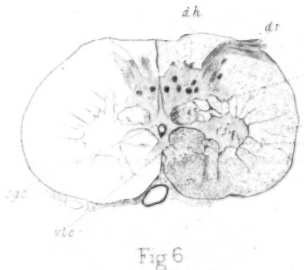


Fig 2

Fig 4

Fig 5×2



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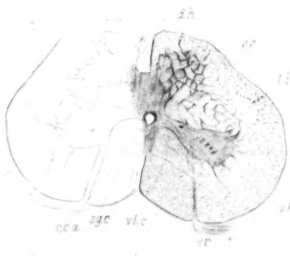
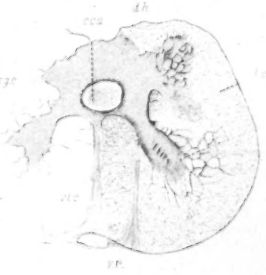


Fig 7



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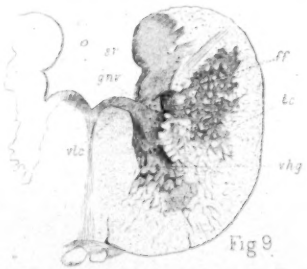

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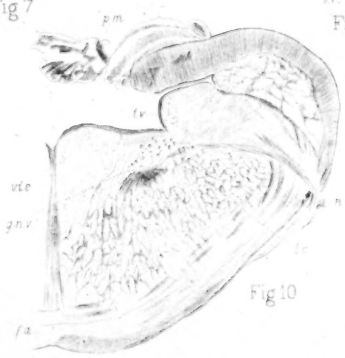


Fig 10

A. SANDERS. DEL. CAMBRIDGE SCIENTIFIC INSTRUMENT COMPANY, Ltd.

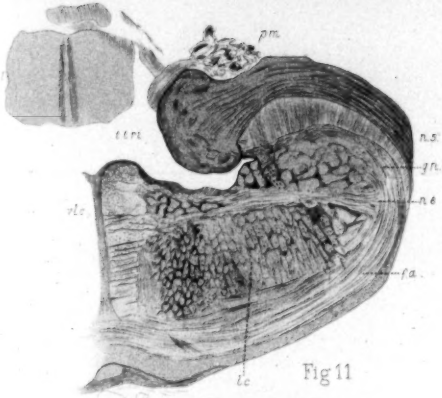


Fig 11

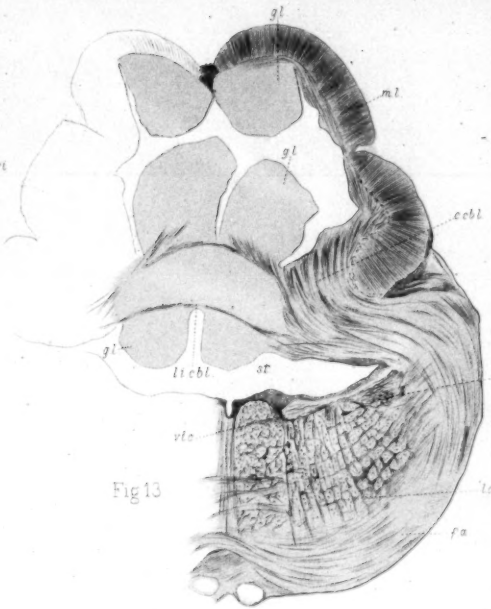


Fig 13

Fig 12

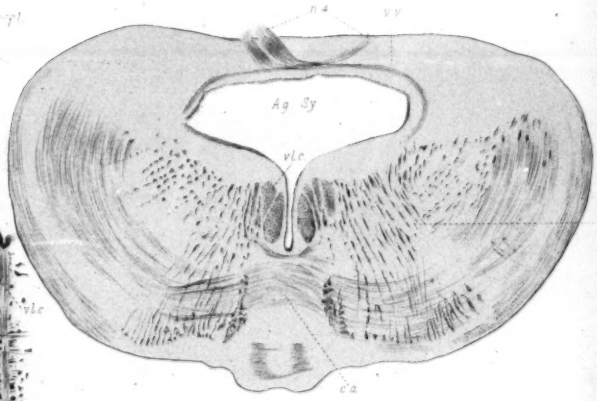


Fig 14

A. SANDERS. DEL. CAMBRIDGE SCIENTIFIC INSTRUMENT COMPANY, Lith.

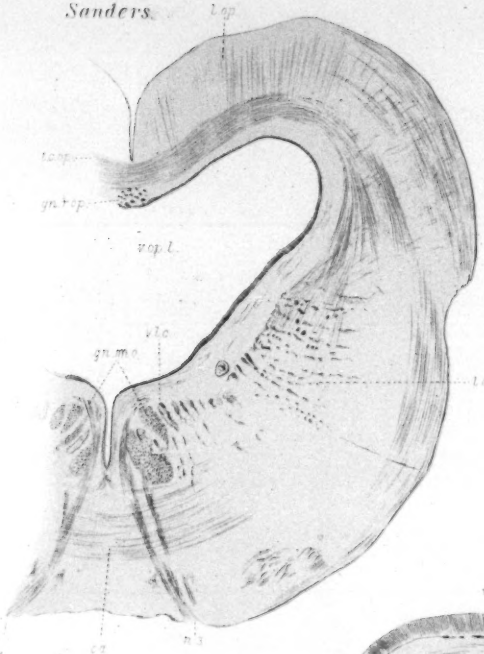


Fig 15

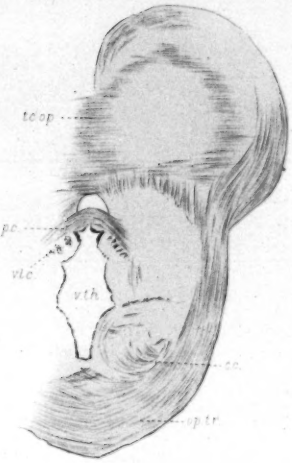
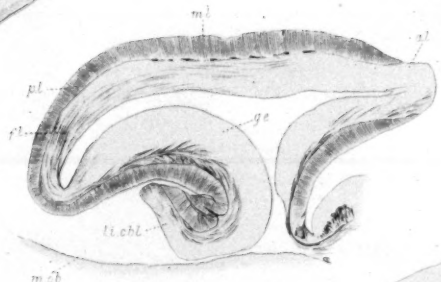


Fig 16



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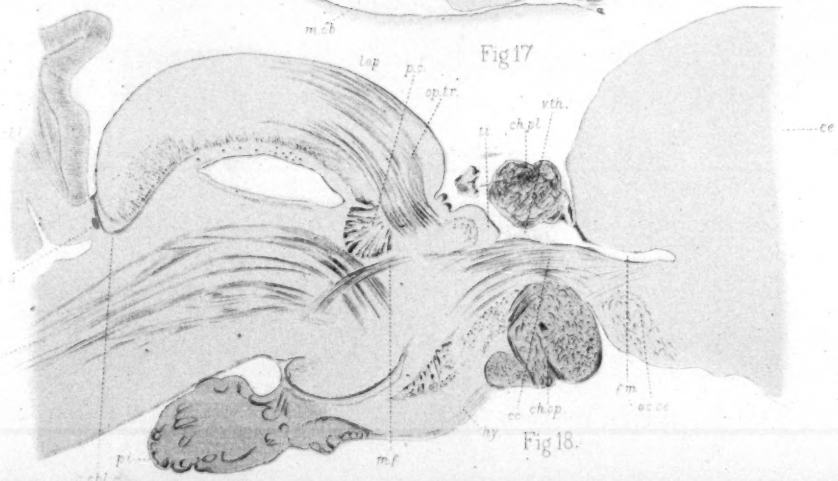

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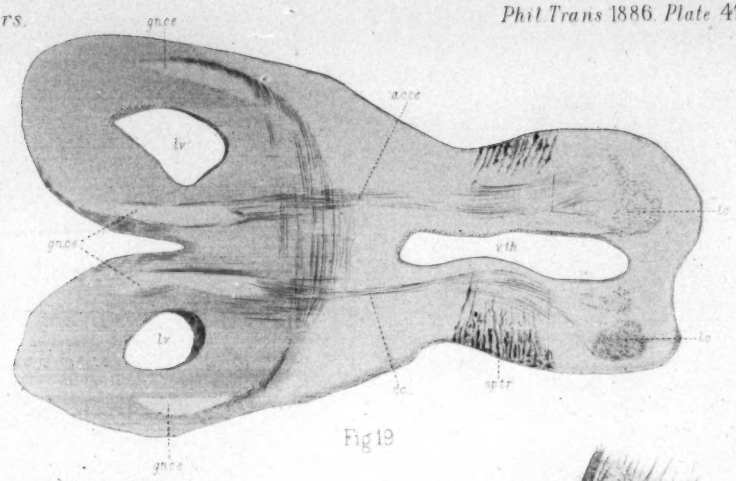

Fig 19

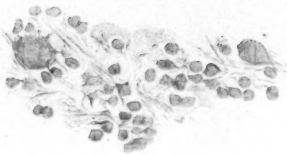

Fig 20

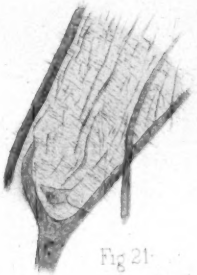

Fig 21

Fig 22

Fig 23.

A. SANDERS DEL.

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Fig. 1.

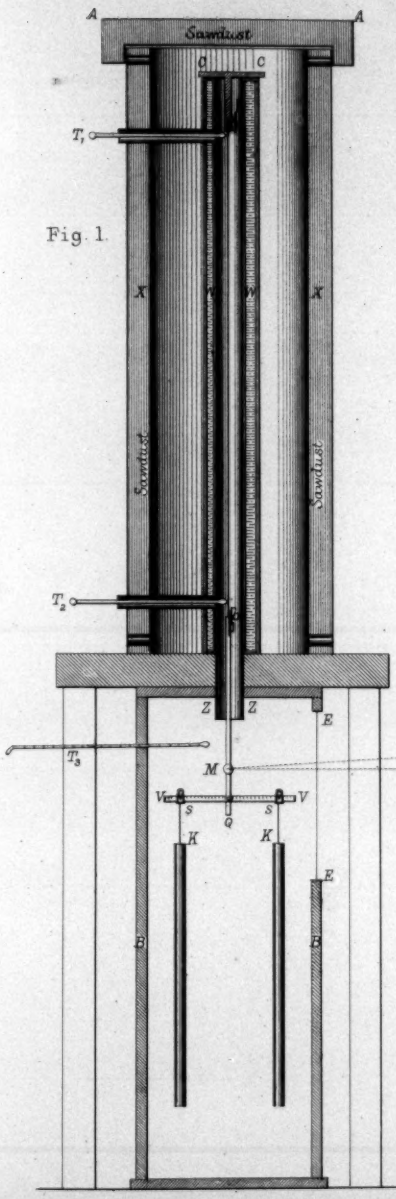
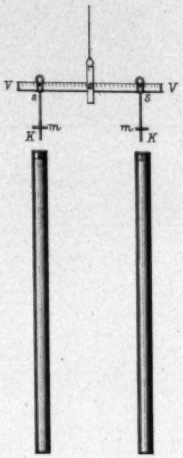


Fig. 3.



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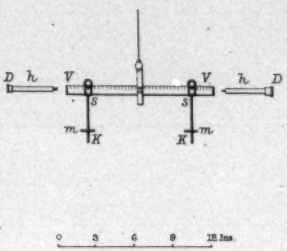
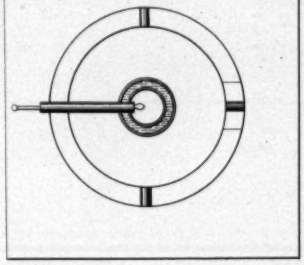


Fig 2.



West, Newman & Co. lith.

Fig. 2.

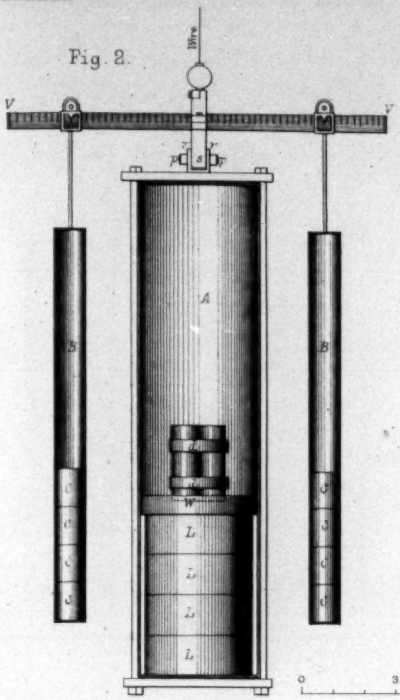
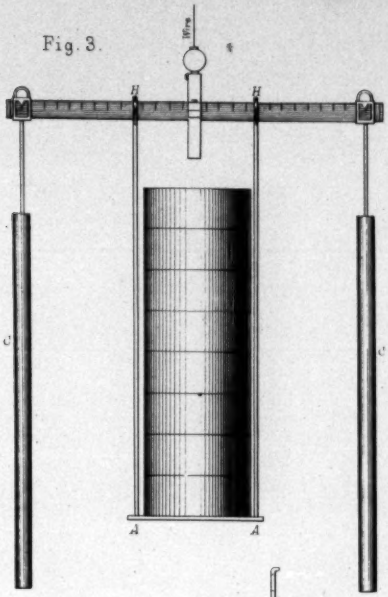


Fig. 3.



Scale of Figs. 2 & 3.

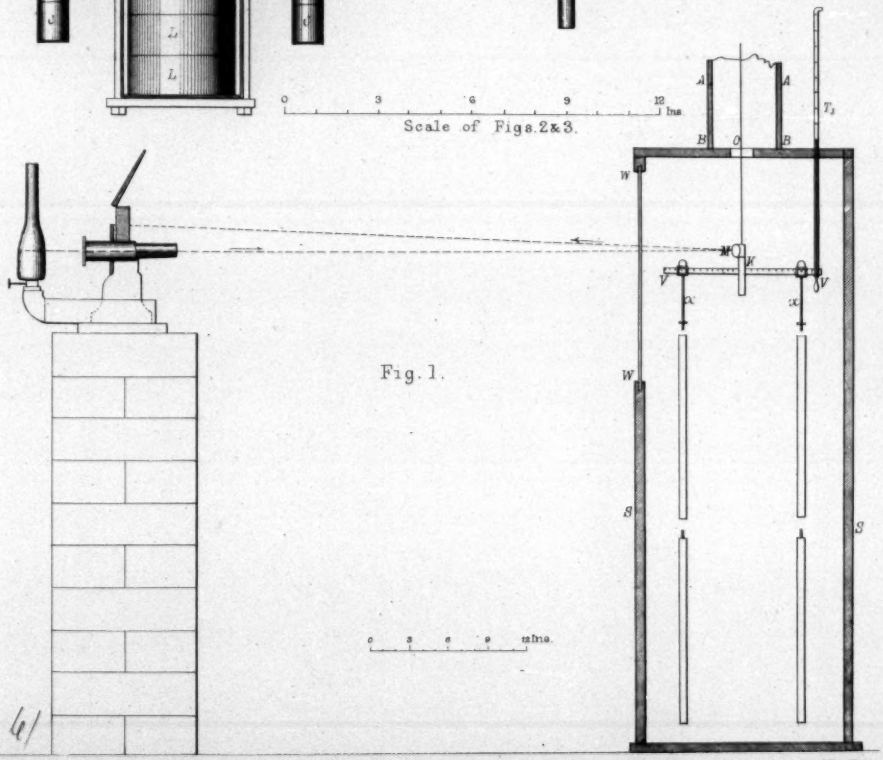


Fig. 1.

West, Newman & Co. lith.